Hydrodynamic limit of symmetric simple exclusion processes with random conductances on point processes - I

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Plan

Monday:

- Random environments and random graphs
- Homogenization results for a random walk

Thursday:

- The symmetric simple exclusion process
- Hydrodynamic limit
- Examples

Transport in disordered media

Target: Large scale limits to study transport in disordered media.

Disorder: $\begin{cases} \text{random jump rates of the interacting particles} \\ \text{random microscopic geometry} \end{cases}$

Environment ω

- ω: environment, modeling the disordered medium and describing all sources of microscopic randomness
- $\omega \in \Omega$, $(\Omega, \mathcal{F}, \mathcal{P})$ probability space
- Particles will lie on the vertexes of a random weighted graph $\mathcal{G}(\omega)$. Much studied cases:
  - $\mathbb{Z}^d$;
  - supercritical percolation cluster in $\mathbb{Z}^d$;

  with random weights (conductances).

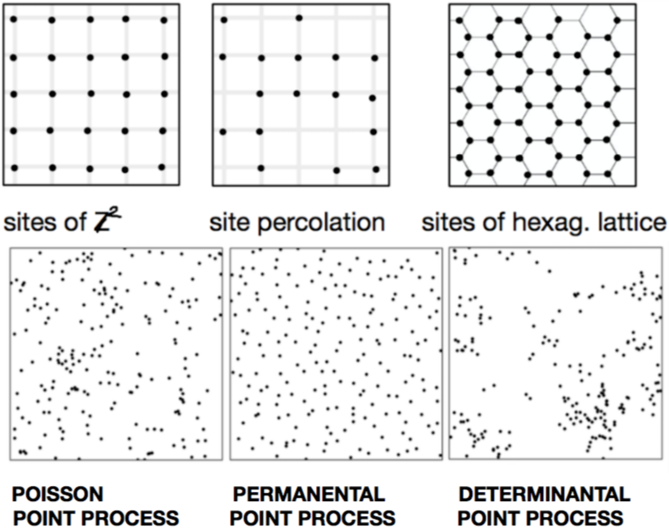
The random weighted graph $\mathcal{G}(\omega)$

Simple point process $\hat{\omega}$

We fix a **simple point process**, i.e.

$$\Omega \ni \omega \mapsto \hat{\omega} \in \{ \text{ locally finite subsets of } \mathbb{R}^d \}$$

Simple point process: examples



Conductance field

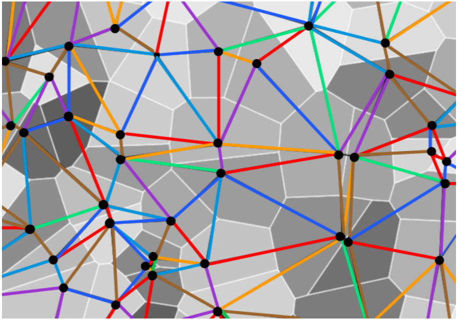
We fix a **conductance field**

$$c : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \ni (\omega, x, y) \mapsto c_{x,y}(\omega) \in [0, +\infty)$$

- $\triangleright$ $c_{x,y}(\omega) = c_{y,x}(\omega)$
- Relevant values are for $x \neq y$ in $\hat{\omega}$
- $c_{x,y}(\omega)$ is called **conductance** of the pair $\{x, y\}$

Weighted edges of $\mathcal{G}(\omega)$

- $\{$ **vertexes of $\mathcal{G}(\omega)$** $\} := \hat{\omega}$
- $\{$**edges of $\mathcal{G}(\omega)$** $\} := \{ \{x,y\} \,:\, x \neq y$ in $\hat{\omega}\,,\; c_{x,y}(\omega) > 0 \}$
- **weight of the edge $\{x,y\}$** $:=$ conductance $c_{x,y}(\omega)$



Statistical homogeneity and ergodicity of the medium

- We deal with media which are

disordered at **microscopic level**,

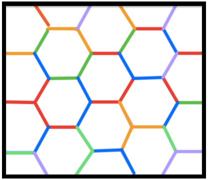
homogeneous at **macroscopic level**.

