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Probability Theory

A Bismut type formula for the Hessian of heat semigroups
Une formule de Bismut pour la hessienne d'un semigroupe de la chaleur

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Abstract

We obtain an intrinsic version of a Bismut type formula for the Hessian of heat semigroups, resp. harmonic functions, by computing second order directional derivatives of families of martingales, along with filtering of redundant noise. As applications we provide a Hessian estimate in the general case as well as a slightly improved one in the radially symmetric situation. ***To cite this article: M. Arnaudon et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On écrit une formule de Bismut intrinsèque pour la hessienne d'un semigroupe de la chaleur ou d'une fonction harmonique sur une variété, en calculant des dérivées secondes directionnelles de familles de martingales, et en filtrant ensuite le bruit superflu. Cela nous permet d'obtenir des estimées de la hessienne dans un cadre très général. Avec des hypothèses de symétrie radiale de la variété, on améliore encore ces estimées. ***Pour citer cet article : M. Arnaudon et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit M une variété de dimension n, munie d'une connexion ∇. On considère la solution sur M d'une EDS d'Itô non-dégénérée $\mathrm{d}^\nabla X(x) = A(X(x))\,\mathrm{d}Z$, $X_0(x) = x$, où $A \in \Gamma(\mathrm{Hom}(\mathbb{R}^m, TM))$ et Z est un mouvement brownien dans $\mathbb{R}^m$. On désigne par $W_s : T_xM \to T_{X_s}M$ le transport parallèle déformé le long des trajectoires de $X_.(x)$. On choisit deux ouverts relativement compacts D_1 et D_2 de M tels que $x \in D_1$, $\overline{D}_1 \subset D_2$, $\overline{D}_2 \neq M$. Pour $0 < t_1 < t_2 \leqslant t$, on définit les temps d'arrêt $\sigma := \inf\{s \geqslant 0\colon\ X_s(x) \notin D_1\} \wedge t_1$ et $\tau := \inf\{s \geqslant 0\colon\ X_s(x) \notin D_2\} \wedge t_2$.

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Notre résultat principal est le suivant.

Théorème 0.1. *Soit ∇ satisfaisant la condition de Le Jan–Watanabe et F une fonction harmonique sur M au sens où $F(s, X_s(x))$, $s \geqslant 0$, est une martingale locale pour tout $x \in M$. Posant $F_s(x) = F(s, x)$, on a la formule* (8) *ci-dessous, où K, L sont adaptés par rapport à $\mathscr{F}^{X(x)}$, d'énergie L^2 finie, vérifiant $K_s = v$ pour $s \leqslant \sigma$, $K_s = 0$ pour $s \geqslant \tau$, $L_0 = w$ et $L_s = 0$ pour $s \geqslant \sigma$.*

On s'intéresse essentiellement aux deux exemples suivants. (1) (cas elliptique) $F_s(x) = F(x)$ où F est $\mathscr{L}$-harmonique sur M ; et (2) (cas parabolique) $F_s(x) = (P_{t-s} f)(x)$ où f est mesurable bornée sur M. Ici $P_t f(x) = \mathbb{E}[f(X_t(x))1_{\{t<\zeta(x)\}}]$ est le semi-groupe minimal associé au générateur $\mathscr{L} = \frac{1}{2}\Delta_M + V = \frac{1}{2}\sum_{i=1}^n (A(\cdot)e_i)^2$, où $V = \mathrm{tr}(\nabla A \otimes A)$, et $\zeta(x)$ est le temps de vie de $X.(x)$.

Le Théorème 0.1 permet d'obtenir des estimées de la hessienne de F_0.

Théorème 0.2. *Avec les hypothèses du Théorème* 0.1, *on a*

$$|\mathrm{Hess}_x F_0| \equiv \sup_{v \in T_x M, |v|=1} \left|\mathrm{Hess}_x F_0(v, v)\right| \leqslant C_{\mathrm{Hess}}\Big(\sup_{D_2} |F_t| \vee \sup_{[0,t]\times\partial D_2} |F|\Big),$$

où la constante C_{Hess} est définie en (10), *ou au Corollaire* 3.2.

Notre dernière application concerne les fonctions harmoniques dans les variétés possédant une symétrie sphérique par rapport à un point.

