Calculation of complete hairy graph cohomology with the connecting differential

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Introduction

We study hairy graph complex $\mathrm{fHGC}_{-1,-1}$:

- Vertices odd of degree -1,
- Edges even of degree 2,
- Hairs odd of degree 1,
- Edge orientation is odd,
- $d = -v + 2e + h + 1$.

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Standard differential δ splits vertices.

$$(\mathrm{fHGC}, \delta) = \left(\mathrm{H}^{\geq 1}\mathrm{fHGC}, \delta\right) \oplus (\mathrm{fGC}_{-1}, \delta)$$

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Our goal today: study $\mathcal{H}\left(\mathrm{fHGC}, \Delta\right)$.

Splitting into subcomplexes

The differential Δ does not change the number of vertices v, and does not change $a := e + h$, so

$$(\mathrm{fHGC}, \Delta) = \bigoplus_{v} (\mathrm{V}^{v}\mathrm{fHGC}, \Delta) = \bigoplus_{v,a} (\mathrm{V}^{v}\mathrm{A}^{a}\mathrm{fHGC}, \Delta)$$

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In any such subcomplex $d = -v + 2e + h + 1 = (-v + 2a + 1) - h$.

Step 1

Proposition

$\mathcal{H}^d\left(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC},\Delta\right)=0$ *unless* $d=-v+2a+1$ *(*$h=0$*) or* $d=-v+2a$ *(*$h=1$*).*

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Strategy of the proof

Taking cohomology commutes with the action of permuting vertices.
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There is a homotopy β such that $\Delta_0\beta+\beta\Delta_0=0$.
$\Rightarrow$ The homology is 0 unless $C=0$, what is possible only if $h=0$ or $h=1$.

Table of results

We need to find numbers

$$\mathrm{H}^0\mathrm{V}^v\mathrm{A}^a := dim\left(\mathcal{H}^{-v+2a+1}(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC},\Delta)\right)$$

$$\mathrm{H}^1\mathrm{V}^v\mathrm{A}^a := dim\left(\mathcal{H}^{-v+2a}(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC},\Delta)\right)$$

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$$\mathrm{H}^1\mathrm{V}^v\mathrm{A}^a := dim\left(\mathcal{H}^{-v+2a}\left(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC}, \Delta\right)\right)$$

Let us arrange them in a table:

	0		1		2		3		4		5		6		7		8	
1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	.	.	1	1	.	.	1	1	.	.	1	1	.	.	1	1	.
4	.	.	.	.	.	.	.	1	1	.	.	1	1	1	1	2	2	1
5	.	.	.	.	.	1	1	.	.	2	2	.	.	8	8	2	2	16
6	.	.	.	.	.	.	.	.	.	2	2	1	1	9	9	10	10	38
7	.	.	.	.	.	.	.	1	1	.	.	3	3	2	2	27	27	40
8	.	.	.	.	.	.	.	.	.	.	.	2	2	1	1	22	22	42
9	.	.	.	.	.	.	.	.	.	1	1	.	.	3	3	4	4	51
10	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	1	1	30
11	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	3	3	4
12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	1

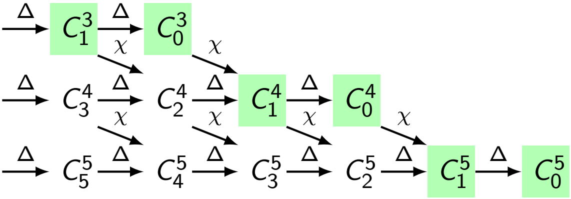
Step 2

We introduce another differential χ, adding a hair.

Lemma

- $\chi^2 = 0 \quad \Rightarrow \quad \chi$ *is a differential.*
- $\Delta\chi + \chi\Delta = 0 \quad \Rightarrow \quad \Delta + \chi$ *is a differential.*
- (fHGC, χ) *is acyclic* $\quad \Rightarrow \quad (\mathrm{fHGC}, \Delta + \chi)$ *is acyclic.*

We set up a spectral sequence on $a = e + h$:



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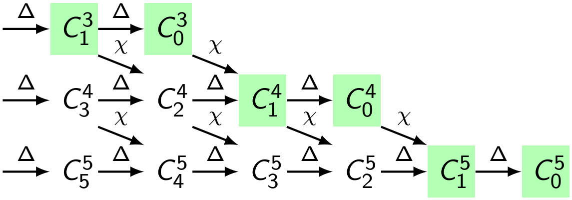

Proposition

$\mathrm{H}^0\mathrm{V}^v\mathrm{A}^a = \mathrm{H}^1\mathrm{V}^v\mathrm{A}^{a+1}$

Step 3

Characters of the action of symmetric groups can be used to calculate number of graphs with chosen number of vertices, edges and hairs. (This is the extension of the method used by WZ in 2015.)

