A shorter note on shorter pants

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Abstract. This note is about variations on a theorem of Bers about short pants decompositions of surfaces. It contains a version for surfaces with boundary but also a slight improvement on the best known bound for closed surfaces.

1. Introduction

A theorem of Bers asserts that any closed hyperbolic surface admits a short pants decomposition. More precisely, Bers exhibited the existence of a constant, that only depends on the topology of the surface, which bounds the length of the shortest pants decomposition of *any* hyperbolic surface with the given topology [3,4]. Quite a bit of effort has gone into quantifying these constants in terms of topology, including in more general cases such as surfaces with cusps or boundary and for Riemaniann surfaces [1,2,5,6,8,9].

The main goal of this note is to show the following.

Theorem 1.1. *Let X be a hyperbolic surface, possibly with geodesic boundary, and of finite area. Then X admits a pants decomposition where each curve is of length at most*

$$\max\{\ell(\partial X), \text{area}(X)\}.$$

While the context is slightly different (here we allow boundary), the techniques are very close to [12]. The main novelty is that the proof has been simplified to its bare essentials and, in the case where the surface is closed, the above statement is a slight improvement on the best known bounds.

2. Setup

Our surfaces will be orientable, finite-type and hyperbolic. They will be either closed or with boundary geodesics, where we use the convention that a cusp is a boundary geodesic of length 0. If X is a hyperbolic surface, and γ is a non-trivial homotopy class of closed curve on X, the quantity $\ell_X(\gamma)$ is the length of the unique shortest closed geodesic freely homotopic to γ. If the surface X is implicit, this will just be written as $\ell(\gamma)$.

We're interested in pants decompositions of surfaces, that is maximal collections of disjoint simple closed geodesics. They decompose a surface into pants (topologically three-holed

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spheres). The length of a pants decomposition is, by convention, the maximal length of the curves in the decomposition. A hyperbolic pair of pants has area 2π, so a hyperbolic surface X has area $\text{area}(X)$ equal to 2π times the number of pairs of pants needed to build it.

This note relies on two tools. The first is well-known and concerns surfaces with non-empty boundary.

Lemma 2.1 (Length expansion lemma)**.** *Let X be a finite-type hyperbolic surface with boundary geodesics of length $(\ell_1, \dots, \ell_n)$ and let $\varepsilon > 0$. Then there exists a hyperbolic surface $X' \cong X$ with boundary geodesics of length $(\ell_1 + \varepsilon, \dots, \ell_n)$ and such that any non-trivial simple closed curve $\gamma \subset \Sigma$ satisfies*

$$\ell_{X'}(\gamma) > \ell_X(\gamma).$$

This result, claimed in [13], continues to be used in various forms (see for example [7], [11] for a direct proof and [10] for a stronger version).

The second tool is already used in [12] and we state it here in the form of a lemma about pants. We provide a short proof idea for convenience.

Lemma 2.2. *Let Y be a hyperbolic pair of pants with geodesic boundary curves α, β, γ and geodesic seams c and h as in Figure 1. If $\ell(c) \leq 2\,\text{arcsinh}(1)$ then $\ell(\alpha) + \ell(\beta) > \ell(\gamma)$. If however $\ell(h) \leq 2\,\text{arcsinh}(1)$ then $\ell(\alpha) + \ell(\beta) < \ell(\gamma)$.*

