Tilings and dissections

Rectangles and triangles

Tiling is a covering of a geometric objects by disjoint copies of a given set or pattern. By tilings we usually mean domino tilings, where we would like to cover a set by rectangles of size 1×2 (or $1 \times k$). Let us see an example.

Take a regular chessboard of size 8×8 *and remove two opposite corner squares. Is it true that the remaining part can be covered by* 1×2 *dominoes?*

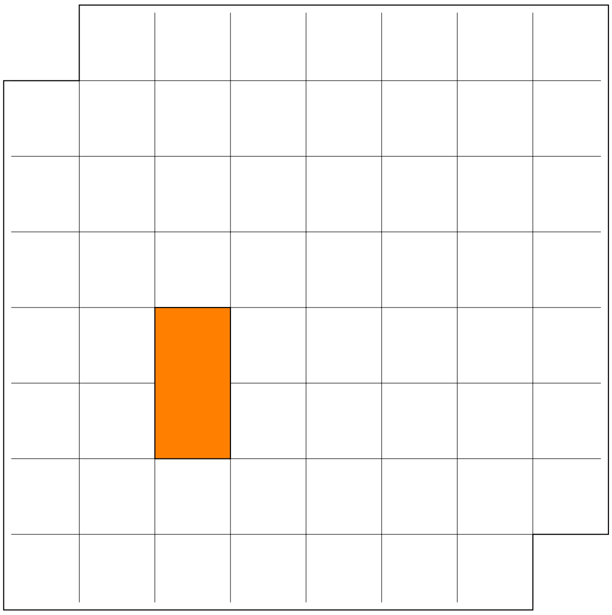

Figure 1 – A chessboard after removing two opposite corners

The answer to this question is NO. The solution is based on the *Invariant Principle* which is basically the following:

Find a quantity which does not change during the process of a tiling and which is different to the one that we want to cover. Then there is no such a covering.

This quantity can be found easily when we color the chessboard by two colors, black and white as usual (see Figure 2). The picture also shows that we removed two black corners from the original chessboard.

Every domino in a possible tiling covers a black and a white square. Thus the number of the black and the number of the white squares is the same in each domino covering.

On the other hand, after removing two black squares the number of the black squares is smaller than the number of the white ones. This shows that the remaining area (Figure 1) cannot be covered by dominoes.

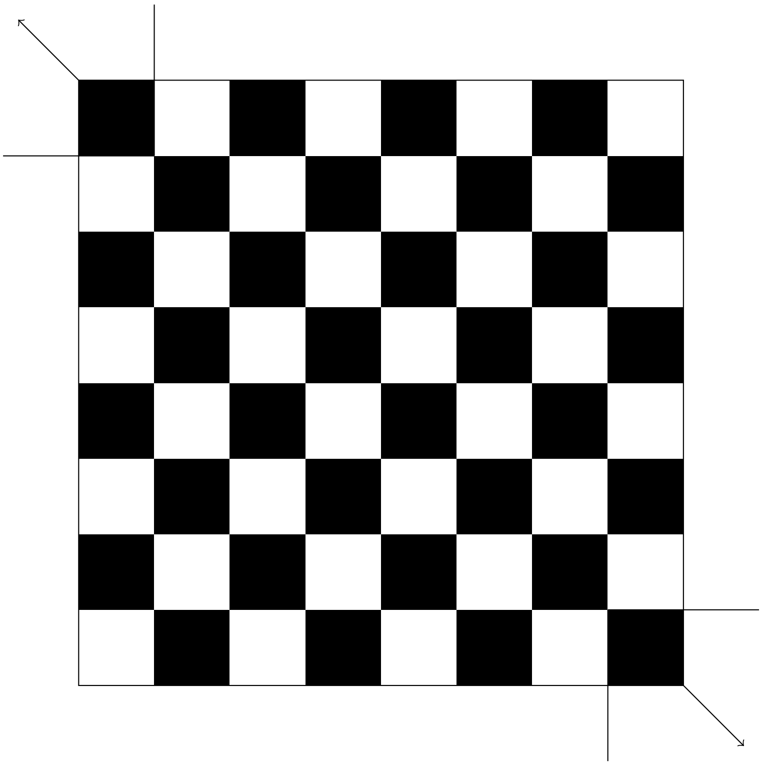

Figure 2 – The regular coloring of a chessboard

If you want to tile a given geometric shape/figure, try to do it first with a smaller grid. In the exercises try to find a particular tiling for 2×2, 4×4 or 5×5 grids instead of 8×8. It can give you good intuition how to solve the question for a bigger grid.
Generally, a *dissection* of a set means a decomposition into some smaller sets, which cover the original set. So the main difference between tiling and dissection is that in tiling we usually use the same tile for a covering while in a dissection the pieces can be arbitrary. Usually we deal with dissections into squares or triangles.