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$\rightarrow$ Euler characteristics $E\left(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC},\Delta\right)$.

$$E\left(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC},\Delta\right)=\mathrm{H}^0\mathrm{V}^v\mathrm{A}^a-\mathrm{H}^1\mathrm{V}^v\mathrm{A}^a$$

The result

	0		1		2		3		4		5		6		7		8		9	
1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	.	.	1	1	.	.	1	1	.	.	1	1	.	.	1	1	.	.	1
4	.	.	.	.	.	.	.	1	1	.	.	1	1	1	1	2	2	1	1	3
5	.	.	.	.	.	1	1	.	.	2	2	.	.	8	8	2	2	16	16	9
6	.	.	.	.	.	.	.	.	.	2	2	1	1	9	9	10	10	38	38	54
7	.	.	.	.	.	.	.	1	1	.	.	3	3	2	2	27	27	40	40	170
8	.	.	.	.	.	.	.	.	.	.	.	2	2	1	1	22	22	42	42	230
9	.	.	.	.	.	.	.	.	.	1	1	.	.	3	3	4	4	51	51	154
10	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	1	1	30	30	83
11	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	3	3	4	4	64
12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	1	1	32
13	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	3	3	4
14	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	1
15	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	3
16	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
17	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.
18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
19	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

At least 1-valent vertices

- Step 1: $\mathcal{H}^d\left(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC}^{\geq 1}, \Delta\right) = 0$ unless $h = 0$ or $h = 1$.
- Step 2: $\left(\mathrm{fHGC}^{\geq 1}, \chi\right)$ is almost acyclic
  $\Rightarrow$ $\left(\mathrm{fHGC}^{\geq 1}, \Delta + \chi\right)$ is almost acyclic
  $\Rightarrow$ $(\mathrm{H}^0\mathrm{V}^v\mathrm{A}^a)^{\geq 1}$ is almost equal to $(\mathrm{H}^1\mathrm{V}^v\mathrm{A}^{a+1})^{\geq 1}$.
- Step 3: Number of graphs in $\mathrm{fHGC}^{\geq 1}$
  $\rightarrow$ $E\left(\mathrm{V}^v\mathrm{A}^a\mathrm{fHGC}^{\geq 1}, \Delta\right)$

The result for at least 1-valent vertices

	0		1		2		3		4		5		6		7		8		9	
1	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	.	.	.	1	.	.	1	1	.	.	1	1	.	.	1	1	.	.	1
4	.	.	.	.	.	1	.	.	.	.	.	.	.	1	1	1	1	1	1	2
5	.	.	.	.	.	.	1	.	.	2	2	.	.	7	7	1	1	15	15	7
6	.	.	.	.	.	.	.	1	.	.	.	1	1	2	2	9	9	23	23	47
7	.	.	.	.	.	.	.	.	1	.	.	2	2	.	.	18	18	17	17	123
8	.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	4	4	25	25	107
9	.	.	.	.	.	.	.	.	.	.	1	.	.	2	2	.	.	26	26	47
10	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	4	4	36
11	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	2	2	.	.	28
12	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	4
13	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	2	2	.
14	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1
15	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	2
16	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.
17	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.
18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
19	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Results on further complexes

	Step 1	Step 2	Step 3
All graphs	Done	Done	Done
At least 1-valent vertices	Done	Done	Done
At least 2-valent vertices	Doable	Done	Done
At least 3-valent vertices	Doable	Doable	Doable
Connected graphs	Maybe	Done	Done
Conn., at least 1-val.	Maybe	Done	Done
Conn., at least 2-val.	Maybe	Done	Done
Conn., at least 3-val.	Maybe	Doable	Doable

Conjectured result for connected graphs

	0		1		2		3		4		5		6		7		8		9	
1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
3	.	.	.	.	.	.	.	1	1	.	.	1	1	.	.	1	1	.	.	1
4	.	.	.	.	.	.	.	.	.	1	1	.	.	2	2	1	1	2	2	2
5	.	.	.	.	.	.	.	.	.	1	1	1	1	6	6	3	3	15	15	9
6	.	.	.	.	.	.	.	.	.	.	.	2	2	3	3	15	15	24	24	62
7	.	.	.	.	.	.	.	.	.	.	.	.	.	2	2	13	13	31	31	130
8	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	3	3	29	29	108
9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	9	9	59
10	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	19
11	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Thank you for your attention!