Corollaire 0.3. *On suppose que M a une symétrie sphérique par rapport à un point x. Soit u une fonction harmonique définie sur M. Au point x, on a les estimées $|\mathrm{grad}_x u| \leqslant C_{\mathrm{grad}}(r)u(x)$ et $|\mathrm{Hess}_x u| \leqslant C_{\mathrm{Hess}}(r)u(x)$, où $C_{\mathrm{Hess}} \equiv C_{\mathrm{Hess}}(r)$ est définie au Corollaire* 3.2 (*pour $t > 0$ arbitraire*) *et $C_{\mathrm{grad}}(r)$ est définie en* (13).

1. The basic formula

Let M be a n-dimensional differentiable manifold endowed with a torsionfree linear connection ∇. Let $X(x)$ be defined as solution to the Itô SDE on M,

$$\mathrm{d}^\nabla X(x) = A\big(X(x)\big)\,\mathrm{d}Z, \quad X_0(x) = x, \tag{1}$$

where $A \in \Gamma(\mathrm{Hom}(\mathbb{R}^m, TM))$ and Z is $\mathrm{BM}(\mathbb{R}^m)$. We assume that SDE (1) is non-degenerate (elliptic) in the sense that $A(x): \mathbb{R}^m \to T_x M$ is surjective for each $x \in M$. Then there exists a Riemannian metric on TM such that $A^*(x): T_x M \to \mathbb{R}^m$ is an isometric embedding, i.e., $A(x)A^*(x) = \mathrm{id}_{T_x M}$ for each $x \in M$.

Let F: $[0, t] \times M \to M$ be a space-time harmonic function in the sense that $F(s, X_s(x))$, $s \geqslant 0$, is a continuous local martingale for any $x \in M$. Write $F_s(x) = F(s, x)$. We have mainly two examples in mind: (1) (Elliptic case) $F_s(x) = F(x)$ where F is $\mathscr{L}$-harmonic on M; and (2) (Parabolic case) $F_s(x) = (P_{t-s} f)(x)$ where f is bounded measurable on M. Herein $P_t f(x) = \mathbb{E}[f(X_t(x))1_{\{t<\zeta(x)\}}]$ is the minimal heat semigroup with respect to the generator $\mathscr{L} = \frac{1}{2}\sum_{i=1}^n (A(\cdot)e_i)^2 = \frac{1}{2}\Delta_M + V$ where $V = \mathrm{tr}(\nabla A \otimes A)$, and $\zeta(x)$ denotes the lifetime of $X.(x)$.

We fix $x \in M$. Denoting by $W_s: T_x M \to T_{X_s} M$ the damped transport along the paths of $X.(x)$, cf. [5], p. 295, or [7] (1.5), we have the local martingale (cf. [1], Lemma 1):

$$N_s := (\mathrm{d}F_s)_{X_s(x)} W_s L_s - F_s\big(X_s(x)\big) \int_0^s \big\langle W_r \dot{L}_r, A\big(X_r(x)\big)\,\mathrm{d}Z_r\big\rangle, \tag{2}$$

where L_s may be any adapted finite energy process with paths in the Cameron–Martin space $\mathbb{H}([0, t], T_x M)$ such that $L_0 = w \in T_x M$ and $L_{\sigma+.} = 0$ for some stopping time $0 < \sigma < \zeta(x) \wedge t$. We write $L_s = w + \int_0^s W_r^{-1} A(X_r(x))\ell_r\,\mathrm{d}r$ with $\ell_s \in L^2_{\mathrm{loc}}(Z)$. Note that, invoking the decomposition $\mathbb{R}^m = \ker A(X_r(x)) \oplus$

$\ker A(X_r(x))^\perp$, we may assume that already $\ell_r \in \ker A(X_r(x))^\perp$ a.s. Let $X_{s*} \equiv TX_s$ denote the differential of the map $x' \to X_s(x')$. Computing the covariant derivative of N_s in direction $v \in T_xM$ yields the local martingale:

$$\nabla_v N_s = \nabla \mathrm{d}F_s(X_{s*}v, W_sL_s) + (\mathrm{d}F_s)_{X_s(x)} \nabla W_s(v, L_s) - (\mathrm{d}F_s)_{X_s(x)} X_{s*}v \int_0^s \langle W_r\dot{L}_r, A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle$$
$$- F_s\big(X_s(x)\big) \int_0^s \langle \nabla W_r(v, \dot{L}_r), A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle - F_s\big(X_s(x)\big) \int_0^s \langle W_r\dot{L}_r, \nabla A\big(X_{r*}v\big)\,\mathrm{d}Z_r\rangle.$$