For the following exercises, you do not need to understand deep mathematical theorems, just draw and think.

1. Albert has a tablet of chocolate which can be seen as a 4×6 rectangular grid of squares as in Figure 3.

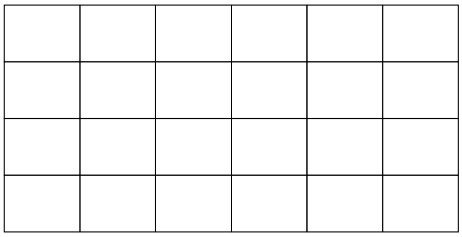

Figure 3 – Tablet of chocolate

In each step he takes one pieces of chocolate and he breaks it into two parts along a horizontal or a vertical line that can be seen in Figure 3. What is minimum number of breaks to get 24 pieces of chocolate?

2. Show that the square can be dissected into n smaller squares for all $n \geq 6$. Prove that cannot be done for $n = 5$.

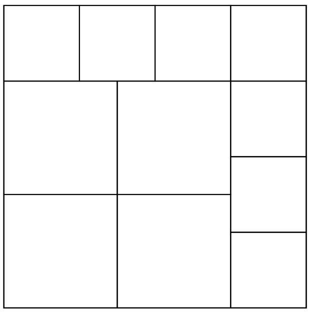

Figure 4 – Dissection of a square into 11 smaller squares

3. Remove any unit square of a 8×8 chessboard. Show that the remaining part can be tiled by L-shape dominoes (see Figure 5).

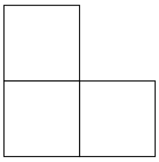

Figure 5 – An L-shape domino

4. Remove a corner of the the chessboard. Is it possible to tile this chessboard using 1×3 dominoes (see Figure 6)?

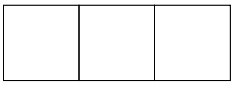

Figure 6 – A 1×3 domino

5. Show that a 10×6 rectangle cannot be tiled by 2×3 L-shapes dominoes (see Figure 7).

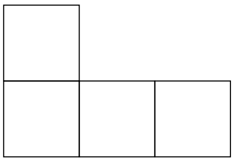

Figure 7 – A 2×3 domino

6.

a. Show that any triangle can be dissected into 4 copies of the same triangles.

b. Show that any triangle can be dissected into 9 copies of the same triangles.

c. Give an example that a triangle can be dissected into 5 copies of the same triangles.

7. Is it possible to reach the position where all of the elements are signed by "+" from the initial situation that can seen in Figure 8 if in each step we change the sign of every element of a row, a column or a diagonal?

+	+	-	+
+	+	+	+
+	+	+	+
+	+	+	+

Figure 8

8. We will call a chessboard *deficient* if one (arbitrarily chosen) square has been removed. We also say that a set can *be tiled by L-shape* if it can be covered by tiles of Figure 5. By following the steps below, prove that all $n \times n$ deficient chessboard can be tiled by L-shapes if and only if n is not divisible by 3 and $n \neq 5$.

a. Show that for a 5×5 deficient chessboard, it depends on the position of the removed square if it can be tiled by L-shapes.

b. Show that any $2k \times 3l$ board with $k, l \in \mathbb{N}$ can be tiled by L-shapes. (No square removed).

c. Show that no deficient board of size $n \times n$ can be tiled by L-shapes if n is divisible by 3.

d. Show that the 7×7 and the 11×11 deficient boards can be tiled by L-shapes.

e. Show that any $6k \times l$ board with $k, l \in \mathbb{N}$ and $l \geq 2$ can be tiled by L-shapes. (No square removed).

f. Show that every deficient board of size $(m+6k) \times (m+6k)$ can be tiled by L-shapes for every $k \in \mathbb{N}$ if $m \geq 11$ and $3 \nmid m$.

ANY QUESTION? JUST ASK!