KNIZHNIK-ZAMOLODCHIKOV EQUATIONS FOR POSITIVE GENUS AND KRICHEVER-NOVIKOV ALGEBRAS

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Abstract. We give a global operator approach to the WZWN theory for compact Riemann surfaces of arbitrary genus with marked points. Globality means here that we use Krichever-Novikov algebras of gauge and conformal symmetries (i.e. algebras of global symmetries) instead of loop and Virasoro algebras (which are local in this context). The basic elements of this global approach are described in a previous article of the authors (Russ. Math. Surv., (**54**)(1)). In the present article we construct the conformal blocks and the projectively flat connection on the bundle constituted by them.

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Contents

1. Introduction

In this article, we consider the following problem of two-dimensional conformal field theory: induced by gauge and conformal symmetry construct a vector bundle equipped with a protectively flat connection on the moduli space of punctured Riemann surfaces[1]so that the fiber of the bundle is equal to the space of coinvariants of gauge symmetries.

This problem originates in the well-known article of V.Knizhnik and A.Zamolodchikov [9], where the case of genus zero is considered. There the following remarkable system of differential equations is obtained:

$$\left(k\frac{\partial}{\partial z_p} - \sum_{r\neq p}\frac{t^a_p t^a_r}{z_p - z_r}\right)\Psi = 0 \ , \qquad p = 1, \dots, N \ . \tag{1.1}$$

Here $z_1, \dots, z_N$ are arbitrary (generic) marked points on the Riemann sphere. For $i = 1, \dots, N$, representations t_i of a certain reductive Lie algebra $\mathfrak{g}$ are given (t^a_i being a representation matrix for the a^{th} generator of $\mathfrak{g}$) and k is a constant. A summation over a is assumed in (1.1).

Nowadays, the equations are known as the Knizhnik-Zamolodchikov (KZ) equations. They can be interpreted as horizontality conditions with respect to

[1]See Section 4 for the precise definition of what me mean by "moduli space of punctured Riemann surfaces".

the *Knizhnik-Zamolodchikov connection*. From the point of view of physics, (1.1) are equations for the N-point correlation functions in Wess-Zumino-Witten-Novikov models.

Further developments of the Knizhnik-Zamolodchikov ideas are briefly outlined in [26]. There more references can be found. Until 1987 these developments were inseparably linked with the conception of gauge and conformal symmetries based on Kac-Moody and Virasoro algebras. The higher genus generalizations in the frame of this conception are initiated by D. Bernard [1], the complete theory (conformal field theory on families of stable algebraic curves) is given by A. Tsuchiya, K. Ueno, Y. Yamada [34]. The construction of these authors locally reproduces the Knizhnik-Zamolodchikov construction.

N. Hitchin [6] proposed another (outside the frame of gauge theory) approach to the problem. By means of the geometric quantization technique, using Hamiltonian reduction, he proved the existence of a projectively flat connection in the case of closed curves (i.e. compact curves without marked points). In the preface to his article, he points out that a generalization of his technique to curves with marked points would require significant modifications, thus he did not consider this question. Later on, the required modifications were done by Y. Laszlo for *one* marked point. Observe that from the physical point of view, considering marked points is crucial, since the desired horizontal sections with respect to the connection are correlation functions of fields located at those points. Also for topological reasons taking in account marked points is of fundamental importance, since only in the presence of marked points the relation to the theory of knots and links in "handle-bodies" appears. Thus, around 1989-90 two directions in conformal field theory took shape: the Wess-Zumino-Witten-Novikov theory and the Hitchin theory. Later on, also other branches appeared: the infinite-dimensional analogue of the Borel-Weyl-Bott theory, and the noncommutative theory of theta-functions. The present work is devoted to the first of the pointed-out directions, the Wess-Zumino–Witten-Novikov theory.

The principal question which we pose in this article is the following: what is the generalization of the Knizhnik-Zamolodchikov equations for Riemann surfaces of positive genus with several marked points? This question remains in the center of current interest and is discussed in a number of articles ([3], [4] and others). The fundamental article of A. Tsuchiya, K. Ueno, Y. Yamada does not close the discussion. Attempts to clarify their approach are going on, e.g. see [3]. Our work develops the method of Tsuchiya, Ueno and Yamada. The use of Krichever-Novikov type algebras is the new ingredient we contribute.

In [10, 11, 12], I.M. Krichever and S.P. Novikov defined the basic objects of two-dimensional conformal field theory (like the energy-momentum tensor) as global meromorphic objects on a Riemann surface. They pointed out another choice for the basic gauge and conformal symmetries which are of a global nature and satisfy the *Krichever-Novikov algebras*. Krichever-Novikov algebras are higher genus algebraic-geometrical analogues of the affine Kac-Moody algebras and the Virasoro algebra, and contain them as subclasses. Their definition is based on the algebraic-geometrical data of the same type which is widely used in soliton theory. The data includes a Riemann surface with marked points and fixed jets of local coordinates there. These algebras and their *local* central extensions admit an *almost graded structure* which allows to develop a theory of representations generated by a vacuum, or, physically speaking, to incorporate the formalism of second quantization.

It is far from evidence that the construction due to Tsuchya, Ueno and Yamada can be generalized to Krichever-Novikov algebras. In the present article we resolve this problem. Actually, what we are doing is not just proving the (abstract) existence of the connection; we give an explicit construction of it and resolve the attendant geometric and deformation problems.

Krichever and Novikov dealt with the two-point case and mainly with the case of an abelian finite-dimensional Lie algebra. The corresponding multipoint algebras were introduced in [19], [22]. In [25, 26] we generalized the Krichever-Novikov results to the non-abelian multi-point case and developed our *global operator approach* to the Wess-Zumino-Witten-Novikov model. In [26], we basically formulated our approach including the general form of the Knizhnik-Zamolodchikov connection for arbitrary finite genus g and the particular forms for lower genera ($g = 0$ or $g = 1$). However, we were not able to show that our connection is well-defined on *conformal blocks* in that article, hence, we also omitted the proof of its projective flatness. Filling up these gaps is one of the goals of the present article. In particular, based on the explicit formulas (obtained in this article) for the infinitesimal deformations of Krichever-Novikov functions and vector-fields under the deformation of the complex structure, we show that the connection is well-defined. We would like to stress the fact that there still exists the problem of identifying our approach with the known ones, as well as the known ones between themselves.

We believe that the global operator approach simplifies the theory and makes more transparent its geometry and relations. It enables us to describe explicitly the Kuranishi tangent space of the moduli space in terms of Krichever-Novikov basis elements, hence to give explicitly the equations of the generalized Knizhnik-Zamolodchikov system. It is well-known that the higher genus theories are related to the representations of the fundamental group of the punctured Riemann surface ("twists" in the early terminology of conformal field theory [4]). In our approach, representations of the fundamental group naturally arise as parameters giving representations of gauge Krichever-Novikov algebras. Thus, the global geometric Langlands correspondence appears in the very beginning of the theory. One more intriguing relation can be easily formulated in the framework of our approach, namely, the relation between Knizhnik-Zamolodchikov systems and quadratic Hitchin integrals [6]. Both the Knizhnik-Zamolodchikov operator and the corresponding quantized Hitchin integral correspond to a Kuranishi tangent vector. On one hand, consider a certain pull-back e of such vector to the Krichever-Novikov vector field algebra. On the other hand, consider the corresponding derivation on the moduli space. In both cases subtract the Sugawara operator corresponding to e. In the first case this yields the quantized Hitchin integral [33], and in the second case the Knizhnik-Zamolodchikov operator. It remains beyond the intended scope of this article to give further details on this relation and to discuss the associated questions.

Two important problems remain untouched in the present article, these are the problem of unitarity of the constructed connection and the problem of its extension to the compactification of the moduli space. The problem of unitarity is posed by Hitchin in the above cited article and was resistant to the attempts of resolving it within the framework of the approaches he proposed. In the axiomatics of Hermitian tensor categories the problem was later considered by A. Kirillov (Jr.). To the best of our knowledge, the statement about the uniqueness of the unitary projectively flat connection is still missing in the literature (though many people think that it is true). As for resolving the problem of extension to the compactification, it requires investigations of the behavior of Krichever-Novikov algebras and their representations under degenerations of curves. Certain steps in this direction were done in [21], see also recent joint works of A. Fialowski and M. Schlichenmaier [36].

The present paper is organized as follows. In Section 2 and Section 3 the necessary setup is revisited from the point of view of the recent progress in the theory of multi-point Krichever-Novikov algebras [23, 24], and their representations [30, 31, 32]. Certain results are extended to be applicable to the situations considered here.

Section 4 contains the main results, namely the construction of the generalized Knizhnik-Zamolodchikov connection on the conformal block bundle on an open dense part of the moduli space, and the proof of the projective flatness of this connection.

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2. The algebras of Krichever-Novikov type

For the general set-up developed in [10] for Riemann surfaces with two marked points, and in [20], [17, 18, 19] for many points we refer to [26]. Let us introduce here some notation.

Let Σ be a compact Riemann surface of genus g, or in terms of algebraic geometry, a smooth projective curve over $\mathbb{C}$ respectively. Let

$$I = (P_1, \ldots, P_N), \quad N \geq 1$$

be a tuple of ordered, distinct points ("marked points" "punctures") on Σ and P_∞ a distinguished marked point on Σ different from P_i for every i. The points in I are called the *in-points*, and the point P_∞ the *out-point*. Let $A = I \cup \{P_\infty\}$ as a set. In [19], [20], the general case where there exists also a finite set of out-points is considered. The results presented in this section are also valid in the more general context.

2.1. The Lie algebras $\mathcal{A}$, $\overline{\mathfrak{g}}$, $\mathcal{L}$, $\mathcal{D}^1$ and $\mathcal{D}^1_{\mathfrak{g}}$.

First let $\mathcal{A} := \mathcal{A}(\Sigma, I, P_\infty)$ be the associative algebra of meromorphic functions on Σ which are regular except at the points $P \in A$. Let $\mathfrak{g}$ be a complex finite-dimensional Lie algebra. Then

$$\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A} \tag{2.1}$$

is called the *Krichever-Novikov current algebra* [10, 27, 28, 29]. The Lie bracket on $\overline{\mathfrak{g}}$ is given by the relations

$$[x \otimes A, y \otimes B] = [x, y] \otimes AB. \tag{2.2}$$

We will often omit the symbol $\otimes$ in our notation.

Let $\mathcal{L}$ denote the Lie algebra of meromorphic vector fields on Σ which are allowed to have poles only at the points $P \in A$ [10, 11, 12].

For the Riemann sphere ($g = 0$) with quasi-global coordinate z, $I = \{0\}$ and $P_\infty = \infty$, the algebra $\mathcal{A}$ is the algebra of Laurent polynomials, the current algebra $\overline{\mathfrak{g}}$ is the loop algebra, and the vector field algebra $\mathcal{L}$ is the Witt algebra. Sometimes, we refer to this case as the *classical situation* for short.

The algebra $\mathcal{L}$ operates on the elements of $\mathcal{A}$ by taking the (Lie) derivative. This enables us to define the Lie algebra $\mathcal{D}^1$ of first order differential operators as the semi-direct sum of $\mathcal{A}$ and $\mathcal{L}$. As vector space $\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{L}$. The Lie structure is defined by

$$[(g, e), (h, f)] := (e\,.\,h - f\,.\,g, [e, f]), \quad g, h \in \mathcal{A},\ e, f \in \mathcal{L}. \tag{2.3}$$

Here $e\,.\,f$ denotes taking the Lie derivative. In local coordinates $e_| = \tilde{e}\frac{d}{dz}$ and $e\,.\,f_| = \tilde{e} \cdot \frac{df}{dz}$.

Consider at last the Lie algebra $\mathcal{D}^1_{\mathfrak{g}}$ of differential operators associated to $\overline{\mathfrak{g}}$, (i.e. the *algebra of Krichever-Novikov differential operators*). As a linear space $\mathcal{D}^1_{\mathfrak{g}} = \overline{\mathfrak{g}} \oplus \mathcal{L}$. The Lie structure is given by the Lie structures on $\overline{\mathfrak{g}}$, on $\mathcal{L}$, and the additional definition

$$[e, x \otimes A] := -[x \otimes A, e] := x \otimes (e\,.\,A). \tag{2.4}$$

In particular, for $\mathfrak{g} = \mathfrak{gl}(1)$ one obtains, as a special case, $\mathcal{D}^1_{\mathfrak{g}} = \mathcal{D}^1$.

2.2. Meromorphic forms of weight λ and Krichever-Novikov duality.

Let $\mathcal{K}$ be the canonical line bundle. Its associated sheaf of local sections is the sheaf of holomorphic differentials. Following the common practice we will usually not distinguish between a line bundle and its associated invertible sheaf of local sections. For every $\lambda \in \mathbb{Z}$ we consider the bundle $\mathcal{K}^\lambda := \mathcal{K}^{\otimes\lambda}$. Here we follow the usual convention: $\mathcal{K}^0 = \mathcal{O}$ is the trivial bundle, and $\mathcal{K}^{-1} = \mathcal{K}^*$ is the holomorphic tangent line bundle. Indeed, after fixing a theta characteristics, i.e. a bundle S with $S^{\otimes 2} = \mathcal{K}$, it is possible to consider $\lambda \in \frac{1}{2}\mathbb{Z}$. Denote by $\mathcal{F}^\lambda$ the (infinite-dimensional) vector space of global meromorphic sections of $\mathcal{K}^\lambda$ which are holomorphic on $\Sigma \setminus A$. The elements of $\mathcal{F}^\lambda$ are called (meromorphic) forms or tensors of weight λ.

Special cases, which are of particular interest, are the functions ($\lambda = 0$), the vector fields ($\lambda = -1$), the 1-forms ($\lambda = 1$), and the quadratic differentials ($\lambda = 2$). The space of functions is already denoted by $\mathcal{A}$, and the space of vector fields by $\mathcal{L}$.

By multiplying sections with functions we again obtain sections. In this way the $\mathcal{F}^\lambda$ become $\mathcal{A}$-modules. By taking the Lie derivative of the forms with respect to the vector fields the vector spaces $\mathcal{F}^\lambda$ become $\mathcal{L}$-modules. In local coordinates the Lie derivative is given as

$$(e\,.g)_| := (\tilde{e}(z)\frac{d}{dz})\,.\,(\tilde{g}(z)\,dz^\lambda) := \left(\tilde{e}(z)\frac{d\tilde{g}}{dz}(z) + \lambda\,\tilde{g}(z)\frac{d\tilde{e}}{dz}(z)\right)\,dz^\lambda\ . \tag{2.5}$$

The vector spaces $\mathcal{F}^\lambda$ become $\mathcal{D}^1$-modules by the canonical definition $(g + e)\,.v = g \cdot v + e\,.v$. Here $g \in \mathcal{A}$, $e \in \mathcal{L}$ and $v \in \mathcal{F}^\lambda$. By universal constructions algebras of differential operators of arbitrary degree can be considered [20, 22].

