Applications of a theorem of Singerman about Fuchsian groups

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Abstract. Assume that we have a (compact) Riemann surface S, of genus greater than 2, with $S = \mathbb{D}/\Gamma$, where $\mathbb{D}$ is the complex unit disc and Γ is a surface Fuchsian group. Let us further consider that S has an automorphism group G in such a way that the orbifold S/G is isomorphic to $\mathbb{D}/\Gamma'$ where Γ' is a Fuchsian group such that $\Gamma \lhd \Gamma'$ and Γ' has signature σ appearing in the list of non-finitely maximal signatures of Fuchsian groups of Theorems 1 and 2 in [S]. We establish an algebraic condition for G such that if G satisfies such a condition then the group of automorphisms of S is strictly greater than G, i. e., the surface S is more symmetric that we are supposing. In these cases, we establish analytic information on S from topological and algebraic conditions.

1 Introduction

Let $\mathbb{D}$ be the complex unit disc and $\mathcal{G}$ be the group of analytic automorphisms of $\mathbb{D}$. A Fuchsian group is a discrete subgroup Γ of $\mathcal{G}$ with compact quotient space. If Γ is such a group then its algebraic structure and the geometric structure of the quotient analytical orbifold $\mathbb{D}/\Gamma$ is given by the signature $(g; [m_1, ..., m_r])$.

The orbit space $\mathbb{D}/\Gamma$ is an orbifold with underlying surface of genus g with r cone points, i.e., there are r orbits containing the fixed points of transformations of Γ and the orbifold charts on such points are branched coverings. The integers m_i are called the proper periods of Γ. They are the orders of the cone points of $\mathbb{D}/\Gamma$ and correspond to the maximal order of the elliptic elements with a fixed point in such an orbit (the orbifold charts on such a point are branched coverings with branched index m_i on the cone point). We shall call type of the orbifold $\mathbb{D}/\Gamma$ the signature of the group Γ.

Let $\sigma = (g; [m_1, ..., m_r])$ be a given signature and T_σ be the Teichmüller space of Fuchsian groups with signature σ (see [MS]). The space T_σ is analytically equivalent to a complex ball of dimension $3g + r - 3$.

Given two signatures σ, σ' we shall denote $\sigma \subseteq \sigma'$ if there exists a Fuchsian group Γ, with signature σ, and another Γ' with signature σ', such that $\Gamma \leq \Gamma'$. If $\sigma \subseteq \sigma'$, then there is a natural emdedding $T_{\sigma'} \to T_\sigma$ and $\dim T_{\sigma'} \leq \dim T_\sigma$.

In 1971, D. Singerman obtains in [S] the complete list of pairs of signatures $\sigma \subseteq \sigma'$ with $\dim T_\sigma = \dim T_{\sigma'}$. We shall refer to this list of signatures as *the list of Singerman's Theorem.* Note that if $\sigma \subseteq \sigma'$ is a pair of signatures that is not in the list, then there are Fuchsian groups with signature σ that are finitely maximal, i. e., Fuchsian groups that are not finite index subgroups of other Fuchsian groups (see [G]). This was the original motivation for Singerman's work.

The results of [S] have been extensively used since their publication. One of the main applications is to establish if a finite group of homeomorphisms of a surface can be represented as the full group of automorphisms of a Riemann surface. The method is as follows: the orbit space of the surface by the action of a given finite group of homeomorphisms has an orbifold structure determining a Fuchsian group signature. If such a signature does not appear in the list of [S] then there is a Riemann surface with its full group of automorphisms acting topologically as the given group. For this application of the result, the important point is to *not* be in the list. In the present work, we shall consider an application of the results in [S] extracting information from the signature pairs $\sigma' \subseteq \sigma$ that *do* appear in the list.