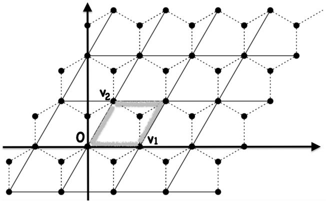
- To formalize that, we need another **MAIN INGREDIENT**: **Group** $\mathbb{G} = \mathbb{R}^d, \mathbb{Z}^d$ acting on
  - the Euclidean space $\mathbb{R}^d$
  - the probability space $(\Omega, \mathcal{F}, \mathcal{P})$

Action of $\mathbb{G}$ on the Euclidean space $\mathbb{R}^d$

- $(\tau_g)_{g\in\mathbb{G}}$, $\tau_g : \mathbb{R}^d \to \mathbb{R}^d$ translation
- Just for simplicity, here: $\tau_g x = x + g$
- In general, $\tau_g x = x + Vg$ with V invertible $d \times d$ matrix

General case with $\mathbb{G} = \mathbb{Z}^d$: relevant for graphs built on crystal lattices





In this case $V = [\mathbf{v}_1|\mathbf{v}_2]$ and $\tau_g x = x + Vg = x + g_1\mathbf{v}_1 + g_2\mathbf{v}_2$

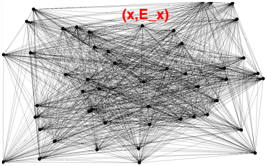
Action of $\mathbb{G}$ on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$

- Action of $\mathbb{G}$ on the probability space: $(\theta_g)_{g \in \mathbb{G}}$,

$$\theta_g : \Omega \to \Omega\,, \quad \theta_0 = \mathbb{1}\,, \quad \theta_g \circ \theta_{g'} = \theta_{g+g'} \text{ for all } g, g' \in \mathbb{G}$$

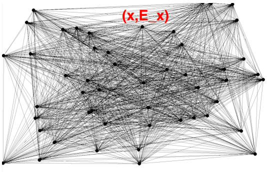
- **Paradigm**: $\theta_g \omega$ describes the updated environment when we perform a translation τ_{-g} on the medium.
- When we make a translation on the Euclidean space, **we assume to move accordingly also all sources of microscopic randomness (slot machines, coins, dice, roulette wheels,...) attached to the Euclidean space**.

Example: Mott variable range hopping



- $\omega = \{(x, E_x)\}$, $\hat{\omega} = \{x\}$.
- $c_{x,y}(\omega) := \exp\Big\{ -|x-y| - \beta(|E_x| + |E_y| + |E_x - E_y|)\Big\}$.
- $\mathcal{G}(\omega)$: complete graph on $\hat{\omega}$ with weights $c_{x,y}(\omega)$.
- Warning: figure is with $|\hat{\omega}| < +\infty$ but in the modelization $|\hat{\omega}| = +\infty$.

Example: Mott variable range hopping



- $\omega = \{(x, E_x)\}$, $\hat{\omega} = \{x\}$.
- $\mathbb{G} = \mathbb{R}^d$, $\tau_g x := x + g$.
- Recall: $\theta_g \omega$ describes the updated environment when we perform a translation τ_{-g} on the medium.
- $\theta_g \omega := \{(\tau_{-g} x, E_x)\}$

Action of $\mathbb{G}$ on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$

- **Paradigm**: $\theta_g \omega$ describes the updated environment when we perform a translation τ_{-g} on the medium.
- This is formalized by assuming some simple **covariant relations between the two $\mathbb{G}$–actions**:

For all $\omega \in \Omega$, $g \in \mathbb{G}$ and $x, y \in \widehat{\omega}$, it holds

$$\widehat{\theta_g \omega} = \tau_{-g}(\hat{\omega}) \,,$$
$$c_{\tau_{-g}x, \tau_{-g}y}(\theta_g \omega) = c_{x,y}(\omega) \,.$$

Equivalently: For all $\omega \in \Omega$, $g \in \mathbb{G}$

$$\mathcal{G}(\theta_g \omega) = \tau_{-g} \mathcal{G}(\omega) \qquad \text{as weighted graphs} \,.$$

Stationarity and ergodicity

- $\mathcal{P}$ is called $\mathbb{G}$–stationarity if $\mathcal{P}(\theta_g A) = \mathcal{P}(A)$ for all $A \in \mathcal{F}$, $g \in \mathbb{G}$.
- $A \in \mathcal{F}$ is called $\mathbb{G}$–invariant if $\theta_g A = A \ \forall g \in \mathbb{G}$.
- $\mathcal{P}$ is called $\mathbb{G}$–ergodic if $\mathcal{P}(A) \in \{0, 1\}$ for all $\mathbb{G}$–invariant $A \in \mathcal{F}$.