Let ρ be a meromorphic differential which is holomorphic on $\Sigma \setminus A$ with exact pole order 1 at the points in A and given positive residues at I and given negative residues at P_∞ (of course obeying the restriction $\sum_{P\in I}\operatorname{res}_P(\rho) + \operatorname{res}_{P_\infty}(\rho) = 0$) and purely imaginary periods. There exists exactly one such ρ (see [16, p.116]). For $R \in \Sigma \setminus A$ a fixed point, the function $u(P) = \operatorname{Re}\int_R^P \rho$ is a well-defined harmonic function. The family of level lines $C_\tau := \{p \in M \mid u(P) = \tau\}$, $\tau \in \mathbb{R}$ defines a fibration of $\Sigma \setminus A$. Each C_τ separates the points in I from the point P_∞. For $\tau \ll 0$ ($\tau \gg 0$) each level line C_τ is a disjoint union of deformed circles C_i around the points P_i, $i = 1, \dots, N$ (a deformed circle C_∞ around the point P_∞). We will call any such level line or any cycle homologous to such a level line a separating cycle C_S.

Definition 2.1. The *Krichever-Novikov pairing* (*KN pairing*) is the pairing between $\mathcal{F}^\lambda$ and $\mathcal{F}^{1-\lambda}$ given by

$$\begin{gathered}\mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \ \to\ \mathbb{C}, \\ \langle f, g\rangle := \frac{1}{2\pi \mathrm{i}}\int_{C_S} f \otimes g = \sum_{P\in I}\operatorname{res}_P(f \otimes g) = -\operatorname{res}_{P_\infty}(f \otimes g),\end{gathered} \tag{2.6}$$

where C_S is any separating cycle.

The last equality follows from the residue theorem. Note that in (2.6) the integral does not depend on the separating cycle chosen. From the construction of special dual basis elements in the next subsection it follows that the KN pairing is non-degenerate.

2.3. **Krichever-Novikov bases.**

Krichever and Novikov introduced special bases (*Krichever-Novikov bases*) for the vector spaces of meromorphic tensors on Riemann surfaces with two marked points. For $g = 0$ the Krichever-Novikov bases coincide with the Laurent bases. The multi-point generalization of these bases is given by one of the authors in [19, 20] (see also Sadov [15] for some results in similar direction). We define here the Krichever-Novikov type bases for tensors of arbitrary weight λ on Riemann surfaces with N marked points as introduced in [19, 20].

For fixed λ and for every $n \in \mathbb{Z}$, and $p = 1, \ldots, N$ a certain element $f^\lambda_{n,p} \in \mathcal{F}^\lambda$ is exhibited. The basis elements are chosen in such a way that they fulfill the duality relation

$$\langle f^\lambda_{n,p}, f^{1-\lambda}_{m,r} \rangle = \delta^m_{-n} \cdot \delta^r_p \tag{2.7}$$

with respect to the pairing (2.6). In particular, this implies that the pairing is non-degenerate. Additionally, the elements fulfill

$$\mathrm{ord}_{P_i}(f^\lambda_{n,p}) = (n + 1 - \lambda) - \delta^p_i, \quad i = 1, \ldots, N. \tag{2.8}$$

The recipe for choosing the order at the point P_∞ is such that up to a scalar multiplication there is a unique such element which fulfills (2.7). To this end, for $g \geq 2$, $\lambda \neq 0, 1$ and A consisting of generic points (and without any additional requirement for $g = 0$), we require

$$\mathrm{ord}_{P_\infty}(f^\lambda_{n,p}) = -N \cdot (n + 1 - \lambda) + (2\lambda - 1)(g - 1)\ . \tag{2.9}$$

After choosing local coordinates z_p at the points P_p the scalar can be fixed by requiring

$$f^\lambda_{n,p|}(z_p) = z_p^{n-\lambda}(1 + O(z_p))\,(dz_p)^\lambda\ , \quad p = 1, \ldots, N. \tag{2.10}$$

By Riemann-Roch type arguments, it is shown in [17] that there exists only one such element. For the necessary modification for other cases see [26], [19, 20].

For the basis elements $f^\lambda_{n,p}$, explicit descriptions in terms of rational functions (for $g = 0$), the Weierstraß σ-function (for $g = 1$), and prime forms and theta functions (for $g \geq 1$) are given in [18]. For $g = 0$ and $g = 1$, such a description can be found also in [26, §§2,7]. For a description using Weierstraß $\wp$-function, see [14], [21]. The existence of such a description is necessary in our context because we want to consider the above algebras and modules over the configuration space, respectively, the moduli space of curves with marked points. In particular, one observes from the explicit representation that the basis elements vary "analytically" when the complex structure of the Riemann surface is deformed.

For the following special cases we introduce the notation:

$$A_{n,p} := f^0_{n,p}, \quad e_{n,p} := f^{-1}_{n,p}, \quad \omega^{n,p} := f^1_{-n,p}, \quad \Omega^{n,p} := f^2_{-n,p}\ . \tag{2.11}$$

For $g = 0$ and $N = 1$ the basis elements constructed coincide with the standard generators of the Witt and loop algebras, respectively. For $g \geq 1$ and $N = 1$ these elements coincide up to an index shift with those given by Krichever and Novikov [10, 11, 12].

2.4. Almost graded structure, triangular decompositions.

For $g = 0$ and $N = 1$ the Lie algebras introduced in Section 2.1 are graded. A grading is a necessary tool for developing their structure theory and the theory of their highest weight representations. For the higher genus case (and for the multi-point situation for $g = 0$) there is no grading. It is a fundamental observation due to Krichever and Novikov [10, 11, 12] that a weaker concept, an almost grading, is sufficient to develop a suitable structure and representation theory in this more general context.

An (associative or Lie) algebra is called *almost-graded* if it admits a direct decomposition as a vector space $\mathcal{V} = \bigoplus_{n\in\mathbb{Z}} \mathcal{V}_n$, with (1) $\dim \mathcal{V}_n < \infty$ and (2) there are constants R and S such that

$$\mathcal{V}_n \cdot \mathcal{V}_m \quad \subseteq \bigoplus_{h=n+m-R}^{n+m+S} \mathcal{V}_h, \qquad \forall n, m \in \mathbb{Z} \ . \tag{2.12}$$

The elements of $\mathcal{V}_n$ are called *homogeneous elements of degree* n. Let $\mathcal{V} = \oplus_{n\in\mathbb{Z}}\mathcal{V}_n$ be an almost-graded algebra and M an $\mathcal{V}$-module. The module M is called an *almost-graded* $\mathcal{V}$-module if it admits a direct decomposition as a vector space $M = \bigoplus_{m\in\mathbb{Z}} M_m$, with (1) $\dim M_m < \infty$ and (2) there are constants T and U such that

$$V_n \,.\, M_m \quad \subseteq \bigoplus_{h=n+m-T}^{n+m+U} M_h, \qquad \forall n, m \in \mathbb{Z} \ . \tag{2.13}$$

The elements of M_n are called *homogeneous elements of degree* n.

In case of $\mathcal{F}^\lambda$ the homogeneous subspaces $\mathcal{F}^\lambda_n$ are defined as the subspace of $\mathcal{F}^\lambda$ generated by the elements $f^\lambda_{n,p}$ for $p = 1, \ldots, N$. Then $\mathcal{F}^\lambda = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}^\lambda_n$.

Proposition 2.2. [19, 20] *With respect to the introduced degree, the vector field algebra $\mathcal{L}$, the function algebra $\mathcal{A}$, and the differential operator algebra $\mathcal{D}^1$ are almost-graded and the $\mathcal{F}^\lambda$ are almost-graded modules over them.*

The algebra $\mathcal{A}$ can be decomposed (as vector space) as follows:

(2.14)
$$\begin{gathered}\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_{(0)} \oplus \mathcal{A}_-,\\ \mathcal{A}_+ := \langle A_{n,p} \mid n \geq 1, p = 1, \ldots, N\rangle, \quad \mathcal{A}_- := \langle A_{n,p} \mid n \leq -K-1, p = 1, \ldots, N\rangle \ ,\\ \mathcal{A}_{(0)} := \langle A_{n,p} \mid -K \leq n \leq 0, p = 1, \ldots, N\rangle \ ,\end{gathered}$$

and the Lie algebra $\mathcal{L}$ as follows:

(2.15)
$$\begin{gathered}\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_{(0)} \oplus \mathcal{L}_-,\\ \mathcal{L}_+ := \langle e_{n,p} \mid n \geq 1, p = 1, \ldots, N\rangle, \quad \mathcal{L}_- := \langle e_{n,p} \mid n \leq -L-1, p = 1, \ldots, N\rangle \ ,\\ \mathcal{L}_{(0)} := \langle e_{n,p} \mid -L \leq n \leq 0, p = 1, \ldots, N\rangle \ .\end{gathered}$$

We call (2.14), (2.15) the *triangular decompositions*. In a similar way we obtain a triangular decomposition of $\mathcal{D}^1$.

Due to the almost-grading the subspaces $\mathcal{A}_\pm$ and $\mathcal{L}_\pm$ are subalgebras but the subspaces $\mathcal{A}_{(0)}$, and $\mathcal{L}_{(0)}$ in general are not. We use the term *critical strip* for them.

Note that $\mathcal{A}_+$, resp. $\mathcal{L}_+$ can be described as the algebra of functions (vector fields) having a zero of at least order one (two) at the points $P_i, i = 1, \ldots, N$. These algebras can be extended by adding all elements which are regular at all P_i's. This can be achieved by moving the set of basis elements $\{A_{0,p}, p = 1, \ldots, N\}$, (resp. $\{e_{0,p}, e_{-1,p}, i = 1, \ldots, N\}$ from the critical strip to these algebras. We denote the enlarged algebras by $\mathcal{A}_+^*$, resp. by $\mathcal{L}_+^*$.

On the other hand $\mathcal{A}_-$ and $\mathcal{L}_-$ could also be extended so that they contain all elements which are regular at P_∞. This is explained in detail in [26]. We obtain $\mathcal{A}_-^*$ and $\mathcal{L}_-^*$ respectively. In the same way for every $p \in \mathbb{N}_0$ let $\mathcal{L}_-^{(p)}$ be the subalgebra of vector fields vanishing of order $\ge p+1$ at the point P_∞, and $\mathcal{A}_-^{(p)}$ the subalgebra of functions respectively vanishing of order $\ge p$ at the point P_∞. We obtain a decomposition

(2.16)
$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_{(0)}^{(p)} \oplus \mathcal{L}_-^{(p)}, \quad \text{for } p \ge 0, \quad \text{and} \quad \mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_{(0)}^{(p)} \oplus \mathcal{A}_-^{(p)}, \quad \text{for } p \ge 1,$$

with "critical strips" $\mathcal{L}_{(0)}^{(p)}$ and $\mathcal{A}_{(0)}^{(p)}$, which are only subspaces. Of particular interest to us is $\mathcal{L}_{(0)}^{(1)}$ which we call reduced critical strip. For $g \ge 2$ its dimension is

$$\dim \mathcal{L}_{(0)}^{(1)} = N + N + (3g-3) + 1 + 1 = 2N + 3g - 1 \ . \tag{2.17}$$

The first two terms here correspond to the dimensions of $\mathcal{L}_0$ and $\mathcal{L}_{-1}$. The intermediate term comes from the vector fields in the basis which have poles both at the $P_i, i = 1, \ldots, N$ and at P_∞. The $1+1$ corresponds to the vector fields in the basis with exact order zero (one) at P_∞.

The almost-grading can easily be extended to the *higher genus current algebra* $\overline{\mathfrak{g}}$ by setting $\deg(x \otimes A_{n,p}) := n$. We obtain a triangular decomposition as above

$$\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_+ \oplus \overline{\mathfrak{g}}_{(0)} \oplus \overline{\mathfrak{g}}_-, \quad \text{with} \quad \overline{\mathfrak{g}}_\beta = \mathfrak{g} \otimes \mathcal{A}_\beta, \quad \beta \in \{-, (0), +\} \ , \tag{2.18}$$

In particular, $\overline{\mathfrak{g}}_\pm$ are subalgebras. The corresponding is true for the enlarged subalgebras. Among them, $\overline{\mathfrak{g}}^r := \overline{\mathfrak{g}}_-^{(1)} = \mathfrak{g} \otimes \mathcal{A}_-^{(1)}$ is of special importance. It is called the regular subalgebra.

The finite-dimensional Lie algebra $\mathfrak{g}$ can naturally be considered as subalgebra of $\overline{\mathfrak{g}}$. It lies in the subspace $\overline{\mathfrak{g}}_0$. To see this we use $1 = \sum_{p=1}^N A_{0,p}$, see [26, Lemma 2.6].

2.5. **Central extensions and 2-cohomologies.**

Let $\mathcal{V}$ be a Lie algebra and γ a Lie algebra 2-cocycle on $\mathcal{V}$, i.e. γ is an antisymmetric bilinear form obeying

$$\gamma([f,g],h) + \gamma([g,h],f) + \gamma([h,f],g) = 0, \quad \forall f,g,h \in \mathcal{V}. \tag{2.19}$$

On $\widehat{\mathcal{V}} = \mathbb{C} \oplus \mathcal{V}$ a Lie algebra structure can be defined by (with the notation $\widehat{f} := (0, f)$ and $t := (1, 0)$)

$$[\widehat{f}, \widehat{g}] := \widehat{[f, g]} + \gamma(f, g) \cdot t, \quad [t, \widehat{\mathcal{V}}] = 0. \tag{2.20}$$

The element t is a central element. Up to equivalence central extensions are classified by the elements of $\mathrm{H}^2(\mathcal{V}, \mathbb{C})$, the second Lie algebra cohomology space with values in the trivial module $\mathbb{C}$. In particular, two cocycles γ_1, γ_2 define equivalent central extensions if and only if there exist a linear form ϕ on $\mathcal{V}$ such that

$$\gamma_1(f, g) = \gamma_2(f, g) + \phi([f, g]). \tag{2.21}$$

Definition 2.3. Let $\mathcal{V} = \bigoplus_{n\in\mathbb{Z}} \mathcal{V}_n$ be an almost-graded Lie algebra. A cocycle γ for $\mathcal{V}$ is called local (with respect to the almost-grading) if there exist $M_1, M_2 \in \mathbb{Z}$ with

$$\forall n, m \in \mathbb{Z} : \quad \gamma(\mathcal{V}_n, \mathcal{V}_m) \neq 0 \implies M_2 \leq n + m \leq M_1. \tag{2.22}$$

The constants M_1 and M_2 are called upper and lower bounds respectively for the local cocycle γ.