For instance the first appearing pair of signatures is $(0; [2, 2, 2, 2, 2, 2]) \subseteq (2; [-])$ which says that every Riemann surface of genus 2 is hyperelliptic. Note that, in this example, a topological condition, the genus of the surface, gives information of analytic nature: the existence of the hyperelliptic involution. We shall obtain similar results to the above example for all signature pairs $\sigma' \subseteq \sigma$ in the list of Theorem 1 from [S]. More concretely, assume that we have a Riemann surface S admitting an automorphism group G in such a way that the orbifold type of S/G is given by a signature σ such that $\sigma' \subseteq \sigma$ is in the list. There is an algebraic condition for G such that if G satisfies this condition then the group of automorphisms of S is strictly greater than G, i. e., the surface S is more symmetric that we are supposing (Theorem 3.1). Theorem 3.4 offers an example of how to use the list of signatures corresponding to non-normal subgroups (Theorem 2 of [S]) in order to obtain similar results to Theorem 3.1.

The article is organized as follows. In Section 2, we reproduce Theorem 1 and consequences of Theorem 2 from [S] which we shall need in the sequel. Section 3 is dedicated to proving Theorems 3.1 and 3.4, as well as examples of applications.

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2 Interpretation of Singerman's theorem using orbifolds

If the signature pair $\sigma \subseteq \sigma'$ satisfies $\dim T_\sigma = \dim T_{\sigma'}$, then the embedding $T_{\sigma'} \to T_\sigma$ is onto and every Fuchsian group Γ with signature σ is contained in a Fuchsian group Γ' with signature σ'. In other words there is an orbifold covering $\mathbb{D}/\Gamma \to \mathbb{D}/\Gamma'$. Furthermore if $\Gamma \lhd \Gamma'$ then the covering $\mathbb{D}/\Gamma \to \mathbb{D}/\Gamma'$ is regular and there is a group of automorphisms F of $\mathbb{D}/\Gamma$ such that $\mathbb{D}/\Gamma/F = \mathbb{D}/\Gamma'$. Hence Theorem 1 of [S] tells us which *types* of analytic orbifolds automatically have non-trivial automorphism groups.

Theorem 2.1 (Theorem 1 of [S])

Let O be an analytic orbifold. Assume that the orbifold type σ of O is in the first column of the Table 2.1. Then the orbifold O has a group of automorphisms F as shown in the second column in the row corresponding to σ and the orbifold type of O/F is in the third column.

Note that, for example, the possibility $F = C_2$ for the orbifold type $(0, [t, t, t])$ is not listed separately as it is a special case of $(0, [t, t, u])$. The same remark will also apply to Tables 3.1 and 3.2 in the following Section.

Orbifold type of O	The group F	Orbifold type of O/F
$(2, [-])$	C_2	$(0, [2, 2, 2, 2, 2, 2])$
$(1, [t, t])$	C_2	$(0, [2, 2, 2, 2, t])$
$(1, [t])$	C_2	$(0, [2, 2, 2, 2t])$
$(0, [t, t, u, u])$	C_2	$(0, [2, 2, t, u])$
$(0, [t, t, u])$	C_2	$(0, [2, t, 2u])$
$(0, [t, t, t, t])$	D_2	$(0, [2, 2, 2, t]$
$(0, [t, t, t])$	C_3	$(0, [3, 3, t])$
$(0, [t, t, t])$	D_3	$(0, [2, 3, 2t])$

Table 2.1

In a similar way, Theorem 2 of [S] tell us which types of analytic orbifolds are automatically irregular coverings of another orbifold. For instance, Theorem 2 implies the following result which we shall use explicitly in the sequel:

Proposition 2.2

Every analytic orbifold O of type $(0; [n, 2n, 2n])$ is an index 4 irregular covering of an analytic orbifold of signature $(0; [2, 4, 2n])$.

3 Application: Groups of automorphisms of Riemann surfaces automatically extendable

We want to present surfaces having a group of automorphisms G in such a way that, if G satisfies an *algebraic* condition and the action of G on S satisfies a *topological* condition, then "automatically" the surface has more symmetry (a *geometric* property), i. e., the surface has a group of automorphisms $H \gneqq G$.

Theorem 3.1

Let S be a Riemann surface and G a group of automorphisms of S. Assume that the orbifold S/G has a signature σ_i, $i = 1, ..., 8$, appearing in column 1 of the Table 3.1. Then the following holds.