Assumptions (A1),..,(A4)

(A1) $\mathcal{P}$ is $\mathbb{G}$-stationary $\mathbb{G}$-and ergodic;

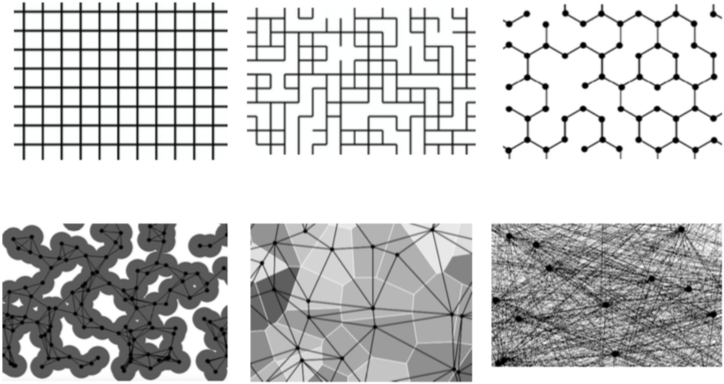
(A2) mean density m of $\hat{\omega}$ is finite and positive;

(A3) covariant relations for the two $\mathbb{G}$–actions;

(A4) for $\mathcal{P}$–a.a. ω the weighted graph $\mathcal{G}(\omega)$ is connected [it can be relaxed].

- $\mathbb{G}$-stationarity in (A1) formalizes that the medium is macroscopically homogeneous (from a statistical viewpoint)
- $m = \lim_{\ell \to +\infty} \frac{\sharp(\hat{\omega} \cap [-\ell,\ell]^d)}{(2\ell)^d}$ $\mathcal{P}$–a.s.
- It must be $\mathcal{P}(|\hat{\omega}| = +\infty) = 1$.

Examples



Palm distribution

To simplify our formulas:

$$\begin{cases} \text{(i) } \tau_g x = x + g, \\ \text{(ii) if } \mathbb{G} = \mathbb{Z}^d, \text{ then } \hat{\omega} \subset \mathbb{Z}^d \end{cases}$$

- $\mathcal{P}_0$: **Palm distribution** associated to $\mathcal{P}$ and the simple point process
- $\boxed{\mathcal{P}_0 := \mathcal{P}(\cdot|0 \in \hat{\omega})}$
- When $\mathbb{G} = \mathbb{R}^d$, $\mathcal{P}(0 \in \hat{\omega}) = 0$ and one deals with regular conditional probabilities as in [DV].

[DV] D.J. Daley, D. Vere-Jones; *An Introduction to the Theory of Point Processes.*

Palm distribution and ergodicity

$\mathcal{P}_0$ is related to ergodicity:

$$\lim_{\ell \uparrow \infty} \frac{1}{m(2\ell)^d} \sum_{x \in \hat{\omega} \cap [-\ell, \ell]^d} f(\theta_x \omega) = \int d\mathcal{P}_0(\omega) f(\omega) \qquad \mathcal{P}\text{–a.s.}$$

Warning

From now we suppose that Assumptions (A1), (A2), (A3), (A4) are satisfied without further mention

Random walk of a single particle

- Convention: $c_{x,x}(\omega) = 0$
- **Random conductance model on $\mathcal{G}(\omega)$**: continuous-time random walk $(X_t^\omega)_{t\geq 0}$ on $\hat{\omega}$ with jump rates $c_{x,y}(\omega)$.
- It waits at x an exponential random time of parameter $c_x(\omega) := \sum_{y\in\hat{\omega}} c_{x,y}(\omega)$, afterwards it jumps to $y \in \hat{\omega}$ with probability $\frac{c_{x,y}(\omega)}{c_x(\omega)}$.
- It is **well defined** if $c_x(\omega) < +\infty$ for all $x \in \hat{\omega}$ and a.s. it has no explosion.