By defining $\deg(t) := 0$ the central extension $\widehat{\mathcal{V}}$ is almost-graded if and only if it is given by a local cocycle γ. In this case we call $\widehat{\mathcal{V}}$ an *almost-graded central extension* or a *local central extension*.

In the following we consider cocycles of geometric origin. First we deal with $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{D}^1$. A thorough treatment for them is given in [23]. The proofs of the following statements and more details can be found there.

For the abelian Lie algebra $\mathcal{A}$ any antisymmetric bilinear form will be a 2-cocycle. Let C be any (not necessarily connected) differentiable cycle in $\Sigma \setminus A$ then

$$\gamma_C^{(f)} : \mathcal{A} \times \mathcal{A} \to \mathbb{C}, \quad \gamma_C^{(f)}(g, h) := \frac{1}{2\pi \mathrm{i}} \int_C g dh \tag{2.23}$$

is antisymmetric, hence a cocycle. Note that replacing C by any homologous (differentiable) cycle one obtains the same cocycle. The above cocycle is $\mathcal{L}$-*invariant*, i.e.

$$\gamma_C^{(f)}(e.g, h) = \gamma_C^{(f)}(e.h, g), \quad \forall e \in \mathcal{L}, \ \forall g, h \in \mathcal{A}. \tag{2.24}$$

For the vector field algebra $\mathcal{L}$ we generalize the standard Virasoro-Gelfand-Fuks cocycle to higher genus. To this end, we have first to choose a projective connection. It will allow us to add a counter term to the integrand to obtain a well-defined 1-differential. Let R be a global holomorphic projective connection (see e.g. [26] for the definition), C be an arbitrary cycle. Assign with them a cycle defined by

$$\gamma_{C,R}^{(v)}(e, f) := \frac{1}{24\pi \mathrm{i}} \int_C \left(\frac{1}{2}(\tilde{e}'''\tilde{f} - \tilde{e}\tilde{f}''') - R \cdot (\tilde{e}'\tilde{f} - \tilde{e}\tilde{f}') \right) dz \ . \tag{2.25}$$

Here $e_| = \tilde{e}\frac{d}{dz}$ and $f_| = \tilde{f}\frac{d}{dz}$ with local meromorphic functions $\tilde{e}$ and $\tilde{f}$. A different choice of the projective connection (even if we allow meromorphic projective connections with poles only at the points in A) yields a cohomologous cocycle, hence an equivalent central extension.

These two types of cocycles can be extended to cocycles on the whole $\mathcal{D}^1$ by setting them to be zero if one of the entries is from the complementary space. For the vector field cocycles this is true without any additional assumption, for the function algebra cocycles the $\mathcal{L}$-invariance (2.24) is necessary and sufficient. But there are other independent types of cocycles which mix functions with vector fields. To define them we first have to fix an affine connection T which is holomorphic outside A and has at most a pole of order one at P_∞. For the definition and existence of an affine connection, see [20], [31], [23]. Now

$$\gamma^{(m)}_{C,T}(e,g) := -\gamma^{(m)}_{C,T}(g,e) := \frac{1}{2\pi \mathrm{i}}\int_C (\tilde{e}\cdot g'' + T\cdot(\tilde{e}\cdot g'))\,dz \tag{2.26}$$

is a 2-cocycle. Again, the cohomology class does not depend on the chosen affine connection.

Next we consider cocycles obtained by integrating over a separating cycle C_S. Instead of γ_{C_S} we will use γ_S. Clearly, these cocycles can be expressed via residues at the points in I or equivalently at the point P_∞.

Proposition 2.4. [10, 20] *The above cocycles if integrated over a separating cycle C_S are local. In each case their upper bounds are equal to zero.*

If we replace R or T by other meromorphic connections which have poles only at A, the cocycles still will be local. The upper and lower bounds might change. If the poles are of at most order two for the projective connection, or order one for the affine connection at the points in I, then the upper bounds will remain zero. Note that such a change of the connection can always be given by adding elements from $\mathcal{F}^2$ or $\mathcal{F}^1$ to R and T respectively.

By locality, geometric cocycles give almost-graded central extensions $\widehat{\mathcal{A}}$, $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{D}}^1$. By the vanishing of the cocycles (at least if R is holomorphic) on the subalgebras $\mathcal{A}_\pm$ and $\mathcal{L}_\pm$ the subalgebras can be identified in a natural way with the subalgebras $\widehat{\mathcal{A}}_\pm$ and $\widehat{\mathcal{L}}_\pm$ of $\widehat{\mathcal{A}}$, resp. $\widehat{\mathcal{L}}$.

One of the main results of [23] is

Theorem 2.5. *(a) Every local cocycle of $\mathcal{A}$ which is $\mathcal{L}$-invariant is a multiple (over $\mathbb{C}$) of the cocycle $\gamma^{(f)}_S$. The cocycle $\gamma^{(f)}_S$ is cohomologically non-trivial.*
(b) Every local cocycle of $\mathcal{L}$ is cohomologous to a scalar multiple of $\gamma^{(v)}_{S,R}$. Moreover, the cocycle $\gamma^{(v)}_{S,R}$ defines a non-trivial cohomology class, and for every cohomologically non-trivial local cocycle a meromorphic projective connection R' which is holomorphic outside of A can be chosen such that the cocycle is equal to a scalar multiple of $\gamma^{(v)}_{S,R'}$.
(c) Every local cocycle for $\mathcal{D}^1$ is a linear combination of the above introduced

cocycles $\gamma_S^{(f)}$, $\gamma_{S,R}^{(v)}$ *and* $\gamma_{S,T}^{(m)}$ *up to coboundary, i.e.*

$$\gamma = r_1\gamma_S^{(f)} + r_2\gamma_{S,T}^{(m)} + r_3\gamma_{S,R}^{(v)} + \textit{coboundary}, \quad r_1, r_2, r_3 \in \mathbb{C}. \tag{2.27}$$

The three basic cocycles are linearly independent in the cohomology space. If the coefficients r_2 *and* r_3 *in the linear combination are non-zero, then a meromorphic projective connection* R' *and an affine connections* T'*, both holomorphic outside* A*, can be found such that* $\gamma = r_1\gamma_S^{(f)} + r_2\gamma_{S,T'}^{(m)} + r_3\gamma_{S,R'}^{(v)}$.

2.6. Affine algebras.

Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra. Above, we introduced the current algebra $\overline{\mathfrak{g}}$ together with its almost-grading. In this subsection we study central extensions of $\overline{\mathfrak{g}}$. Given an invariant, symmetric bilinear form $\alpha(.,.)$, i.e. a form obeying $\alpha([x,y],z) = \alpha(x,[y,z])$ we define as generalization of the Kac-Moody algebras of affine type the *higher genus (multi-point) affine Lie algebra*. We call it also a *Krichever-Novikov algebra of affine type*. It is the Lie algebra based on the vector space $\widehat{\mathfrak{g}} = \overline{\mathfrak{g}} \oplus \mathbb{C}\,t$ equipped with Lie structure

$$[\widehat{x\otimes f}, \widehat{y\otimes g}] = \widehat{[x,y]\otimes (fg)} + \alpha(x,y)\cdot\gamma_{C_S}(f,g)\cdot t, \qquad [\,t,\widehat{\mathfrak{g}}] = 0\ , \tag{2.28}$$

where

$$\gamma_{C_S}(f,g) = \frac{1}{2\pi \mathrm{i}}\int_{C_S} f dg \tag{2.29}$$

is the geometric cocycle for the function algebra obtained by integration along a separating cycle C_S. We denote this central extension by $\widehat{\mathfrak{g}}_{\alpha,S}$. It depends on the bilinear form α. As usual we set $\widehat{x\otimes f} := (0, x\otimes f)$. The cocycle defining the central extension $\widehat{\mathfrak{g}}_{\alpha,S}$ is local. Hence we can extend our almost-grading to the central extension by setting $\deg t := 0$ and $\deg(\widehat{x\otimes A_{n,p}}) := n$. Again we obtain a triangular decomposition

$$\widehat{\mathfrak{g}}_{\alpha,S} = \widehat{\mathfrak{g}}_+ \oplus \widehat{\mathfrak{g}}_{(0)} \oplus \widehat{\mathfrak{g}}_- \quad \text{with} \quad \widehat{\mathfrak{g}}_\pm \cong \overline{\mathfrak{g}}_\pm \quad \text{and} \quad \widehat{\mathfrak{g}}_{(0)} = \overline{\mathfrak{g}}_{(0)} \oplus \mathbb{C}\cdot t\ . \tag{2.30}$$

The corresponding is true for the enlarged subalgebras. Among them,

$$\widehat{\mathfrak{g}}^r := \widehat{\mathfrak{g}}_-^{(1)} = \overline{\mathfrak{g}}_-^{(1)} = \mathfrak{g}\otimes\mathcal{A}_-^{(1)}, \qquad \widehat{\mathfrak{g}}_+^{*,ext} = \overline{\mathfrak{g}}_+^* \oplus \mathbb{C}\,t = (\mathfrak{g}\otimes\mathcal{A}_+^*)\oplus\mathbb{C}\,t\ .$$

are of special interest.

Instead integrating over a separating cycle in (2.29) we could integrate over any other cycle C and obtain in this way another (in general, non-equivalent and even non-isomorphic) central extension $\widehat{\mathfrak{g}}_{\alpha,C}$. In addition, there is no a priori reason why every cocycle defining a central extension of $\overline{\mathfrak{g}}$ should be of this type, i.e. should be obtained by choosing an invariant symmetric bilinear form α and integrating the differential fdg over a cycle.

Before we can formulate the results needed in our context we have to extend the definition of $\mathcal{L}$-invariance of cocycles to $\overline{\mathfrak{g}}$.

Definition 2.6. A cocycle γ of $\overline{\mathfrak{g}}$ is called $\mathcal{L}$-*invariant* if

$$\gamma(x(e.g), y(h)) + \gamma(x(g), y(e.h)) = 0, \qquad \forall x,y\in\mathfrak{g},\ e\in\mathcal{L},\ g,h\in\mathcal{A}. \tag{2.31}$$

The cocycles introduced above are obviously $\mathcal{L}$-invariant.

Theorem 2.7. [24, Thm. 3.13, Cor. 3.14] *(a) Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, then every local cocycle of the current algebra $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{A}$ is cohomologous to a cocycle given by*

$$\gamma(x \otimes f, y \otimes g) = r \cdot \frac{\beta(x,y)}{2\pi \mathrm{i}} \int_{C_S} f dg, \quad \text{with } r \in \mathbb{C}, \tag{2.32}$$

and with β the Cartan-Killing form of $\mathfrak{g}$. In particular, every local cocycle is cohomologous to a local and $\mathcal{L}$-invariant cocycle.
(b) If the cocycle is already local and $\mathcal{L}$-invariant, then it coincides with the cocycle (2.32) with $r \in \mathbb{C}$ suitable chosen.
(c) For $\mathfrak{g}$ simple, up to equivalence and rescaling of the central element there is a unique non-trivial almost-graded central extension $\widehat{\mathfrak{g}}$ of its higher genus multi-point current algebra $\overline{\mathfrak{g}}$. It is given by the cocycle (2.32).

Next, let $\mathfrak{g}$ be an arbitrary complex reductive finite-dimensional Lie algebra, and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_M \tag{2.33}$$

be its decomposition into its abelian ideal $\mathfrak{g}_0$ and simple ideals $\mathfrak{g}_1, \cdots, \mathfrak{g}_M$. For the corresponding current algebra we obtain $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_0 \oplus \overline{\mathfrak{g}}_1 \oplus \cdots \oplus \overline{\mathfrak{g}}_M$.

Theorem 2.8. [24, Thm. 3.20] *(a) Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra. Given a cocycle γ for $\overline{\mathfrak{g}}$ which is local, and whose restriction to $\overline{\mathfrak{g}}_0$ is $\mathcal{L}$-invariant, there exists a symmetric invariant bilinear form α for $\mathfrak{g}$ such that γ is cohomologous to*

$$\gamma'_{\alpha,S}(x \otimes f, y \otimes g) = \frac{\alpha(x,y)}{2\pi \mathrm{i}} \int_{C_S} f dg. \tag{2.34}$$

Vice versa, every such α determines a local cocycle.
(b) If the cocycle γ is already local and $\mathcal{L}$-invariant for the whole $\overline{\mathfrak{g}}$, then it coincides with the cocycle $\gamma'_{a,S}$.
(c)

$$\dim \mathrm{H}^2_{loc,\mathcal{L}}(\overline{\mathfrak{g}}, \mathbb{C}) = \frac{n(n+1)}{2} + M. \tag{2.35}$$

Here $\mathrm{H}^2_{loc,\mathcal{L}}(\overline{\mathfrak{g}}, \mathbb{C})$ denotes the subspace of those cohomology classes which have a local and $\mathcal{L}$-invariant cocycle as representative.

2.7. Central extensions of $\mathcal{D}^1_{\mathfrak{g}}$.

Recall that $\mathcal{D}^1_{\mathfrak{g}} = \overline{\mathfrak{g}} \oplus \mathcal{L}$ with $[e, xA] = x(e.A)$, see (2.4). There is a short exact sequence of Lie algebras

$$0 \longrightarrow \overline{\mathfrak{g}} \xrightarrow{i_1} \mathcal{D}^1_{\mathfrak{g}} \xrightarrow{p_2} \mathcal{L} \longrightarrow 0. \tag{2.36}$$

First note again that by the almost-grading of $\mathcal{L}$ and $\overline{\mathfrak{g}}$ and by the fact that $\mathcal{A}$ is an almost-graded $\mathcal{L}$-module the algebra $\mathcal{D}^1_{\mathfrak{g}}$ is an almost-graded algebra. Local cocycles and central extensions of this algebra, as well as of the above mentioned algebras, are described in [24] (the absence of any mixing type cocycle in the semi-simple case is observed in [31]) By restricting a local cocycle for $\mathcal{D}^1_{\mathfrak{g}}$ to the subalgebra $\overline{\mathfrak{g}}$ one obtains a local cocycle for $\overline{\mathfrak{g}}$. In the semi-simple case we obtain

Theorem 2.9. *(a) Let $\mathfrak{g}$ be a semi-simple Lie algebra and γ a local cocycle of $\mathcal{D}^1_{\mathfrak{g}}$. Then there exists a symmetric invariant bilinear form α for $\mathfrak{g}$ such that γ is cohomologous to a linear combination of the local cocycle $\gamma_{\alpha,S}$ given by (2.34) and of the local cocycle $\gamma^{(v)}_{S,R}$ (2.25) for $C = C_S$ of the vector field algebra $\mathcal{L}$.*
(b) If $\mathfrak{g}$ is a simple Lie algebra, then $\gamma_{\alpha,S}$ is a multiple of the standard cocycle (2.32) for $\mathfrak{g}$.
(c) $\dim \mathrm{H}^2_{loc}(\mathcal{D}^1_{\mathfrak{g}}, \mathbb{C}) = M + 1$, where M is the number of simple ideals of $\mathfrak{g}$.