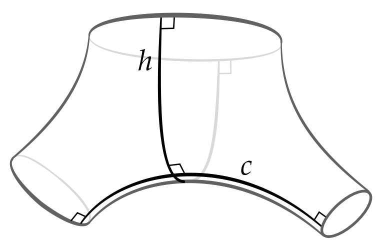

Figure 1: The paths c and h

Proof. Both statements follow from standard trigonometry computations. We first consider the hexagon with non-adjacent sides of length $\ell(\alpha)/2, \ell(\beta)/2$ and $\ell(\gamma)/2$, two copies of which form the pair of pants. In the first case, we can use $\ell(c) \leq 2\,\text{arcsinh}(1)$ and so $\cosh(\ell(c)) \leq 3$ to obtain

$$\begin{aligned}\cosh \frac{\ell(\gamma)}{2} &= \sinh \frac{\ell(\alpha)}{2} \sinh \frac{\ell(\beta)}{2} \cosh \ell(c) - \cosh \frac{\ell(\alpha)}{2} \cosh \frac{\ell(\beta)}{2} \\ &\leq \sinh \frac{\ell(\alpha)}{2} \sinh \frac{\ell(\beta)}{2} 3 - \cosh \frac{\ell(\alpha)}{2} \cosh \frac{\ell(\beta)}{2} \\ &< \cosh \left(\frac{\ell(\alpha)}{2} + \frac{\ell(\beta)}{2} \right).\end{aligned}$$

The second statement follows from splitting the hexagon into two pentagons along the perpendicular path of length $\ell(h)/2$ between c and γ (see Figure 2). Let x_α and x_β be the

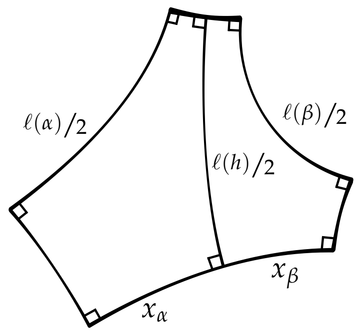

Figure 2: The hexagon and pentagons in the second case

lengths of the two subpaths of γ as indicated in the figure. Now using $h \leq 2\,\text{arcsinh}(1)$ we have

$$\begin{aligned} \sinh x_\alpha &\leq \sinh x_\alpha \sinh \tfrac{h}{2} &= \cosh \frac{\ell(\alpha)}{2} \\ \sinh x_\beta &\leq \sinh x_\beta \sinh \tfrac{h}{2} &= \cosh \frac{\ell(\beta)}{2} \end{aligned}$$

and thus

$$\frac{\ell(\gamma)}{2} = x_\alpha + x_\beta < \frac{\ell(\alpha) + \ell(\beta)}{2}$$

as claimed. □

3. Proof of Theorem 1.1

We begin with the case when X is not closed. We argue by induction on the number of pairs of pants needed to construct X. The initial step of the induction, when X is a pair of pants, holds by definition.

Now if $\ell(\partial X) < \text{area}(X)$, then by Lemma 2.1 we can increase the boundary length while increasing the length of all simple closed geodesics until the length is equal to $\text{area}(X)$. As we are proving an upper bound on curve lengths, if the statement holds for the resulting surface, it will also hold for the initial surface. Hence we can suppose that $\ell(\partial X) \geq \text{area}(X)$.

For $r > 0$, consider an r-neighborhood of ∂X. Provided r is small enough, this neighborhood is embedded and has area $\sinh(r)\ell(\partial X) < \text{area}(X)$. If we set r_0 to be the supremum of all values of r where the neighborhood is embedded we have

$$r_0 < \text{arcsinh}\left(\frac{\text{area}(X)}{\ell(\partial X)}\right) \leq \text{arcsinh}(1).$$

(Note the strict inequality still holds because the closed neighborhood cannot entirely cover the surface.) In this limit case, we have a non-trivial geodesic arc of length $2r_0 \leq 2\,\mathrm{arcsinh}(1)$ either between distinct boundary curvesor from one boundary curve to itself (see the left and right illustrations in Figure 3).