Stochastic homogenization

- $\mathcal{P}_0$=Palm distribution associated to $\mathcal{P}$. Roughly, $\mathcal{P}_0 = \mathcal{P}(\cdot|0 \in \hat{\omega})$
- $\lambda_k(\omega) := \sum_{x\in\hat{\omega}} c_{0,x}(\omega)|x|^k$ for ω with $0 \in \hat{\omega}$

Fact: $\lambda_0 \in L^1(\mathcal{P}_0) \Longrightarrow$ the rw $(X_t^\omega)_{t\geq 0}$ is well defined for $\mathcal{P}$–a.a. ω.

Theorem (A.F. arXiv:2009.08258, to appear on AIHP)

If $\lambda_0, \lambda_2 \in L^1(\mathcal{P}_0)$, then $\mathcal{P}$–a.s. we have homogenization for the massive Poisson equation $\gamma u_\varepsilon - \mathbb{L}_\omega^\varepsilon u_\varepsilon = f_\varepsilon$ towards $\gamma u - \nabla \cdot D\nabla u = f$, where $\mathbb{L}_\omega^\varepsilon$ is the generator of $(\varepsilon X^\omega_{\varepsilon^{-2}t})_{t\geq 0}$.

- massive = $\gamma > 0$
- D = effective homogenized matrix
- The above result implies several weak forms of CLT

Stochastic homogenization

- $\mu_\omega^\varepsilon := \varepsilon^d \sum_{x\in\hat\omega} \delta_{\varepsilon x}$
- $\lim_{\varepsilon\downarrow 0} \mu_\omega^\varepsilon = m dx$ $\mathcal{P}$-a.s.
- Given $f_\varepsilon \in L^2(\mu_\omega^\varepsilon)$, $\exists! u_\varepsilon \in L^2(\mu_\omega^\varepsilon)$ s.t. $\gamma u_\varepsilon - \mathbb{L}_\omega^\varepsilon u_\varepsilon = f_\varepsilon$
- Given $f \in L^2(m dx)$, $\exists! u \in L^2(m dx)$ s.t. $\gamma u - \nabla \cdot D \nabla u = f$

Homogenization:

$$f_\varepsilon \to f \implies u_\varepsilon \to u$$

Effective homogenized matrix

Definition

We define the **effective homogenized matrix** D as the $d \times d$ nonnegative symmetric matrix such that, for all $a \in \mathbb{R}^d$,

$$a \cdot Da = \inf_{f \in L^\infty(\mathcal{P}_0)} \frac{1}{2} \int d\mathcal{P}_0(\omega) \sum_{x \in \hat{\omega}} c_{0,x}(\omega) \left(a \cdot x - \nabla f(\omega, x)\right)^2 ,$$

where $\nabla f(\omega, x) := f(\theta_x \omega) - f(\omega)$.

- D is well defined since $\lambda_2 \in L^1(\mathcal{P}_0)$
- D can be degenerate and non zero.

Weak forms of CLT

For $\mathcal{P}$–a.a. ω

$$\lim_{\varepsilon \downarrow 0} \varepsilon^d \sum_{x \in \varepsilon \hat{\omega}} \left| P^{\varepsilon}_{\omega,t} f(x) - P_t f(x) \right| = 0 \,.$$

- $P^{\varepsilon}_{\omega,t}$: Markov semigroup of $\varepsilon X^{\omega}_{\varepsilon^{-2} t}$
- P_t: Markov semigroup of limiting process, i.e. BM with diffusion matrix $2D$

For $\mathcal{P}$–a.a. ω,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^d \sum_{x \in \varepsilon \hat{\omega}} \left| R^{\varepsilon}_{\omega,\lambda} f(x) - R_\lambda f(x) \right| = 0 \,.$$

- $R^{\varepsilon}_{\omega,\lambda}$, R_λ resolvents,
- $R^{\varepsilon}_{\omega,\lambda} f = \int_0^\infty e^{-\lambda t} P^{\varepsilon}_{\omega,t} f dt, \quad R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f dt$