In the reductive case it turns out that a cocycle of $\mathcal{D}^1_{\mathfrak{g}}$ restricted to the abelian summand $\overline{\mathfrak{g}}_0$ is $\mathcal{L}$-invariant. In generalization of the mixing cocycle for $\mathcal{D}^1$ we obtain for every linear form $\phi \in \mathfrak{g}^*$ which vanishes on $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_M$ a local cocycle given by

$$\gamma_{\phi,S}(e, x(g)) := \frac{\phi(x)}{2\pi\,\mathrm{i}} \int_{C_S} (\tilde{e} \cdot g'' + T \cdot (\tilde{e} \cdot g'))\, dz, \tag{2.37}$$

Here again T is a meromorphic affine connection with poles only at the points in A.

Theorem 2.10. [24, Thm. 4.11] *(a) Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra. For every local cocycle γ for $\mathcal{D}^1_{\mathfrak{g}}$ there exists a symmetric invariant bilinear form α on $\mathfrak{g}$, and a linear form ϕ of $\mathfrak{g}$ which vanishes on $\mathfrak{g}'$, such that γ is cohomologous to*

$$\gamma' = \gamma_{\alpha,S} + \gamma_{\phi,S} + r\gamma^{(v)}_{S,R}, \tag{2.38}$$

with $r \in \mathbb{C}$ and a current algebra cocycle $\gamma_{\alpha,S}$ given by (2.34), a mixing cocycle $\gamma_{\phi,S}$ given by (2.37) and the vector field cocycle $\gamma^{(v)}_{S,R}$ given by (2.25). Vice versa, any such α, ϕ, $r \in \mathbb{C}$ determine a local cocycle.
(b) The space of local cocycle classes $\mathrm{H}^2_{loc}(\mathcal{D}^1_{\mathfrak{g}}, \mathbb{C})$ is $\dfrac{n(n+1)}{2} + n + M + 1$ dimensional.

2.8. **Local cocycles for $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n)$.**
Consider $\mathfrak{sl}(n)$, the Lie algebra of trace-less complex $n \times n$ matrices. Up to multiplication with a scalar the Cartan-Killing form $\beta(x, y) = \mathrm{tr}(xy)$ is the unique symmetric invariant bilinear form on it. From the Theorems 2.7 and 2.9 follows

Proposition 2.11. *(a) Every local cocycle for the current algebra $\overline{\mathfrak{sl}}(n)$ is cohomologous to*

$$\gamma(x(g), y(h)) = r \cdot \frac{\mathrm{tr}(xy)}{2\pi\,\mathrm{i}} \int_{C_S} g dh, \quad r \in \mathbb{C}. \tag{2.39}$$

(b) Every $\mathcal{L}$-invariant local cocycle equals the cocycle (2.39) with a suitable r.
(c) Every local cocycle for the differential operator algebra $\mathcal{D}^1_{\mathfrak{sl}(n)}$ is cohomologous to a linear combination of (2.39) and the standard local cocycle $\gamma^{(v)}_{S,R}$ for the vector field algebra. In particular, there exist no cocycles of mixing type.

Next, we deal with $\mathfrak{gl}(n)$, the Lie algebra of all complex $n\times n$-matrices. It can be decomposed (cf. (2.33)) into the direct sum of its 1-dimensional center $\mathfrak{s}(n)$ and a simple ideal $\mathfrak{sl}(n)$: $\mathfrak{gl}(n)=\mathfrak{s}(n)\oplus\mathfrak{sl}(n)\cong\mathbb{C}\oplus\mathfrak{sl}(n)$ where $\mathfrak{s}(n)$ consists of scalar $n\times n$ matrices.

The space of symmetric invariant bilinear forms for $\mathfrak{gl}(n)$ is two-dimensional. A basis is given by the forms

$$\alpha_1(x,y)=\mathrm{tr}(xy),\quad\text{and}\quad\alpha_2(x,y)=\mathrm{tr}(x)\mathrm{tr}(y). \tag{2.40}$$

The form α_1 is the "standard" extension for the Cartan-Killing form for $\mathfrak{sl}(n)$ to $\mathfrak{gl}(n)$ and is also $\mathfrak{gl}(n)$ invariant. From the Theorems 2.8 and 2.10 follows

Proposition 2.12. *(a) A cocycle γ for $\overline{\mathfrak{gl}}(n)$ is local and restricted to $\overline{\mathfrak{s}(n)}$ is $\mathcal{L}$-invariant if and only if it is cohomologous to a linear combination of the following two cocycles*

$$\gamma_1(x(g),y(h))=\frac{\mathrm{tr}(xy)}{2\pi\,\mathrm{i}}\int_{C_S}gdh,\qquad\gamma_2(x(g),y(h))=\frac{\mathrm{tr}(x)\mathrm{tr}(y)}{2\pi\,\mathrm{i}}\int_{C_S}gdh. \tag{2.41}$$

(b) If the local cocycle γ on $\overline{\mathfrak{gl}}(n)$ is $\mathcal{L}$-invariant then γ is equal to a linear combination of the cocycles (2.41).

Proposition 2.13. *(a) Every local cocycle γ for $\mathcal{D}^1_{\mathfrak{gl}(n)}$ is cohomologous to a linear combination of the cocycles γ_1 and γ_2 of (2.41), of the mixing cocycle*

$$\gamma_{3,T}(e,x(g))=\frac{\mathrm{tr}(x)}{2\pi\,\mathrm{i}}\int_{C_S}\big(\tilde{e}g''+T\tilde{e}g'\big)dz, \tag{2.42}$$

and of the standard local cocycle $\gamma^{(v)}_{S,R}$ for the vector field algebra, i.e.

$$\gamma=r_1\gamma_1+r_2\gamma_2+r_3\gamma_{3,T}+r_4\gamma^{(v)}_{S,R}+\textit{coboundary}, \tag{2.43}$$

with suitable $r_1,r_2,r_3,r_4\in\mathbb{C}$.
(b) If the cocycle γ is local and restricted to $\overline{\mathfrak{gl}}(n)$ is $\mathcal{L}$-invariant, and $r_3,r_4\neq 0$ then there exist an affine connection T and a projective connection R holomorphic outside A such that in the combination (2.43) there will be no additional coboundary.
(c) $\dim\mathrm{H}^2_{loc}(\mathcal{D}^1_{\mathfrak{gl}(n)},\mathbb{C})=4$.

It turns out that for a cocycle of the differential operator algebra, its restriction to $\overline{\mathfrak{gl}}(n)$ is cohomologous to an $\mathcal{L}$-invariant cocycle. Moreover, its restriction to $\mathfrak{s}(n)$ *is* $\mathcal{L}$-invariant [24, Prop. 4.10].

3. Representations of the multi-point Krichever-Novikov algebras

3.1. Projective $\mathcal{D}^1_{\mathfrak{g}}$-modules.

Let $\mathcal{V}$ be an arbitrary Lie algebra, V a vector space and $\pi : \mathcal{V} \to End(V)$ a linear map. The space V is called a *projective $\mathcal{V}$-module* if for all pairs $f, g \in \mathcal{V}$ there exists $\gamma(f, g) \in \mathbb{C}$ such that

$$\pi([f, g]) = [\pi(f), \pi(g)] + \gamma(f, g) \cdot id. \tag{3.1}$$

In this case π is also called a projective action. Often we write simply fv instead $\pi(f)(v)$.

If V is a projective $\mathcal{V}$-module then γ is necessarily a Lie algebra 2-cocycle. Via Equation (2.20) the cocycle, hence the projective action π, defines a central extension $\widehat{\mathcal{V}}_\gamma = \widehat{\mathcal{V}}_\pi$ of $\mathcal{V}$. Obviously, by defining $\pi(t) = id$, the projective action can be extended to a honest Lie action of $\widehat{\mathcal{V}}_\gamma$ on V. Usually, we omit the indices π, γ if they are clear from the context. One should keep in mind that the cocycle defining $\widehat{\mathcal{V}}$ and also the equivalence and even isomorphy class of the central extension will depend on the projective action given. In certain cases considered below we will be able to identify the cocycle.

Let $\mathcal{V}$ be an almost-graded Lie algebra. A projective $\mathcal{V}$-module V is called *admissible* if for each $v \in V$ there exists a $k(v) \in \mathbb{N}$ such that for all $f \in \mathcal{V}$ with $\deg f \geq k(v)$ we have $fv = 0$. By a projective almost-graded module V over an almost-graded algebra $\mathcal{V}$ we understand an almost-graded module structure on V with respect to the projective action of $\mathcal{V}$ such that the corresponding cocycle is local (see Definition 2.3). We call an almost-graded module V a *highest weight (vacuum) module* if an element $|0\rangle \in V$ exists such that $f|0\rangle = 0$ for every f with $\deg f > 0$ and $V = U(\mathcal{V})|0\rangle$ where $U(\cdot)$ denotes the universal enveloping algebra. The element $|0\rangle$ is called a vacuum vector. Obviously, vacuum modules are admissible.

Below, we will assume V to be an admissible projective almost-graded $\overline{\mathfrak{g}}$-module (we will write *projective representation* as well). In our main example (the *fermion representations* which will be discussed in the next subsection) we even have a structure of a $\mathcal{D}^1_{\mathfrak{g}}$-module. Moreover, these modules are generated by vacuum vectors over $\mathcal{D}^1_{\mathfrak{g}}$ and over $\overline{\mathfrak{g}}$ as well.

In fact, we will be mainly interested in the case $\mathfrak{g} = \mathfrak{gl}(n)$. In particular, the fermion modules (of fixed charge) are irreducible in this case. Observe that for the constructions of the following Section 4 we need only a (projective) $\overline{\mathfrak{g}}$-module structure on V, and we will assume the irreducibility of this module.

3.2. Fermion representations.

In this paragraph, we briefly outline the construction of a projective fermion $\mathcal{D}^1_{\mathfrak{g}}$-module [30, 31, 33]. It uses the Krichever-Novikov bases in the spaces of sections of holomorphic vector bundles as an important ingredient. These bases are introduced in [13] for the two-point case, and, combining the approaches of [13] and Section 2.3 (going back to [17, 18]) generalized in [33] to the multi-point case. We refer to the cited works for the details. Here we only mention that these bases are given by asymptotic behavior of their elements at the marked points and their properties are similar to those described in Section 2.3. In particular, these bases are almost graded with respect to the action of Krichever-Novikov algebras.

Consider a holomorphic bundle F on Σ of rank r and degree $g \cdot r$ (g is the genus of the Riemann surface Σ). Let $\Gamma(F)$ denote the space of meromorphic sections of F holomorphic except at $P_1, \ldots, P_N, P_\infty$. Let τ be a finite-dimensional representation of $\mathfrak{g}$ with representation space V_τ. Set $\Gamma_{F,\tau} := \Gamma(F) \otimes V_\tau$.

We define a $\overline{\mathfrak{g}}$-action on $\Gamma_{F,\tau}$ as follows:

$$(x \otimes A)(s \otimes v) = (A \cdot s) \otimes \tau(x) v \quad \text{for all} \quad x \in \mathfrak{g}, A \in \mathcal{A}, s \in \Gamma(F), v \in V_\tau. \tag{3.2}$$

In order to define an $\mathcal{L}$-action on $\Gamma(F)$ we choose a meromorphic (therefore flat) connection ∇ on F which has logarithmic singularities at $P_1, \ldots, P_N$ and P_∞ (see [31] for more details). By flatness, $\nabla_{[e,f]} = [\nabla_e, \nabla_f]$ for all $e, f \in \mathcal{L}$. Hence, ∇ defines a representation of $\mathcal{L}$ in $\Gamma(F)$.

From the definition of a connection, for any $s \in \Gamma(F)$, $e \in \mathcal{L}$ and $A \in \mathcal{A}$ we have $\nabla_e(As) = (e.A)s + A\nabla_e s$ where $e.A$ is the Lie derivative. Hence, $[\nabla_e, A] = e.A$, i.e. the mapping $e + A \to \nabla_e + A$ gives rise to a representation of $\mathcal{D}^1$ in $\Gamma(F)$.

Define the corresponding $\mathcal{L}$-action on $\Gamma_{F,\tau}$ by

$$e(s \otimes v) = \nabla_e s \otimes v \quad \text{for all} \quad e \in \mathcal{L}, s \in \Gamma(F), v \in V_\tau. \tag{3.3}$$

It can be verified directly that (3.2) and (3.3) give a representation of $\mathcal{D}^1_{\mathfrak{g}}$ in $\Gamma_{F,\tau}$.

Choose a Krichever-Novikov basis in $\Gamma(F)$ [33] and a weight basis $\{v_i | 1 \le i \le \dim V_\tau\}$ in V_τ. Each basis element in $\Gamma(F)$ is determined by its *degree* $n \in \mathbb{Z}$, the number p of a marked point: $1 \le p \le N$, and an integer j: $0 \le j \le r-1$. Denote the element corresponding to a triple (n, p, j) by $\psi_{n,p,j}$. Introduce $\psi^i_{n,p,j} = \psi_{n,p,j} \otimes v_i$. Enumerate the elements $\psi^i_{n,p,j}$ linearly in ascending lexicographical order of the quadruples n, p, j, i. In this way we set $\psi_M = \psi^i_{n,p,j}$ where $M = M(n, p, j, i) \in \mathbb{Z}$.

Lemma 3.1. [33] *With respect to the index M, the module $\Gamma_{F,\tau}$ is an almost-graded $\mathcal{D}^1_{\mathfrak{g}}$-module.*

The proof of the almost-gradedness is similar to that in the two-point case [30, 31].

The final step of the construction of the fermion representation corresponding to the pair (F, τ) is passing to the space of the semi-infinite monomials on $\Gamma_{F,\tau}$.

