Inequalities

We describe here a few methods/tips to prove inequalities.

1. A square is always non-negative. For any real number a, we know $a^2 \geq 0$. As strange as it may seem, some interesting inequalities can easily be proved by simply stating that some squared expression is non-negative. Of course, the name of the game for these problems is to find a suitable expression to be squared in order to end up with the result. This can be tricky. You can always do some attempt and see what you get.

2. Use the monotony of some functions. Let f be a function that takes a real number x and send it to another real number $f(x)$. For instance f_1 sending x to $3x - 13$, f_2 sending x to $x^3 + 1$, or "$\tan$" the tangent map sending an angle θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ to $\tan(\theta) := \frac{\sin(\theta)}{\cos(\theta)}$. A function f is *increasing* (respectively *decreasing*) if for all real numbers x and y verifying $x \leq y$, we have $f(x) \leq f(y)$ (respectively $f(x) \geq f(y)$). For instance f_1, f_2 and $\tan$ defined above are increasing functions.
Sometimes, to prove an inequality, you can simply write it as $f(x) \leq f(y)$ such that f is an increasing function and $x \leq y$. If you do this, make sure you can prove that x is indeed inferior to y!

3. Prove the inequality in some particular cases. When you don't know where to begin to prove your inequality, you can try to fix some variables or parameters. Dealing with the particular case might lead to the general case proof!

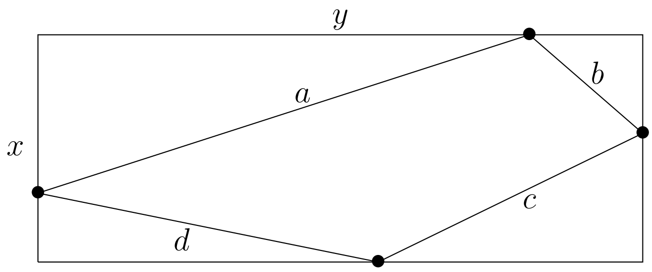
Inequality 1 (Arithmetic mean-Geometric mean inequality)**.** Let a and b be positive real numbers. Prove that

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

Inequality 2. Let a, b and c be positive real numbers. Prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$$

Inequality 3. Consider a rectangle whose sides are of lengths x and y. On each side of the rectangle chose one point. Let a, b, c and d be the lengths of segments joining the points like in the figure.



Show that

$$1 \leq \frac{a^2+b^2+c^2+d^2}{x^2+y^2} \leq 2.$$

For which choice(s) of points does the left inequality become an equality? Same question for the right?

Inequality 4 (Cauchy-Schwarz inequality)**.** Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be real numbers. Prove that

$$(a_1^2+\cdots+a_n^2)(b_1^2+\cdots+b_n^2) \geq (a_1b_1+\cdots+a_nb_n)^2.$$

Furthermore prove that there is equality if and only if there is a positive real number r such that $a_i = rb_i$ for all i or $b_i = ra_i$ for all i.

Inequality 5. Show that among any set of 5 real numbers, you can always find two numbers a and b such that

$$0 \leq \frac{a-b}{1+ab} \leq 1.$$

ANY QUESTION? JUST ASK!