1. The group G has a presentation having as generators the elements in the entry $(i, 2)$ of the Table 3.1 and the set of relations contains the ones appearing in the entry $(i, 3)$.

2. If the group G admits the action α_i of a group F_i as described in the entry $(i, 2)$ of Table 3.2 then the group of automorphisms of the surface S has a subgroup isomorphic to $G \rtimes_{\alpha_i} F_i$.

<u>Notation:</u> In column three of Table 3.1, we use the notation $[a, b] = aba^{-1}b^{-1}$.

Orbifold type of S/G	Generators of G	Some of the relations of G
$(2,[-])$	a_1, b_1, a_2, b_2	$[a_1, b_1][a_2, b_2] = 1$
$(1,[t,t])$	a, b, x_1, x_2	$[a,b]x_1x_2 = 1$; $x_1^t = 1$; $x_2^t = 1$
$(1,[t])$	a, b, x	$[a,b]x = 1$; $x^t = 1$
$(0,[t,t,u,u])$	x_1, x_2, x_3, x_4	$x_1x_2x_3x_4 = 1$ $x_1^t = 1$, $x_2^t = 1$, $x_3^u = 1$; $x_4^u = 1$
$(0,[t,t,u])$	x_1, x_2, x_3	$x_1x_2x_3 = 1$ $x_1^t = 1$, $x_2^t = 1$, $x_3^u = 1$
$(0,[t,t,t,t])$	x_1, x_2, x_3, x_4	$x_1x_2x_3x_4 = 1$ $x_1^t = 1$, $x_2^t = 1$, $x_3^t = 1$; $x_4^t = 1$
$(0,[t,t,t])$	x_1, x_2, x_3	$x_1x_2x_3 = 1$ $x_1^t = 1$, $x_2^t = 1$, $x_3^t = 1$
$(0,[t,t,t])$	x_1, x_2, x_3	$x_1x_2x_3 = 1$ $x_1^t = 1$, $x_2^t = 1$, $x_3^t = 1$

Table 3.1

Orbifold type of S/G	Group F_i and action $\alpha_i$of $F_i$on G
$(2,[-])$	$F_1 = C_2 = \langle g : g^2 = 1 \rangle$ $\alpha_1(g)(a_1) = a_1^{-1}$; $\alpha_1(g)(b_1) = b_1^{-1}$ $\alpha_1(g)(a_2) = b_1^{-1}a_1^{-1}a_2^{-1}b_1a_1$; $\alpha_1(g)(b_2) = a_1^{-1}b_1^{-1}b_2^{-1}a_1b_1$
$(1,[t,t])$	$F_2 = C_2 = \langle g : g^2 = 1 \rangle$ $\alpha_2(g)(a) = a^{-1}$; $\alpha_2(g)(b) = b^{-1}$ $\alpha_2(g)(x_1) = b^{-1}a^{-1}x_2ab$; $\alpha_1(g)(x_2) = a^{-1}b^{-1}x_1ba$
$(1,[t])$	$F_3 = C_2 = \langle g : g^2 = 1 \rangle$ $\alpha_3(g)(a) = a^{-1}$; $\alpha_3(g)(b) = b^{-1}$ $\alpha_3(g)(x) = b^{-1}a^{-1}xab$
$(0,[t,t,u,u])$	$F_4 = C_2 = \langle g : g^2 = 1 \rangle$ $\alpha_4(g)(x_1) = x_2$; $\alpha_4(g)(x_2) = x_1$ $\alpha_4(g)(x_3) = x_1x_4x_1^{-1}$; $\alpha_4(g)(x_4) = x_2x_3x_2^{-1}$
$(0,[t,t,u])$	$F_5 = C_2 = \langle g : g^2 = 1 \rangle$ $\alpha_5(g)(x_1) = x_2$; $\alpha_5(g)(x_2) = x_1$; $\alpha_5(g)(x_3) = x_2x_3x_2^{-1}$
$(0,[t,t,t,t])$	$F_6 = C_2 \oplus C_2 = \langle g_1 : g_1^2 = 1 \rangle \oplus \langle g_2 : g_2^2 = 1 \rangle$ $\alpha_6(g_1)(x_1) = x_2$; $\alpha_6(g_1)(x_2) = x_1$; $\alpha_6(g_1)(x_3) = x_1x_4x_1^{-1}$; $\alpha_6(g_1)(x_4) = x_2x_3x_2^{-1}$ $\alpha_6(g_2)(x_1) = x_3$; $\alpha_6(g_2)(x_2) = x_4$; $\alpha_6(g_2)(x_3) = x_1$; $\alpha_6(g_2)(x_4) = x_2$
$(0,[t,t,t])$	$F_7 = C_3 = \langle g : g^3 = 1 \rangle$ $\alpha_7(g)(x_1) = x_2$; $\alpha_7(g)(x_2) = x_3$; $\alpha_7(g)(x_3) = x_1$
$(0,[t,t,t])$	$F_8 = D_3 = \langle g_1,\ g_2 : g_1^3 = 1;\ g_2^2 = 1;\ g_2g_1g_2 = g_1^{-1} \rangle$ $\alpha_8(g_1)(x_1) = x_2$; $\alpha_8(g_1)(x_2) = x_3$; $\alpha_8(g_1)(x_3) = x_1$ $\alpha_8(g_2)(x_1) = x_2$; $\alpha_8(g_2)(x_2) = x_1$; $\alpha_8(g_2)(x_3) = x_2x_3x_2^{-1}$