Consider the vector space $\mathcal{H}_{F,\tau}$ generated over $\mathbb{C}$ by the formal expressions (*semi-infinite monomials*) of the form $\Phi = \psi_{N_0} \wedge \psi_{N_1} \wedge \ldots$, where the ψ_{N_i} are the above introduced basis elements of $\Gamma_{F,\tau}$, the indices are strictly increasing, i.e. $N_0 < N_1 < \dots$, and for all k sufficiently large $N_k = k + m$ for a suitable m (depending on the monomial). Following [8] we call m *the charge* of the monomial. For a monomial Φ of charge m the degree of Φ is defined as follows:

$$\deg \Phi = \sum_{k=0}^{\infty} (N_k - k - m). \tag{3.4}$$

Observe that there is an arbitrariness in the enumeration of the $\psi^i_{n,p,j}$'s for a fixed n; the just defined degree of a monomial does not depend on this arbitrariness.

We want to extend the action of $\mathcal{D}^1_{\mathfrak{g}}$ on $\Gamma_{F,\tau}$ to $\mathcal{H}_{F,\tau}$. Here, we briefly outline the construction; see [30, 31] for details. Assuming the $\mathcal{D}^1_{\mathfrak{g}}$-action on ψ_{N_i}'s to be known, apply a basis element of $\mathcal{D}^1_{\mathfrak{g}}$ to Φ by the Leibniz rule. If, in the process, a monomial containing the same ψ_N in different positions occurs, it is set to zero. If a pair $\psi_N \wedge \psi_{N'}$ in a wrong order $(N > N')$ occurs then it should be transposed and the sign before the corresponding monomial changes. This is done until all entries are in the strictly increasing order. Due to the almost-gradedness of $\mathcal{D}^1_{\mathfrak{g}}$-action on $\Gamma_{F,\tau}$ and the above mentioned stabilization ($N_k = k+m$, $k \sim \infty$), the result of the above steps is well defined for all except for the finite number basis elements of the algebra. For those, apply the standard *process of regularization* [8, 11, 12, 20] (see in the proof of Lemma 4.5). In this way the action on $\Gamma_{F,\tau}$ can be extended to $\mathcal{H}_{F,\tau}$ as a projective Lie algebra action.

Let $\mathcal{H}^{(m)}_{F,\tau}$ be the subspace of $\mathcal{H}_{F,\tau}$ generated by the semi-infinite monomials of charge m. These subspaces are invariant under the projective action of $\mathcal{D}^1_{\mathfrak{g}}$. This follows, as in the classical situation, from the fact that after the action of $\mathcal{D}^1_{\mathfrak{g}}$ the resulting monomials will have the same "tail" as the monomial one has started with. Hence, for every m the space $\mathcal{H}^{(m)}_{F,\tau}$ is itself a projective $\mathcal{D}^1_{\mathfrak{g}}$-module, and $\mathcal{H}_{F,\tau} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{(m)}_{F,\tau}$ as projective $\mathcal{D}^1_{\mathfrak{g}}$-module. We call the modules $\mathcal{H}_{F,\tau}$,$\mathcal{H}^{(m)}_{F,\tau}$ (projective) *fermion representations.*

Proposition 3.2. *Let $\mathcal{H}^{(m)}_{F,\tau}$ be the submodule of $\mathcal{H}_{F,\tau}$ of charge m.*
(a) With respect to the degree (3.4) the homogeneous subspaces $(\mathcal{H}^{(m)}_{F,\tau})_d$ of degree are finite-dimensional and $\dim(\mathcal{H}^{(m)}_{F,\tau})_d = p(-d)$ *for $d \le 0$ where p is the partition function. If $d > 0$ then $(\mathcal{H}^{(m)}_{F,\tau})_d = 0$.*
(b) The cocycle γ for $\mathcal{D}^1_{\mathfrak{g}}$ defined by the projective representation is local. It is bounded from above by zero.
(c) The module $\mathcal{H}^{(m)}_{F,\tau}$ is an almost-graded projective $\mathcal{D}^1_{\mathfrak{g}}$-module.

Proof. (a) Observe that all summands in (3.4) are non-positive, hence $d \le 0$. The number of monomials of a given degree d is equal, therefore, to the number of partitions of d into negative summands, which implies both statements.
(b) and (c) follow as in the two-point case, [30, 31]. □

From this proposition and the classification results of Theorem 2.10 we obtain

Proposition 3.3. *$\mathcal{H}^{(m)}_{F,\tau}$ is a module over a certain central extension of the Lie algebra $\widehat{\mathcal{D}^1_{\mathfrak{g}}}$ defined via a local cocycle of the form (2.38).*

Consider the central extension of Lie algebra $\overline{\mathfrak{g}}$ induced by its embedding (as a linear space) into $\widehat{\mathcal{D}^1_{\mathfrak{g}}}$. The cocycle of this extension is cohomologous to an $\mathcal{L}$-invariant cocycle, i.e. it is cohomologous to a geometric cocycle of the type (2.28) with a suitable invariant symmetric bilinear form α (see Theorem 2.8). In particular, for $\mathfrak{g} = \mathfrak{gl}(n)$ the cohomology class of the cocycle is given by Proposition 2.12.

Under an admissible representation V of $\widehat{\mathfrak{g}}$ we understand a representation admissible with respect to the almost-grading (in the sense introduced above) as projective representation of $\overline{\mathfrak{g}}$ in which the central element t operates as a scalar $c \cdot id$, $c \in \mathbb{C}$. The number c is called the *level* of the $\widehat{\mathfrak{g}}$-module V. It follows immediately

Proposition 3.4. *The representation of the Lie algebra $\widehat{\mathfrak{g}}$ in $\mathcal{H}^{(m)}_{F,\tau}$ is admissible.*

3.3. **Sugawara representation.**

Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra. We fix an invariant symmetric bilinear form α on $\mathfrak{g}$. Starting from this section we assume that α is non-degenerate. By $\widehat{\mathfrak{g}}$ we denote the standard central extension (depending on the bilinear form α) as introduced in Section 2.6 (see Equations (2.28) and (2.29)) together with its almost-grading. Let V be an admissible representation of level c.

If $\mathfrak{g}$ is abelian or simple then each admissible representation of $\widehat{\mathfrak{g}}$ of non-critical level i.e. for a level which is not the negative of the dual Coxeter number in the simple case, or a level $\neq 0$ in the abelian case, the (affine) Sugawara construction yields a projective representation of the Krichever-Novikov vector field algebra $\mathcal{L}$. This representation is called the *Sugawara representation.* For the two-point case, the abelian version of this construction was introduced in [11]. The nonabelian case was later considered in [2], [25]. In [25] also the multi-point version was given. Observe that every positive level is non-critical. For an arbitrary complex reductive Lie algebra $\mathfrak{g}$ the Sugawara representation is defined as a certain linear combination of Sugawara representations of its simple ideals (V.Kac [7], [8, Lecture 10]).

Due to its importance in our context, we have to describe the construction in more detail. For any $u \in \mathfrak{g}$, $A \in \mathcal{A}$ we denote by $u(A)$ the operator in V corresponding to $u \otimes A$. We also denote an element of the form $u(A_{n,p})$ by $u(n,p)$. We choose a basis u_i, $i = 1, \dots, \dim \mathfrak{g}$ of $\mathfrak{g}$ and the corresponding dual basis u^i, $i = 1, \dots, \dim \mathfrak{g}$ with respect to the form α. The *Casimir element* $\Omega^0 = \sum_{i=1}^{\dim \mathfrak{g}} u_i u^i$ of the universal enveloping algebra $U(\mathfrak{g})$ is independent of the choice of the basis. To simplify notation, we denote $\sum_i u_i(n,p)u^i(m,q)$ by $u(n,p)u(m,q)$.

We define the higher genus *Sugawara operator* (also called *Segal operator* or *energy-momentum tensor*) as

$$T(P) := \frac{1}{2}\sum_{n,m}\sum_{p,s} :u(n,p)u(m,s):\, \omega^{n,p}(P)\omega^{m,s}(P)\ . \tag{3.5}$$

By $:\ldots.:$ we denote some normal ordering. In this section the summation indices n, m run over $\mathbb{Z}$, and p, s over $\{1, \ldots, N\}$. The precise form of the normal ordering is of no importance here. As an example we may take the following "standard normal ordering" ($x, y \in \mathfrak{g}$)

$$:x(n,p)y(m,r): := \begin{cases} x(n,p)y(m,r), & n \le m \\ y(m,r)x(n,p), & n > m\ . \end{cases} \tag{3.6}$$

The expression $T(P)$ can be considered as a formal series of quadratic differentials in the variable P with operator-valued coefficients. Expanding it over the basis $\Omega^{k,r}$ of the quadratic differentials we obtain

$$T(P) = \sum_k \sum_r L_{k,r} \cdot \Omega^{k,r}(P)\ , \tag{3.7}$$

with

$$\begin{aligned} L_{k,r} &= \frac{1}{2\pi \mathrm{i}}\int_{C_S} T(P)e_{k,r}(P) = \frac{1}{2}\sum_{n,m}\sum_{p,s} :u(n,p)u(m,s):\, l_{(k,r)}^{(n,p)(m,s)}, \\ &\text{where} \qquad l_{(k,r)}^{(n,p)(m,s)} := \frac{1}{2\pi \mathrm{i}}\int_{C_S} \omega^{n,p}(P)\omega^{m,s}(P)e_{k,r}(P)\ . \end{aligned} \tag{3.8}$$

Formally, the operators $L_{k,r}$ are infinite double sums. But for given k and m, the coefficient $l_{(k,r)}^{(n,p)(m,s)}$ will be non-zero only for finitely many n. This can be seen by checking the residues of the integrands. After applying the remaining infinite sum to a fixed element $v \in V$, by the normal ordering and admissibility of the representation only finitely many of the operators will operate non-trivially on this element.

The following theorem is proved in [25].

Theorem 3.5. *Let $\mathfrak{g}$ be a finite dimensional either abelian or simple Lie algebra and $2k$ be the eigenvalue of its Casimir operator in the adjoint representation. Let α be the Cartan-Killing form in the simple case or any non-degenerate bilinear form in the abelian case, and $\widehat{\mathfrak{g}}$ be the corresponding central extension. Let V be an admissible almost-graded $\widehat{\mathfrak{g}}$-module of level c. If $c + k \neq 0$ then the rescaled "modes"*

$$L^*_{k,r} = \frac{-1}{2(c+k)}\sum_{n,m}\sum_{p,s} :u(n,p)u(m,s):\, l_{(k,r)}^{(n,p)(m,s)}\ , \tag{3.9}$$

of the Sugawara operator are well-defined operators on V and define an admissible projective representation of $\mathcal{L}$. The corresponding cocycle for the vector field algebra $\mathcal{L}$ is a local cocycle.

Remark. By the locality of the cocycle and in view of Theorem 2.5 the cocycle is a scalar multiple of (2.25) with $C = C_S$. In particular, the central extension $\widehat{\mathcal{L}}$ for which the Sugawara representation is a honest representation, is fixed up to an isomorphism.

Proposition 3.6. *The Sugawara representation endows V with the structure of an almost-graded $\widehat{\mathcal{L}}$-module.*

Proof. We have to show that there exist constants M_1, M_2 such that for every given homogeneous element $\psi_s \in V$ of degree s, and every k we have

$$k + s + M_1 \le \deg(L^*_{k,r}\psi_s) \le k + s + M_2. \tag{3.10}$$

Consider the coefficients $l^{(n,p)(m,s)}_{(k,r)}$ in (3.9) which are given as integrals (3.8). The integral could only be non-vanishing if the integrand has poles at the points in I and at the point P_∞. Using the explicit formulas (2.8) and (2.9) for the orders at these points we obtain

$$k \le n + m \le k + C(g, N), \tag{3.11}$$

with a rational constant $C(g, N) \ge 0$ only depending on the genus g and the number of points N. By the almost-gradedness of V as $\widehat{\mathfrak{g}}$-module, there exist constants c_1 and c_2 such that for all m, r, s

$$m + s + c_1 \le \deg(u(m, r)\psi_s) \le m + s + c_2, \tag{3.12}$$

if $u(m, r)\psi_s \ne 0$. Hence,

$$n + m + s + 2c_1 \le \deg(:u(n, p)u(m, r):\, \psi_s) \le n + m + s + 2c_2, \tag{3.13}$$

if $:u(n, p)u(m, r):\, \psi_s \ne 0$. Using (3.11) we obtain Equation (3.10) if we set $M_1 = 2c_1$ and $M_2 = 2c_2 + C(g, N)$. $\square$

We call the $L^*_{k,r}$, resp. the $L_{k,r}$ the Sugawara operators too. For $e = \sum_{n,p} a_{n,p}e_{n,p} \in \mathcal{L}$ $(a_{n,p} \in \mathbb{C})$ we set $T[e] = \sum_{n,p} a_{n,p}L^*_{n,p}$ and obtain the projective representation T of $\mathcal{L}$. It is called *Sugawara representation* of the Lie algebra $\mathcal{L}$ corresponding to the given admissible representation V of $\widehat{\mathfrak{g}}$.

By the Krichever-Novikov duality the Sugawara operator $T[e]$ assigned to the vector field $e \in \mathcal{L}$ can equivalently be given as

$$T[e] = \frac{-1}{c + k} \cdot \frac{1}{2\pi \mathrm{i}} \int_{C_S} T(P)e(P). \tag{3.14}$$

Let $\mathfrak{g}$ be a reductive Lie algebra with decomposition (2.33). The elements $x \in \mathfrak{g}$ can be decomposed as $x = \sum_{i=0}^{n} x_i$, with $x_i \in \mathfrak{g}_i$. Let α be a symmetric invariant bilinear form on $\mathfrak{g}$. Due to the invariance, we have $\alpha(x_i, x_j) = 0$ for $i \ne j$, thus the decomposition is orthogonal with respect to α. By restricting α to $\mathfrak{g}_i$ we obtain a symmetric invariant bilinear form on $\mathfrak{g}_i$, hence a multiple of the Cartan-Killing form (resp. an arbitrary symmetric bilinear form on $\mathfrak{g}_0$). The form α is called *normalized* if all its restrictions on the simple ideals of $\mathfrak{g}$ are *equal* to the corresponding Cartan-Killing forms.

Let γ be a local cocycle on $\overline{\mathfrak{g}}$ for $\mathfrak{g}$ reductive, $A, B \in \mathcal{A}$, and $x, y \in \mathfrak{g}$. It follows from the cocycle condition (see [24, Lemma 3.11]) that $\gamma(x_i A, y_j B) = 0$ for $i \neq j$ (with the same decomposition for y as for x above). This implies

$$\gamma(xA, yB) = \sum_{i=0}^{M} \gamma(x_i A, y_i B). \tag{3.15}$$

Given an admissible representation V of $\widehat{\mathfrak{g}}$ we obtain representations of $\widehat{\mathfrak{g}}_i$'s in the same space V. We can define the individual (rescaled) Sugawara operators $T_i[e]$, $i = 0, 1, \ldots, M$. We set

$$T[e] := \sum_{i=0}^{M} T_i[e]. \tag{3.16}$$

The following lemma expresses a fundamental property of the Sugawara representation. It was shown in [25] for the abelian and simple case. Here we will show how to extend the result to the general reductive case.