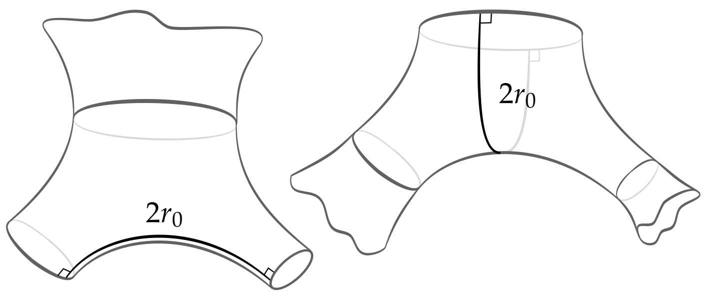

Figure 3: The two topological types for the path

In both cases, associated to this arc we have an embedded pair of pants which has either one or two curves belonging to ∂X. We can apply Lemma 2.2 to this pair of pants, and remove it from X, to obtain a surface X' with one less pair of pants and $\ell(\partial X') < \ell(\partial X)$. By induction, we are done.

We now have to prove the result when X is closed. We will cut X along a shortest simple closed geodesic (a systole) and then refer to the case of a surface with boundary to complete the systole into a full pants decomposition.

Lemma 3.1. *Any closed hyperbolic surface X has a systole of length strictly less than* $\mathrm{area}(X)/2$.

Proof. Let α be a systole and s its length. By a standard cut and paste argument, the $\frac{s}{4}$ neighborhood of α is embedded (otherwise it is easy to construct a non-trivial curve of shorter length). The area of this neighborhood is

$$2\,s\sinh\frac{s}{4} < \mathrm{area}(X).$$

Now if $s \geq 4\,\mathrm{arcsinh}(1)$, the result holds. If not, $s < 4\,\mathrm{arcsinh}(1) < 4 < 2\pi$. And any closed surface X has area at least 4π and so the result follows. □

The main result then follows by cutting along the systole to obtain a surface with boundary of length strictly less than $\mathrm{area}(X)$ (which may be possibly disconnected, but that only makes the result easier).

References

[1] Balacheff, Florent and Parlier, Hugo. Bers' constants for punctured spheres and hyperelliptic surfaces. J. Topol. Anal. 4 (2012), no. 3, 271–296.

[2] Balacheff, Florent, Parlier, Hugo and Sabourau, Stéphane. Short loop decompositions of surfaces and the geometry of Jacobians. Geom. Funct. Anal. 22 (2012), no. 1, 37–73.

[3] Bers, Lipman. Spaces of degenerating Riemann surfaces. Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), pp. 43–55. Ann. of Math. Studies, No. 79, Princeton Univ. Press, Princeton, N.J., 1974.

[4] Bers, Lipman. An inequality for Riemann surfaces. Differential geometry and complex analysis, 87–93, Springer, Berlin, 1985.

[5] Buser, Peter. Geometry and spectra of compact Riemann surfaces. Reprint of the 1992 edition. Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, 2010.

[6] Buser, Peter and Seppälä, Mika. Symmetric pants decompositions of Riemann surfaces. Duke Math. J. 67 (1992), no. 1, 39–55.

[7] Danciger, Jeffrey, Guéritaud, François, and Kassel, Fanny. Margulis spacetimes via the arc complex. Invent. Math. 204 (2016), no. 1, 133–193.

[8] Gendulphe, Matthieu. Constante de Bers en genre 2. (French) [Bers constant of genus 2] Math. Ann. 350 (2011), no. 4, 919–951.

[9] Guth, Larry, Parlier, Hugo and Young, Robert. Pants decompositions of random surfaces. Geom. Funct. Anal. 21 (2011), no. 5, 1069–1090.

[10] Papadopoulos, Athanase and Théret, Guillaume. Shortening all the simple closed geodesics on surfaces with boundary. Proc. Amer. Math. Soc. 138 (2010), no. 5, 1775–1784.

[11] Parlier, Hugo. Lengths of geodesics on Riemann surfaces with boundary. Ann. Acad. Sci. Fenn. Math. 30 (2005), no. 2, 227–236.

[12] Parlier, Hugo. A short note on short pants. Canad. Math. Bull. 57 (2014), no. 4, 870–876.

[13] Thurston, William. A spine for Teichmüller space. *Preprint* (1986).

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