Table 3.2

Proof. Let S be a Riemann surface and G a group of automorphisms of S. Then we have a covering $S \to S/G$ having as monodromy epimorphism:

$$\omega : \pi_1 O(S/G) \to G.$$

We shall prove the theorem in the case that S/G has type $(1, [t, t])$, we can deal with the other cases in a similar way.

Since the orbifold type of S/G is $(1, [t, t])$, then the group $\pi_1 O(S/G)$ is isomorphic to a Fuchsian group with signature $(1, [t, t])$. So the group $\pi_1 O(S/G)$ has a canonical presentation as follows:

$$\left\langle a, b, x_1, x_2 : [a, b]x_1 x_2 = 1; x_1^t = 1; x_2^t = 1 \right\rangle.$$

The monodromy map ω ensures a presentation of G with generators $\omega(a), \omega(b), \omega(x_1), \omega(x_2)$ which contains the relations (as in column 2 and 3 of Table 3.1):

$$[\omega(a), \omega(b)]\omega(x_1)\omega(x_2) = 1,\ \omega(x_1)^t = 1,\ \omega(x_2)^t = 1.$$

As there is no danger of confusion, we shall denote $\omega(y) = y$ in the sequel.

By [S] or Theorem 2.1 the orbifold S/G admits a $C_2 = \left\langle g : g^2 = 1 \right\rangle$ action and $(S/G)/C_2$ has type $(0, [2, 2, 2, 2, t])$. In [CT] conditions are given on G to ensure that the automorphism g lifts to an automorphism $\widetilde{g}$ of S. The conditions are exactly the automorphisms described in column two of the Table 3.2. Hence the group $\langle G, \widetilde{g} \rangle \subset \mathrm{Aut}(S)$ is the subgroup that we are looking for. In order to be as self-contained as possible, we shall give the sketch of a complete proof:

Let $\theta : \pi_1 O((S/G)/C_2) \to C_2$ be the monodromy of the covering $S/G \to (S/G)/C_2$, so $\ker \theta = \pi_1 O(S/G)$.

The group $\pi_1 O((S/G)/C_2)$ has a canonical presentation:

$$\left\langle y_1, y_2, y_3, y_4, y_5 : y_1 y_2 y_3 y_4 y_5 = 1, y_1^2 = y_2^2 = y_3^2 = y_4^2 = y_5^t = 1 \right\rangle,$$

and $\theta(y_1) = \theta(y_2) = \theta(y_3) = g, \theta(y_4) = g, \theta(y_5) = 1$. Remark that since $\ker \theta = \pi_1 O(S/G)$ is an index two subgroup of $\pi_1 O((S/G)/C_2)$, we have

$$\pi_1 O((S/G)/C_2) = \langle y_1, \ker \theta \rangle.$$

Assuming that the group G admits the action α_2 of the group C_2 as described in Table 3.2, we can define $G \rtimes_{\alpha_2} C_2$. We want to construct $\varpi : \pi_1 O((S/G)/C_2) \to G \rtimes_{\alpha_2} C_2$, such that $\ker \varpi = \pi_1(S)$, because in that case we have the action of $G \rtimes_{\alpha_2} C_2$ on S which implies the result.