Lemma 3.7. *Let $\mathfrak{g}$ be a reductive Lie algebra with a chosen normalized form α, and $T[e]$ for every $e \in \mathcal{L}$ the operator as defined above. Then T defines a representation of the centrally extended vector field algebra $\widehat{\mathcal{L}}$ and for any $x \in \mathfrak{g}$, $A \in \mathcal{A}$, $e \in \mathcal{L}$, we have*

$$[T[e], x(A)] = x(e.A). \tag{3.17}$$

Proof. Given $x, y \in \mathfrak{g}$ denote by x_i and y_j its components as above. Then $[x_i(A), y_j(B)] = [x_i, y_j](AB) + \gamma(x_i A, y_j B)\, c \cdot id$. In particular, if $i \neq j$ we obtain that x_i and x_j commute and that the cocycle vanishes. Hence $x_i(A)$, $y_j(B)$ also commute. Let T_k, be the Sugawara representation corresponding to the representation V of $\widehat{\mathfrak{g}_k}$. Since the operators of the representation T_i are expressed via $x_i(A)$'s and operators of T_j via $x_j(A)$'s, T_i and T_j commute for $i \neq j$. Hence, T is a representation of $\widehat{\mathcal{L}}$. Moreover, for $x_i \in \mathfrak{g}_i$, $x_j \in \mathfrak{g}_j$, $A \in \mathcal{A}$, $e \in \mathcal{L}$ we have $[T_j[e], x_i(A)] = 0$. For $\mathfrak{g}$ simple or abelian, (3.17) is shown in [25] (see also [26]). Hence,

$$[T_k[e], x_k(A)] = x_k(e.A), \quad k = 0, 1, \ldots, M.$$

This implies

$$[T[e], x(A)] = [\sum_{k=0}^{M} T_k[e], \sum_{i=0}^{M} x_i(A] = \sum_{i=0}^{M} x_i(e.A) = x(e.A).$$

$\square$

Having only one simple summand for $\mathfrak{g} = \mathfrak{gl}(n)$ the condition that the form α is normalized on its simple summands can always be achieved by rescaling the level. Hence for the admissible (in particular, fermionic) representations of $\widehat{\mathfrak{gl}(n)}$, the relation (3.17) is true.

4. Moduli of curves with marked points, conformal blocks and projectively flat connection

4.1. Moduli space $\mathcal{M}_{g,N+1}^{(k,p)}$ and the sheaf of conformal blocks.

In [26] we described the moduli spaces of curves which typically occur in 2d conformal field theories. Here we will slightly extend the definitions introduced there. We denote by $\mathcal{M}_{g,N+1}^{(k,p)}$ the moduli space of smooth projective curves of genus g (over $\mathbb{C}$) with $N+1$ ordered distinct marked points and fixed k-jets of local coordinates at the first N points and a fixed p-jet of a local coordinate at the last point. The elements of $\mathcal{M}_{g,N+1}^{(k,p)}$ are given as

$$\tilde{b}^{(k,p)} = [\Sigma, P_1, \ldots, P_N, P_\infty, z_1^{(k)} \ldots, z_N^{(k)}, z_\infty^{(p)}] \ , \tag{4.1}$$

where Σ is a smooth projective genus g curve, P_i $(i = 1, \ldots, N, \infty)$ are distinct points on Σ, z_i is a coordinate at P_i with $z_i(P_i) = 0$, and $z_i^{(l)}$ is a l-jet of z_i $(l \in \mathbb{N}_0)$. Here [..] denotes an equivalence class of such tuples in the following sense. Two tuples representing $\tilde{b}^{(k,p)}$ and $\tilde{b}^{(k,p)\prime}$ are equivalent if there exists an algebraic isomorphism $\phi : \Sigma \to \Sigma'$ with $\phi(P_i) = P_i'$ for $i = 1, \ldots, N, \infty$ such that after the identification via ϕ we have

$$z_i' = z_i + O(z_i^{k+1}), \quad i = 1, \ldots, N \quad \text{and} \quad z_\infty' = z_\infty + O(z_i^{p+1}). \tag{4.2}$$

For the following two special cases we introduce the same notation as in [26]: $\mathcal{M}_{g,N+1} = \mathcal{M}_{g,N+1}^{(0,0)}$, and $\mathcal{M}_{g,N+1}^{(1)} = \mathcal{M}_{g,N+1}^{(1,1)}$. By forgetting either coordinates or higher order jets we obtain natural projections

$$\mathcal{M}_{g,N+1}^{(1,p)} \to \mathcal{M}_{g,N+1}, \qquad \mathcal{M}_{g,N+1}^{(k,p)} \to \mathcal{M}_{g,N+1}^{(k',p')} \tag{4.3}$$

for any $k' \leq k$ and $p' \leq p$. In this article (as well as in the previous one [26]) we are only dealing with the local situation in the neighborhood of a moduli point corresponding to a generic curve Σ with a generic marking $(P_1, P_2, \ldots, P_N, P_\infty)$. Let $\widetilde{W} \subseteq \mathcal{M}_{g,N+1}$ be an open subset around such a generic point $\tilde{b} = [\Sigma, P_1, P_2, \ldots, P_N, P_\infty]$. A generic curve of $g \geq 2$ admits no nontrivial infinitesimal automorphism, and we may assume that there exists a universal family of curves with marked points over $\widetilde{W}$. In particular, this says that there is a proper, flat family of smooth curves over $\widetilde{W}$

$$\pi : \mathcal{U} \to \widetilde{W} \ , \tag{4.4}$$

such that for the points $\tilde{b} = [\Sigma, P_1, P_2, \ldots, P_N, P_\infty] \in \widetilde{W}$ we have $\pi^{-1}(\tilde{b}) = \Sigma$ and that the sections defined as

$$\sigma_i : \widetilde{W} \to \mathcal{U}, \quad \sigma_i(\tilde{b}) = P_i, \quad i = 1, \ldots, N, \infty \tag{4.5}$$

are holomorphic. For more background information, see [35, Sect. 1.2, Sect. 1.3], in particular Thm. 1.2.9 of [35].

If we "forget" the point P_∞ we obtain maps

$$\mathcal{M}_{g,N+1} \to \mathcal{M}_{g,N}, \quad \mathcal{M}_{g,N+1}^{(0,p)} \to \mathcal{M}_{g,N}, \quad \mathcal{M}_{g,N+1}^{(k,p)} \to \mathcal{M}_{g,N}^{(k)}. \tag{4.6}$$

Let us fix a holomorphic section $\widehat{\sigma}_\infty$ of the universal family of curves (without marking). In particular, for every curve there is a point chosen in a manner depending analytically on the moduli. (Recall, we are only dealing with the local and generic situation.) The analytic subset

$$W' := \{\tilde{b} = [\Sigma, P_1, P_2, \dots, P_N, P_\infty] \mid P_\infty = \widehat{\sigma}_\infty([\Sigma])\} \quad \subseteq \quad \widetilde{W} \tag{4.7}$$

can be identified with an open subset W of $\mathcal{M}_{g,N}$ via

$$\tilde{b} = [(\Sigma, P_1, P_2, \dots, P_N, \widehat{\sigma}_\infty([\Sigma]))] \;\to\; b = [(\Sigma, P_1, P_2, \dots, P_N)]\ . \tag{4.8}$$

By genericity, the map is one-to-one. By choosing not only a section $\widehat{\sigma}_\infty$ but also a p-th order infinitesimal neighborhood of this section we even get an identification of the open subset W of $\mathcal{M}_{g,N}$ with an analytic subset $W'^{,(p)}$ of $\widetilde{W}^{(0,p)}$ of $\mathcal{M}^{(0,p)}_{g,N+1}$. It is defined in a similar way as W'.

All these considerations can be extended to the case where we allow infinite jets of local coordinates at P_∞. We obtain then the moduli space $\mathcal{M}^{(k,\infty)}_{g,N+1}$.

At generic points, the moduli spaces $\mathcal{M}^{(k,p)}_{g,N+1}$ are smooth. Denote by S the divisor $S = \sum_{i=1}^N P_i$ on Σ. The tangent space $\mathrm{T}_{\tilde{b}^{(1,p)}}\mathcal{M}^{(1,p)}_{g,N+1}$ can be identified with the cohomology space $\mathrm{H}^1(M, T_M(-2S-(p+1)P_\infty))$. As in Prop. 4.4 and Thm. 4.5 of [26] we obtain that there exists a surjective linear map from the Krichever-Novikov vector field algebra $\mathcal{L}$ to the cohomology space

$$\theta = \theta_p : \mathcal{L} \to \mathrm{H}^1(M, T_M(-2S-(p+1)P_\infty) \tag{4.9}$$

such that θ restricted to the following subspaces gives isomorphisms

$$\begin{aligned} \mathcal{L}_0 \oplus \mathcal{L}_{-1} \oplus \mathcal{L}^{(p)}_{(0)} &\cong \mathrm{H}^1(M, T_M(-2S-(p+1)P_\infty) \cong \mathrm{T}_{\tilde{b}^{(1,p)}}\mathcal{M}^{(1,p)}_{g,N+1}, \\ \mathcal{L}_{-1} \oplus \mathcal{L}^{(p)}_{(0)} &\cong \mathrm{H}^1(M, T_M(-S-(p+1)P_\infty)) \cong \mathrm{T}_{\tilde{b}^{(0,p)}}\mathcal{M}_{g,N+1}, \\ \mathcal{L}^{(p)}_{(0)} &\cong \mathrm{H}^1(M, T_M(-(p+1)P_\infty) \cong \mathrm{T}_{[\Sigma,P_\infty]}\mathcal{M}^{(p)}_{g,1}. \end{aligned} \tag{4.10}$$

Again, for the infinite jets we obtain

$$\mathrm{T}_{\tilde{b}^{(1,\infty)}}\mathcal{M}^{(1,\infty)}_{g,N+1} = \lim_{p\to\infty} \mathrm{H}^1(M, T_M(-2S-pP_\infty)) \cong \mathcal{L}_{(0)} \oplus \mathcal{L}_{-}. \tag{4.11}$$

Let us note the dimension formula

$$\dim_{b^{(1,p)}}(\mathcal{M}^{(1,p)}_{g,N+1}) = \begin{cases} 3g-2+2N+p, & g \ge 1 \\ \max(0, N-2)+N+p, & g=0\ . \end{cases} \tag{4.12}$$

For $N \ge 2$ the first expression is valid for any genus.

Let $\tilde{b}^{(1,p)} \in \mathcal{M}^{(1,p)}_{g,N+1}$ be a moduli point. Let $\nu^{(p)} : \mathcal{M}^{(1,p)}_{g,N+1} \to \mathcal{M}_{g,N+1}$ be the map forgetting the coordinates and let $\tilde{b} = \nu^{(p)}(\tilde{b}^{(1,p)})$ be a generic point with open neighborhood $\widetilde{W}$. For $\tilde{b} = [\Sigma, P_1, P_2, \dots, P_N, P_\infty]$ we can construct the Krichever-Novikov objects

$$\mathcal{A}_{\tilde{b}},\ \mathcal{L}_{\tilde{b}},\ \widehat{\mathcal{L}}_{\tilde{b}},\ \overline{\mathfrak{g}}_{\tilde{b}},\ \widehat{\mathfrak{g}}_{\tilde{b}},\ \mathcal{F}^\lambda_{\tilde{b}},\ \text{etc.} \tag{4.13}$$

Recall from [26] that there are sheaf versions of these objects

$$\mathcal{A}_{\widetilde{W}},\ \mathcal{L}_{\widetilde{W}},\ \widehat{\mathcal{L}}_{\widetilde{W}},\ \overline{\mathfrak{g}}_{\widetilde{W}},\ \widehat{\mathfrak{g}}_{\widetilde{W}},\ \mathcal{F}^{\lambda}_{\widetilde{W}}. \tag{4.14}$$

Similarly, we can consider the sheaf versions of the objects introduced in Section 2, and Section 3 of the present paper: e.g. $\mathcal{D}^1_{\mathfrak{g},\widetilde{W}}$ – the sheaf of algebras of Krichever-Novikov differential operators of order ≤ 1, and $V_{\widetilde{W}}$ – the sheaf of fermion modules.

Introduce the *regular subalgebras* of $\overline{\mathfrak{g}}$ and $\mathcal{L}$ as follows. Let $\mathcal{A}^r \subset \mathcal{A}$, and $\mathcal{L}^r \subset \mathcal{L}$ consist of all elements vanishing at P_∞. Introduce $\overline{\mathfrak{g}}^r = \mathfrak{g}\otimes\mathcal{A}^r$. Observe that $\overline{\mathfrak{g}}^r$ is a Lie subalgebra of $\overline{\mathfrak{g}}$ as well as of $\widehat{\mathfrak{g}}$, and $\mathcal{L}^r$ is also a Lie subalgebra of $\widehat{\mathcal{L}}$. Denote by $\overline{\mathfrak{g}}^r_{\widetilde{W}}$, $\mathcal{L}^r_{\widetilde{W}}$ etc. the corresponding sheaves.

Definition 4.1. Let $\mathcal{V}_{\widetilde{W}}$ be a sheaf of (fibrewise) representations of $\widehat{\mathfrak{g}}_{\widetilde{W}}$. The sheaf of *conformal blocks* (associated to the representation sheaf $\mathcal{V}_{\widetilde{W}}$) is defined as the sheaf of coinvariants

$$C_{\widetilde{W}} = \mathcal{V}_{\widetilde{W}}/\overline{\mathfrak{g}}^r_{\widetilde{W}}\mathcal{V}_{\widetilde{W}}. \tag{4.15}$$

For $p\in\mathbb{N}$ or $p=\infty$ let $\widetilde{W}^{(p)} = \left(\nu^{(p)}\right)^{-1}(\widetilde{W})$. By pulling-back the above sheaves over $\widetilde{W}$ via $\nu^{(p)}$ we obtain sheaves on the open subset $\widetilde{W}^{(p)}$ of $\mathcal{M}^{(p)}_{g,N+1}$. Starting from a fibrewise representation $\mathcal{V}_{\widetilde{W}^{(p)}}$, the sheaf of conformal blocks can be defined in the same way as (4.15) by

$$C_{\widetilde{W}^{(p)}} = \mathcal{V}_{\widetilde{W}^{(p)}}/\overline{\mathfrak{g}}^r_{\widetilde{W}^{(p)}}\mathcal{V}_{\widetilde{W}^{(p)}}. \tag{4.16}$$

Clearly, $\nu^{(p)*}(C_{\widetilde{W}}) = C_{\nu^{(p)*}\widetilde{W}}$.

Of special importance is the pull-back to $\widetilde{W}^{(1)}$. Recall that this means that we fix a set of first order jets of coordinates. As shown in [26, Lemma 4.3] this fixes the Krichever-Novikov basis elements uniquely. In particular we can choose in every of the above vector spaces the basis given by these elements. By their explicit form given in [18] it is obvious that they depend analytically on moduli. In this way we see that over $\widetilde{W}^{(1)}$ the sheaves

$$\mathcal{A}_{\widetilde{W}^{(1)}},\ \mathcal{L}_{\widetilde{W}^{(1)}},\ \widehat{\mathcal{L}}_{\widetilde{W}^{(1)}},\ \overline{\mathfrak{g}}_{\widetilde{W}^{(1)}},\ \widehat{\mathfrak{g}}_{\widetilde{W}^{(1)}},\ \mathcal{F}^{\lambda}_{\widetilde{W}^{(1)}}. \tag{4.17}$$

are free sheaves (of sections) of trivial infinite-dimensional vector bundles with trivializations given by the Krichever-Novikov bases. Of course, everything remains true for $\widetilde{W}^{(p)}$ instead $\widetilde{W}^{(1)}$.

Below, we take the sheaf of fermion modules $\mathcal{V}_{\widetilde{W}^{(1)}}$ as a representation sheaf. This will be our model situation. In our description, the basis fermions do not depend on moduli at all, only the Lie algebra action does via the structure constants. We trivialize the sheaf $\mathcal{V}_{\widetilde{W}^{(1)}}$ using these bases and obtain the corresponding trivial vector bundle $V\times\widetilde{W}^{(1)}$, where V is a standard fermion space $\mathcal{H}^{(m)}_{F,\tau}$ (Section 3). Over a generic point of the moduli space, the space $\overline{\mathfrak{g}}^r V$ also does not depend on moduli (the dependence due to the structure constants disappears after taking the linear span of images of the basis fermions under the action of $\overline{\mathfrak{g}}^r$). Hence, locally, the sheaf of conformal blocks is free and defines a vector bundle. Moreover, we will take $\mathfrak{g} = \mathfrak{gl}(n)$, and τ the standard representation of $\mathfrak{g}$ in the n-dimensional vector space, see Section 3. This guarantees the finite-dimensionality of coinvariants[2]. In particular, we obtain in this case that the vector bundle of conformal blocks is of finite rank. Note also that in this case the representation $\mathcal{H}_{F,\tau}^{(m)}$ is irreducible [3].

Observe, that all statements and arguments of the next section remain true under sufficiently extended assumptions about the representation sheaf $\mathcal{V}$ which are not only true for the fermionic modules. In particular, they are fulfilled for the class of representations considered in [26]. For the sheaf of algebras $\widehat{\mathfrak{g}}$ we could assume that $\mathfrak{g}$ is an arbitrary reductive Lie algebra.

4.2. **Projectively flat connection and generalized Knizhnik-Zamolodchikov equations.**

For short, denote by $\mathcal{V}$ a sheaf $\mathcal{V}_{\widetilde{W}^{(1)}}$ fulfilling the conditions given at the end of the last subsection, e.g. a sheaf of fermion representations, and let it be fixed through-out this subsection.

Let $\tau = (\tau_1, \dots, \tau_m)$ denote the local coordinates (moduli parameters, moduli) on $\mathcal{M}_{g,N+1}^{(1)}$. Denote by $\Sigma(\tau)$ the Riemann surface with corresponding conformal structure.

Choose a generic point with moduli parameters τ_0 in $\mathcal{M}_{g,N+1}^{(1)}$. Let τ_0 be represented by the geometric data $(\Sigma(\tau_0), P_1, \dots, P_N, P_\infty, z_1^{(1)} \dots, z_N^{(1)}, z_\infty^{(1)})$. In particular, $\Sigma(\tau_0)$ has a fixed conformal structure representing the algebraic curve corresponding to the moduli parameters τ_0. For τ lying in a small enough neighbourhood of τ_0, the conformal structure on $\Sigma(\tau)$ can be obtained by deforming the conformal structure $\Sigma(\tau_0)$ in the following way.

On the Riemann surface $\Sigma(\tau_0)$, we choose a local coordinate w at P_∞. Rigorously speaking, this amounts to passing (temporary) to the moduli space $\widetilde{W}^{(\infty)}$. Assume that w runs over the disc of radius 2 centered at $w = 0$. Let $U_\infty \subset \mathbb{C}$ be a unit disc with natural coordinate z. After identification of z with w we can think of U_∞ as a subset of the coordinate chart at P_∞. Let $v \in \mathcal{L}$ be a Krichever-Novikov vector field. By restriction, it defines a meromorphic vector field on U_∞. In turn, this vector field defines a family of local diffeomorphisms ϕ_t – the corresponding local flow. For t small enough they map an annulus $U_v \subset U_\infty$ which is bounded by the unit circle from outside to a deformed annulus in the disc with radius 2. We take the set on the coordinate chart at P_∞ which (after the identification of z with w) is the interior complement to $\phi_t(U_v)$, cut it out of the curve and use ϕ_t to define a gluing of U_∞ to the rest of the curve along the subset U_v. In this way, for every t we obtain another conformal structure. Depending on the vector field (and corresponding diffeomorphisms) the equivalence class of the conformal structure will change or not. But, in any case we obtain via this process any conformal structure

[2]We are grateful to B. Feigin for this remark

[3]The irreducibility is well-studied in the graded case [8, Lecture 9]. The proof for the almost graded case is similar. We will give it somewhere else.

which is close to the given one in moduli space (see [5] for further information). Moreover, the deformation in any tangent direction on the moduli spaces $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(\infty)}$ can be realized.

We assign to τ a diffeomorphism d_τ of the above described type, such that it is defined on the annulus $U_\tau \subset U_\infty$, and the gluing function $w = d_\tau(z)$ is giving the corresponding conformal structure; keeping in mind that τ uniquely defines neither d_τ nor U_τ.

In this section, let $\mathcal{A}$ denote the sheaf of Krichever-Novikov function algebras on $\widetilde{W}^{(1)}$, as well as the corresponding infinite-dimensional vector bundle. In this bundle, denote the fibre over $\tau \in \widetilde{W}^{(1)}$ by $\mathcal{A}_\tau$. If A is a section of the bundle we write it as $A(\tau) = A_\tau$, where $A_\tau \in \mathcal{A}_\tau$. Similarly, let $\mathcal{A}^r$, $\widehat{\mathcal{L}}$, $\mathcal{L}$, $\mathcal{L}^r$, $\widehat{\mathfrak{g}}$, $\overline{\mathfrak{g}}^r$ denote the sheaves (respectively the bundles) of the corresponding algebras.

Let $\mathcal{A}^{ann}_\tau$ be the algebra of regular functions on the annulus U_τ. Embed $\mathcal{A}_\tau$ into $\mathcal{A}^{ann}_\tau$ by restricting every function A_τ onto the image of U_τ in $\Sigma(\tau)$ and consider the restriction as a function of the variable w. Denote the result $A_\tau(w)$. On the other hand, restrict A_τ onto the coordinate chart at the point $P_\infty \in \Sigma(\tau)$, and denote the result $\tilde{A}_\tau(z)$. Thus, we assign to every pair $(A_\tau, \Sigma(\tau))$, in a non-unique way, a pair $(\tilde{A}_\tau, d_\tau)$, where $\tilde{A}_\tau(z) = A_\tau(d_\tau(z))$ on U_τ. For $0 < s < 1$ let $\tilde{\mathcal{A}}_s$ be the algebra of regular functions in the annulus $U_s \subset U_\infty$ with boundary circles of radius 1 and s respectively. We set $\tilde{A} = \operatorname{inj}\lim_{s\to 1} \tilde{\mathcal{A}}_s$ with respect to the natural inclusion. Clearly, $\tilde{A}_\tau \subseteq \tilde{\mathcal{A}}$ in the neighborhood of τ_0 (moreover, $\tilde{A}_\tau$ gives an element of the sheaf of germs of meromorphic functions in z at the point $z = 0$). Thus, the correspondence $A_\tau \to \tilde{A}_\tau$ gives an embedding of $\mathcal{A}$ into $\tilde{\mathcal{A}}$. Denote the subsheaf of sections of $\tilde{\mathcal{A}}$ corresponding to germs of analytic functions vanishing at P_∞ by $\tilde{\mathcal{A}}^r$. Let $\tilde{\mathcal{L}}^r$ have the similar meaning with respect to $\mathcal{L}$.

Given a vector field X on $\widetilde{W}^{(1)}$ by $\partial_X A_\tau$ we mean a full derivative in τ of $\tilde{A}_\tau(d_\tau^{-1}(w))$ along the vector field X. We interpret this as a differentiation (in τ) of a Krichever-Novikov function as function in the variable z taking account of its direct dependence on τ and of the dependence of the local coordinate z on τ. After the substitution $w = d_\tau(z)$ we consider $\partial_X A_\tau$ as an element of the sheaf $\tilde{\mathcal{A}}$.

Consider a family of local diffeomorphisms d_τ where τ runs over a disc in the space of moduli parameters. Recall that for every τ the corresponding d_τ is nothing but the gluing function $w = d_\tau(z)$ for $\Sigma(\tau)$. From this point of view, the family d_τ is nothing but a family of (local) functions depending on parameters. To stress this interpretation, define the function $d(z,\tau) = d_\tau(z)$. Define $\partial_X d_\tau$ as follows:

$$(\partial_X d_\tau)(z) = \sum_i X_i(\tau) \frac{\partial d(z,\tau)}{\partial \tau_i}. \tag{4.18}$$

Given a vector field X on $\widetilde{W}^{(1)}$ we can assign to it a local vector field $\rho(X) := d_\tau^{-1} \cdot \partial_X d_\tau$ on $\Sigma(\tau)$ which represents the *Kodaira-Spencer cohomology class* of the corresponding 1-parameter deformation family. By adding suitable coboundary terms (which amounts to composing the diffeomorphism d_τ with a diffeomorphism of the disc) we obtain

$$\rho(X) = d_\tau^{-1} \cdot \partial_X d_\tau \in \mathcal{L}. \tag{4.19}$$

Given the vector field X we can also assign to it via the isomorphism (4.11) a Krichever-Novikov vector field e_X. Note that we consider X as vector field on $\widetilde{W}^{(1)}$. Hence e_X is only fixed up to the addition of elements of $\mathcal{L}_-^{(1)}$. See Section 2.4 for the definition of $\mathcal{L}_-^{(1)}$; its elements correspond to the infinitesimal changes of the coordinate at P_∞. We call every such element e_X a *pull-back* of X. Independently of the pull-back e_X and of the choice of $\rho(X)$ (satisfying (4.19)) we have $e_X - \rho(X) \in \mathcal{L}^r$, hence

$$e_X = \rho(X) + e^r, \tag{4.20}$$

with $e^r \in \mathcal{L}^r$ (e^r depends on both e_X and $\rho(X)$).

Proposition 4.2. *For every section A of the sheaf $\mathcal{A}$, and every local vector field X define A^X by the relation*

$$\partial_X A = -(e_X).A + A^X. \tag{4.21}$$

Then $A^X \in \tilde{\mathcal{A}}$, and, moreover, $A^X \in \tilde{\mathcal{A}}^r$ for $A \in \mathcal{A}^r$.

Proof. According to the above given definition, $\partial_X A_\tau$ is the full derivative of $A_\tau = \tilde{A}_\tau \cdot d_\tau^{-1}$ along X where $\cdot$ denotes a composition of maps. By the chain rule,

$$\partial_X A_\tau = \tilde{\partial}_X \tilde{A}_\tau + \partial_z \tilde{A}_\tau \cdot \partial_X d_\tau^{-1} = \tilde{\partial}_X \tilde{A}_\tau + \left(-\widetilde{\rho(X)} \frac{\partial}{\partial z} \tilde{A}_\tau \right) \cdot d_\tau^{-1}, \tag{4.22}$$

where $\tilde{\partial}_X \tilde{A}_\tau$ is a derivative along X with the assumption of independence of the local coordinate z of τ, $\partial_z \tilde{A}_\tau$ is a differential of $\tilde{A}_\tau$ (in the variable z), and $\widetilde{\rho(X)} \frac{\partial}{\partial z}$ is the first order differential operator corresponding to the vector field $\rho(X) = d_\tau^{-1} \cdot \partial_X d_\tau$. In more detail, $\partial_X d_\tau^{-1} = -d_\tau^{-1} \cdot \partial_X d_\tau \cdot d_\tau^{-1}$, hence $\partial_z \tilde{A}_\tau \cdot \partial_X d_\tau^{-1} = \partial_z \tilde{A}_\tau \cdot (-d_\tau^{-1} \cdot \partial_X d_\tau) \cdot d_\tau^{-1}$. By (4.19), for every τ, the $\rho(X) = d_\tau^{-1} \cdot \partial_X d_\tau$ is an element of $\mathcal{L}_\tau$. Denote the corresponding first order differential operator (in the variable z) by $\widetilde{\rho(X)} \frac{\partial}{\partial z}$. Observe that $\partial_z \tilde{A}_\tau \cdot (d_\tau^{-1} \cdot \partial_X d_\tau) = \widetilde{\rho(X)} \frac{\partial}{\partial z} \tilde{A}$.

Making use of (4.20), we replace $\widetilde{\rho(X)} \frac{\partial}{\partial z}$ in (4.22) with the differential operator corresponding to $e_X - e^r$ and obtain (4.21) with $A_\tau^X = \tilde{\partial}_X \tilde{A}_\tau + e^r A_\tau \in \tilde{\mathcal{A}}$.

Assume, $A_\tau \in \mathcal{A}^r$ for every τ, i.e. $A_\tau(P_\infty) = 0$ for $P_\infty \in \Sigma(\tau)$. We have $e^r \in \mathcal{L}^r$ by definition of the pull-back, hence also $e^r(P_\infty) = 0$. Since $z(P_\infty) = 0$, $A_\tau(P_\infty) = 0$ implies $\tilde{A}_\tau(0) = 0$, for every τ, and, further on, $\tilde{\partial}_X \tilde{A}_\tau(0) = 0$. Therefore, $A_\tau^X(P_\infty) = 0$, hence $A_\tau^X \in (\tilde{\mathcal{A}}^r)_\tau$ for every τ, which completes the proof. ∎

Let $\mathcal{L}$ be the sheaf of Krichever-Novikov vector field algebras, and e be a meromorphic section of it. Turning to the definition of $\partial_X e$, we observe that there is also no conventional one. In analogy with the case of functions, we could define it using local vector fields on U_τ. Another possibility, which we prefer here, is to define it via Leibniz rule: i.e. for every $A \in \mathcal{A}$, by definition,

$$(\partial_X e).A = \partial_X(e.A) - e.\partial_X A.$$

Proposition 4.3. *For every section e of the sheaf $\mathcal{L}$, and every local vector field X define e^X by the relation*

$$\partial_X e = -[e_X, e] + e^X. \tag{4.23}$$

Then $e^X \in \tilde{\mathcal{L}}$, and, moreover, $e^X \in \tilde{\mathcal{L}}^r$ for $e \in \mathcal{L}^r$.