We set $\varpi(y_1) = (1, g)$ and $\varpi(k) = (\omega(k), 1)$ if $k \in \ker\theta$. The above equalities define an epimorphism $\varpi : \pi_1 O((S/G)/C_2) \to G \rtimes_{\alpha_2} C_2$ if the action by conjugation of $\langle y_1 \rangle$ on $\ker\theta$ is compatible with ϖ. To show this, we express the canonical generators of $\pi_1 O(S/G)$ in function of the canonical generators of $\pi_1 O((S/G)/C_2)$:

$$a = y_3 y_1,\ b = y_1 y_4,\ x_1 = bay_5 a^{-1} b^{-1},\ x_2 = y_1 y_5 y_1.$$

Then the action by conjugation of y_1 becomes

$$\begin{array}{l} y_1 a y_1 = a^{-1},\ y_1 b y_1 = b^{-1}, \\ y_1 x_1 y_1 = b^{-1} a^{-1} x_2 ab,\ y_1 x_2 y_1 = a^{-1} b^{-1} x_1 ba. \end{array}$$

The above action is compatible with ϖ by definition of $G \rtimes_{\alpha_2} C_2$.

□

As an example of an application of the above result we have the following Corollary.

Corollary 3.2
Let S be a cyclic $n-$fold covering of an elliptic surface having exactly two branched points. Then the group of automorphisms of S contains the group D_n.
Proof. We can consider $G = C_n$ and then S/C_n has type $(1, [t, t])$. In this case the automorphism given by Table 3.2 is $x \to x^{-1}$, hence the group of automorphisms of S contains $C_n \rtimes_{x \to x^{-1}} C_2 = D_n$.

□

To ensure that the theorem is not indeed void of content, one must show that the conditions of Table 3.2 are not always satisfied.

Example 3.3.
We consider the finite group P with presentation:

$$\left\langle x, y : x^4 = y^4 = 1, y^{-1} x y = x^{-1} \right\rangle (= C_4 \rtimes_{x \to x^{-1}} C_4),$$

and let G be the direct product $P \times C_3 = P \times \langle z \rangle$. Remark that G has order $4 \times 4 \times 3 = 48$.

We consider an orbifold $\mathbb{S}^2_{3,3,4}$ of type $(0, [12, 12, 4])$ and we define the epimorphism:

$$\omega : \pi_1 O(\mathbb{S}^2_{3,3,4}) \to G, \text{ given by } x_1 \to (x,z) \text{ and } x_2 \to (y, z^{-1}).$$

Now $S = \mathbb{D}/\ker\omega$ is a Riemann surface on which G acts, but G has no order two automorphisms such that:

$$(x,z) \to (y,z^{-1}) \text{ and } (y,z^{-1}) \to (x,z^{-1}),$$

since x generates a normal subgroup of P but y does not. Hence we can not apply the theorem.

An application of the list of non-normal inclusions

Theorem 2 of [S] provides a complete list of non-normal inclusions between Fuchsian groups. As a consequence, one can obtain a similar result to Theorem 3.1, this time using this other list. However, the conditions being much more involved, the list would be of considerable length and thus, instead of a a complete list, we give only one example.