Proof. By Proposition 4.2 we have $\partial_X(e.A) = -e_X.(e.A) + (e.A)^X$ where, $(eA)^X = \tilde{\partial}_X(\widetilde{e.A}) + e^r.(e.A)$. The latter follows from the proof of Proposition 4.2. Similarly, $e.(\partial_X A) = e.(-e_X.A + \tilde{\partial}_X \tilde{A} + e^r.A)$. All together

$$(\partial_X e).A = -[e_X, e].A + \tilde{\partial}_X(\widetilde{e.A}) - e.\tilde{\partial}_X \tilde{A} + [e^r, e].A. \tag{4.24}$$

Since the objects e and A are global, we have $\widetilde{e.A} = \tilde{e}.\tilde{A}$. Applying the Leibniz rule again, we obtain $\tilde{\partial}_X(\widetilde{e.A}) - e.\tilde{\partial}_X \tilde{A} = (\tilde{\partial}_X \tilde{e})\tilde{A}$. Since (4.24) is a relation in the sheaf $\tilde{\mathcal{A}}$, we do not distinguish between A and $\tilde{A}$. Hence, (4.24) implies (4.23) where $e^X = \tilde{\partial}_X \tilde{e} + [e^r, e] \in \tilde{\mathcal{L}}$.

If $e \in \mathcal{L}^r$ then $[e^r, e] \in \mathcal{L}^r$ since $e^r \in \mathcal{L}^r$ and $\mathcal{L}^r$ is a subalgebra. Further on, $\tilde{\partial}_X \tilde{e}(0) = 0$ for the same reason as $\tilde{\partial}_X \tilde{A}(0) = 0$ in the proof of Proposition 4.2. Thus, $e^X \in \tilde{\mathcal{L}}^r$ which completes the proof. □

Consider a sheaf of operators on the local sections of the sheaf $\mathcal{V}$. Assume B to be a local section of it. By definition, $\partial_X B = [\partial_X, B]$, where, on the right hand side, ∂_X is a differentiation of sections of the sheaf $\mathcal{V}$. Here the following cases occur: $B = u(A)$, where $u \in \mathfrak{g}$ and $A \in \mathcal{A}$ and $B = T(e)$, the Sugawara operator introduced by (3.14)which was denoted $T[e]$ earlier[4] .

For every pull-back e_X of X we introduce the following first order differential operator on sections of the trivial sheaf $\mathcal{V}$:

$$\nabla_X = \partial_X + T(e_X), \tag{4.25}$$

where $\partial_X = \sum_i X_i(\tau) \frac{\partial}{\partial \tau_i}$.

Proposition 4.4. *∇_X is well-defined on conformal blocks and is independent of a pull-back of X there.*

Before we prove this proposition we have to extend the operators $u(A)$ and $T(e)$ to the case when A or e are local objects. Local vector fields (functions, currents etc.) form completions of the corresponding Krichever-Novikov objects since they have infinite expansions over the corresponding Krichever-Novikov bases [10]. By (4.10) and Proposition 4.3 we can restrict ourselves

[4] In this section there will be no danger of confusion of $T(e)$ with (3.5), hence, we choose the notation $T(e)$ to avoid confusion with the Lie bracket.

with expansions with only a finite pole order at P_∞, i.e. with finitely many components of positive degree. Define $\widetilde{V}$ as the space of formal linear combinations of basis fermions which are infinite in negative direction (with respect to the fermion degree (3.4)). Due to the almost-gradedness, the action of the above operators can be extended on $\widetilde{V}$ with the advantage, that these extensions do exist also for A and e local. Indeed, only finitely many terms of the expansions (of the operator and of the corresponding element in $\widetilde{V}$) contribute in the result to the component of a given degree. Let $\widetilde{\mathfrak{g}}^r$ be the regular subalgebra completed in this way. Obviously, $V/\overline{\mathfrak{g}}^r V = \widetilde{V}/\widetilde{\mathfrak{g}}^r\widetilde{V}$, hence the conformal blocks (Definition 4.1) can be defined via completed objects.

First, we prove the following Lemma.

Lemma 4.5. *For the fermion representations we have*

$$\partial_X u(A) = u(\partial_X A) \tag{4.26}$$

Proof. For every $u \in \mathfrak{g}$, $A \in \mathcal{A}$ and every basis fermion $\Psi = \psi_{i_1} \wedge \psi_{i_2} \dots$ we have $u(A)\Psi = (uA)\psi_{i_1} \wedge \psi_{i_2} \dots + \psi_{i_1} \wedge (uA)\psi_{i_2} \dots + \lambda_1 \cdot \Psi$, where in the expressions $(uA)\psi_{i_k}$ the term with ψ_{i_k} (if there is any) has to be ignored by regularization and the last term is the counter term coming from regularization. Since basis fermions do not depend on moduli, we have

$$\begin{aligned}(\partial_X u(A))\Psi &= \partial_X(u(A)\Psi)\\ &= (u\partial_X A)\psi_{i_1} \wedge \psi_{i_2} \dots + \psi_{i_1} \wedge (u\partial_X A)\psi_{i_2} \dots + (\partial_X\lambda_1)\cdot\Psi\\ &= u(\partial_X A)\Psi + (\partial_X \lambda_1)\cdot\Psi - \lambda_2\cdot\Psi\end{aligned} \tag{4.27}$$

where $\lambda_2 \cdot \Psi$ appears due to regularization of $u(\partial_X A)$. As long as no regularization is necessary, the relation (4.26) follows immediately. The regularization can be easily calculated via the matrix of the operator uA in the space of sections of the holomorphic bundle involved. Let $uA = \sum\limits_{i,j=-\infty}^{\infty} a_{ij}E_{ij}$ where $\{E_{ij} | i,j \in \mathbb{Z}\}$ is the natural basis in the matrix space. By the regularization procedure [8]

$$\lambda_1 = \sum_{i\in N_-} a_{ii} - \sum_{i \in N_+} a_{ii},$$

where N_+ is the set of non-occupied positions, or holes of positive degree in Ψ, i.e. $N_+ = \mathbb{N} \setminus \{i_1, i_2, \cdots\}$, and N_- is the set of occupied positions of degree ≤ 0.[5] Similarly, $\lambda_2 = \sum_{i\in N_-} \partial_X a_{ii} - \sum_{i\in N_+} \partial_X a_{ii}$. Therefore, $\partial_X \lambda_1 - \lambda_2 = 0$ and the claim is true also in this case. □

For more general representations, we take relation (4.26) as an additional requirement.

Proof of Proposition 4.4. Using Lemma 4.5 and Lemma 3.7 we find

$$\begin{aligned}[\nabla_X, u(A)] &= [\partial_X + T(e_X), u(A)] = [\partial_X, u(A)] + [T(e_X), u(A)]\\ &= u(\partial_X A) + u(e_X.A).\end{aligned}$$

[5] Other descriptions are possible, but they are equivalent.

Assume, $A \in \mathcal{A}^r$. Then, by Proposition 4.2, we have

$$u(\partial_X A) = -u(e_X.A) + u(A^X),$$

where $A^X \in \tilde{\mathcal{A}}^r$. Hence, $[\nabla_X, u(A)] = u(A^X)$ which implies $[\nabla_X, u(\mathcal{A}^r)] \subseteq u(\tilde{\mathcal{A}}^r)$, and, further on

$$[\nabla_X, u(\tilde{\mathcal{A}}^r)] \subseteq u(\tilde{\mathcal{A}}^r).$$

Hence $\widetilde{\mathfrak{g}}^r\widetilde{V}$ is a ∇_X-invariant subspace and ∇_X is well-defined on $\widetilde{V}/\widetilde{\mathfrak{g}}^r\widetilde{V}$. $\square$

Lemma 4.6. *For every $X \in T\mathcal{M}^{(1)}_{g,N+1}$ we have*

$$\partial_X T(e) = T(\partial_X e) + \lambda \cdot id,$$

where $\lambda = \lambda(X, e) \in \mathbb{C}$.

Proof. By the fundamental relation (Lemma 3.7) for every $e \in \mathcal{L}$, $u \in \mathfrak{g}$, $A \in \mathcal{A}$

$$[T(e), u(A)] = u(e.A). \tag{4.28}$$

Take the derivative on both sides of the relation (4.28) along a local vector field $X \in \mathrm{T}\widetilde{W}^{(1)}$. By Lemma 4.5 we obtain

$$[\partial_X T(e), u(A)] + [T(e), \partial_X u(A)] = u((\partial_X e).A) + u(e.(\partial_X A)). \tag{4.29}$$

Again, by (4.28) and Lemma 4.5 the second terms on both sides of (4.29) are equal. Therefore,

$$[\partial_X T(e), u(A)] = u((\partial_X e).A).$$

Applying (4.28) once more, we replace the right hand side of the latter relation by $[T(\partial_X e), u(A)]$ (see the remark below). Therefore,

$$[\partial_X T(e) - T(\partial_X e), u(A)] = 0 \tag{4.30}$$

for every $X \in \mathrm{T}\widetilde{W}^{(1)}$, $u \in \mathfrak{g}$, $A \in \mathcal{A}$.

By standard arguments of the theory of highest weight representations (either by the irreducibility of the representation or by uniqueness of the vacuum vector), the commutation relations (4.30) immediately imply the lemma. $\square$

Remark. The $\partial_X e$ is a local vector field on a deformed annulus (an element of the sheaf $\tilde{\mathcal{L}}$ to be more precise). Hence, we need the relation (4.28) for local vector fields to prove (4.30). Due to the definition given after the formulation of Proposition 4.4, the representations $T(e)$ and $u(A)$ are well defined also on local vector fields and functions, respectively, with preserving the relation (4.28). Indeed, for an $A \in \mathcal{A}$, a homogeneous $v \in V$ and an arbitrary n there exists a partial sum $\tilde{e}$ of the expansion for $\partial_X e$ such that $(u((\partial_X e).A)v)_n = (u(\tilde{e}.A)v)_n$ and $([T(\partial_X e), u(A)]v)_n = ([T(\tilde{e}), u(A)]v)_n$ where $(\cdot)_n$ denotes the projection onto the component of degree n in $\overline{V}$. By (4.28) the right hand sides of the last two relations are equal, hence their left hand sides also are equal. This implies the relation (4.28) with $\partial_X e$ instead e.

For the remainder of this section, our goal is to prove the projective flatness of (4.25) and to introduce the corresponding analog of the Knizhnik-Zamolodchikov equations.

Lemma 4.7. *For every pull-backs e_X, e_Y of local vector fields X, Y to $\mathcal{L}$, there exist a pull-back $e_{[X,Y]}$ of $[X,Y]$ such that*

$$e_{[X,Y]} = [e_X, e_Y] + \partial_X e_Y - \partial_Y e_X.$$

Proof. By [34, Lemma 1.3.8]

$$\rho([X,Y]) = [\rho(X), \rho(Y)] + \partial_X \rho(Y) - \partial_Y \rho(X). \tag{4.31}$$

Observe that for every pull-backs e_X, e_Y we have $e_X = \rho(X) + e_1^r$, $e_Y = \rho(Y) + e_2^r$, where $e_1^r e_2^r \in \mathcal{L}^r$, see (4.20). In terms of e_X, e_Y (4.31) reads as

$$\rho([X,Y]) = [e_X, e_Y] + \partial_X e_Y - \partial_Y e_X + e_3^r \tag{4.32}$$

where $e_3^r = [e_1^r, e_2^r] - (\partial_X e_2^r + [e_X, e_2^r]) + (\partial_Y e_1^r + [e_Y, e_1^r])$.

Since $\mathcal{L}^r$ is a Lie subalgebra, we have $[e_1^r, e_2^r] \in \mathcal{L}^r$. By Proposition 4.3, the elements $\partial_X e_2^r + [e_X, e_2^r]$ and $\partial_Y e_1^r + [e_Y, e_1^r]$ are also regular, hence $e_3^r \in \mathcal{L}^r$. Thus, $e_{[X,Y]} = \rho([X,Y]) - e_3^r$ is a pull-back of $[X,Y]$, and the lemma is proved. □

Theorem 4.8. ∇_X *is a projectively flat connection on the vector bundle of conformal blocks.*

Proof.

$$\begin{aligned}[\nabla_X, \nabla_Y] &= [\partial_X + T(e_X), \partial_Y + T(e_Y)] \\ &= [\partial_X, \partial_Y] + [\partial_X, T(e_Y)] - [\partial_Y, T(e_X)] + [T(e_X), T(e_Y)].\end{aligned} \tag{4.33}$$

Since T is a projective representation of $\mathcal{L}$ and due to the relations $[\partial_X, T(e_Y)] = \partial_X T(e_Y)$, $[\partial_Y, T(e_X)] = \partial_Y T(e_X)$, we can rewrite (4.33) in the following form:

$$[\nabla_X, \nabla_Y] = \partial_{[X,Y]} + T(\partial_X e_Y - \partial_Y e_X + [e_X, e_Y]) + \lambda \cdot id. \tag{4.34}$$

Here we used also Lemma 4.6. By Lemma 4.7, this reads as

$$\begin{aligned}[\nabla_X, \nabla_Y] &= \partial_{[X,Y]} + T(e_{[X,Y]}) + \lambda \cdot id \\ &= \nabla_{[X,Y]} + \lambda \cdot id.\end{aligned} \tag{4.35}$$

□

We consider the following equations for horizontal sections of the connection ∇_X as a generalization of Knizhnik-Zamolodchikov equations:

$$\nabla_X \Psi = 0, \quad X \in H^0(\mathcal{U}, \mathrm{T}\mathcal{M}_{g,N+1}^{(1)}) \tag{4.36}$$

where Ψ is a section of the sheaf of conformal blocks. These equations are proposed in [26], and for $g = 0$ and $g = 1$, but for different choices of representations of the affine algebra, the operator ∇_X was explicitly calculated there. Nevertheless, the calculation there is valid in the cases under consideration here.

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