Theorem 3.4.
Let S be a Riemann surface and G be a group of automorphisms of S such that the orbifold S/G has type $(0;[n,2n,2n])$. Then the following holds.
1. The group G has a presentation

$$\langle x_1, x_2, x_3 : x_1x_2x_3 = 1, x_1^{2n} = x_2^{2n} = x_3^{n} = 1, ...\rangle.$$

2. If $H = \langle x_1^2, x_2x_3x_2^{-1}, x_2^2, x_3\rangle$ is an index two subgroup of G having the automorphisms:

$$x_1^2 \to x_2x_3x_2^{-1},\ x_2x_3x_2^{-1} \to x_2^2,\ x_2^2 \to x_3,\ x_3 \to x_1^2,$$
$$x_1^2 \to x_3,\ x_3 \to x_1^2, x_2x_3x_2^{-1} \to x_3^{-1}x_2^2x_3,\ x_2^2 \to x_1^2x_2x_3x_2^{-1}x_1^{-2},$$

then $\mathrm{Aut}(S)$ contains $H \rtimes_\alpha D_4$, and G is an index four subgroup of $H \rtimes_\alpha D_4$.

Proof. Let S be a Riemann surface and G a group of automorphisms of S. Then we have a covering $S \to S/G$ having as monodromy epimorphism:

$$\omega : \pi_1 O(S/G) \to G.$$

Assume that S/G has type $(0;[n,2n,2n])$. The group $\pi_1 O(S/G)$ is isomorphic to the triangular Fuchsian group $(0;[n,2n,2n])$. So the group $\pi_1 O(S/G)$ has a canonical presentation as follows:

$$\langle x_1, x_2, x_3 : x_1x_2x_3 = 1, x_1^{2n} = x_2^{2n} = x_3^{n} = 1\rangle.$$

The monodromy map ω ensures a presentation of G with generators $\omega(x_1), \omega(x_2), \omega(x_3)$ and containing the relations

$$\omega(x_1)\omega(x_2)\omega(x_3) = 1,\ \omega(x_1)^{2n} = 1,\ \omega(x_2)^{2n} = 1,\ \omega(x_3)^n = 1.$$

In the sequel we denote $\omega(x_i) = x_i$, $i = 1, 2, 3$.

Now by Theorem 2 of [S], there is an orbifold covering $S/G \to \mathbb{S}^2_{2,4,2n}$, where $\mathbb{S}^2_{2,4,2n}$ is an orbifold of type $(0, [2, 4, 2n])$.

Since $H = \langle x_1^2, x_2x_3x_2^{-1}, x_2^2, x_3\rangle$ is an index two subgroup of G, we have the following diagram of orbifold coverings:

$$\begin{tikzcd}
 & S/H \arrow[dl] \arrow[d] \\
S/G = S/H/C_2 \arrow[r] & \mathbb{S}^2_{2,4,2n} = (S/H)/D_4
\end{tikzcd}$$

The orbifold covering $S/H \to (S/H)/D_4 = \mathbb{S}^2_{2,4,2n}$ is given by an action of the group D_4. In order to ensure that the composition $S \to S/H \to \mathbb{S}^2_{2,4,2n}$ is a regular covering, we need to construct the product $H \rtimes_\alpha D_4$. In a similar way as in the proof of Theorem 3.1, it is possible to show that the action of $D_4 = \langle r, s : s^2 = r^4 = 1; srs = r^{-1}\rangle$ on $H = \langle x_1^2, x_2x_3x_2^{-1}, x_2^2, x_3\rangle$ must be given by the automorphisms of point 2 in the hypothesis.

$\square$

Example 3.5

Consider a surface S such that there is the cyclic group $C_{2n} = \langle g\rangle$ of order $2n$ acting on S such that S/C_{2n} is an orbifold of type $(0; [n, 2n, 2n])$ and the orbifold covering $S \to S/C_{2n}$ has the monodromy epimorphism

$$\pi_1 O(S/G) = \langle x_1, x_2, x_3 : x_1x_2x_3 = 1, x_1^{2n} = x_2^{2n} = x_3^n = 1\rangle \to C_{2n} : x_1 \to g, x_2 \to g, x_3 \to g^{-2}.$$

In this case, the subgroup H in condition 2 of the Theorem is C_n and the two automorphisms are equal to $z \to z^{-1}$, so the group of automorphisms of S is strictly bigger than C_{2n}.

Remark 3.6.

A similar study can be made for group of automorphisms containing anticonformal transformations using the results in [B] and [EI].

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