Let now K_s be another adapted finite energy process with paths in $\mathbb{H}([0,t], T_xM)$ such that $K|[0,\sigma] \equiv v$ and $K|[\tau, t] \equiv 0$ where τ is a stopping time such that $\sigma < \tau \leqslant \zeta(x) \wedge t$. Exploiting the local martingale property of $N'_s := (\mathrm{d}F_s)_{X_s(x)} X_{s*}K_s - F_s(X_s(x)) \int_0^s \langle X_{r*}\dot{K}_r, A(X_r(x))\,\mathrm{d}Z_r\rangle$, we get the local martingale:

$$n_s = \nabla \mathrm{d}F_s(X_{s*}v, W_sL_s) + (\mathrm{d}F_s)_{X_s(x)} \nabla W_s(v, L_s)$$
$$- F_s\big(X_s(x)\big) \int_0^s \langle \nabla W_r(v, \dot{L}_r), A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle - F_s\big(X_s(x)\big) \int_0^s \langle W_r\dot{L}_r, \nabla A(X_{r*}v)\,\mathrm{d}Z_r\rangle$$
$$- \left[(\mathrm{d}F_s)_{X_s(x)} X_{s*}K_s - F_s\big(X_s(x)\big) \int_0^s \langle X_{r*}\dot{K}_r, A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle\right] \int_0^s \langle W_r\dot{L}_r, A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle.$$

Note that n_s is non-intrinsic in the sense that it is adapted to the filtration $(\mathscr{F}_s^Z)$ generated by Z, whereas N_s itself is $(\mathscr{F}_s^{X(x)})$-adapted (cf. [3]). However, n_s is a good candidate for filtering out redundant noise, since all terms on the r.h.s. depend linearly on just one factor that is non-measurable with respect to $(\mathscr{F}_s^{X(x)})$.

Lemma 1.1. *Let K, L be bounded adapted finite energy processes with paths in $\mathbb{H}([0,t], T_xM)$ such that $K|[0,\sigma] \equiv v$ and $K|[\tau, t] \equiv 0$, resp., $L_0 = w$, $L|[\sigma, t] \equiv 0$, where $0 < \sigma < \tau \leqslant \zeta(x) \wedge t$. Assuming that $(n_s)_{0 \leqslant s \leqslant t}$ is a true martingale, by comparing the expectation at* 0 *and t, the following formula results*:

$$\nabla \mathrm{d}F_0(v, w) = -\mathbb{E}\left[F_\sigma\big(X_\sigma(x)\big) \int_0^\sigma \langle \nabla W_r(v, \dot{L}_r), A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle\right] - \mathbb{E}\left[F_\sigma\big(X_\sigma(x)\big) \int_0^\sigma \langle W_r\dot{L}_r, \nabla A(X_{r*}v)\,\mathrm{d}Z_r\rangle\right]$$
$$+ \mathbb{E}\left[F_\tau\big(X_\tau(x)\big) \int_0^\sigma \langle W_r\dot{L}_r, A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle \int_\sigma^\tau \langle X_{r*}\dot{K}_r, A\big(X_r(x)\big)\,\mathrm{d}Z_r\rangle\right]. \tag{3}$$

Remark 1. If one starts from the non-intrinsic equivalent of N_s (with W_s replaced by X_{s*}) and takes $\sigma = t/2$, $\tau = t$ in the non-explosive situation, the special choice of linearly decaying processes L_s and K_s provides the result in [4], Theorem 3.1, if all martingale assumptions hold true (e.g., for M compact).

Remark 2. Formula (3) relies on the fact that certain local martingales are indeed true martingales. It is easy to see that this can always be achieved by a proper choice of the stopping times σ and τ. For instance, given $x \in M$, we may fix open, relatively compact subsets D_1, D_2 of M such that $x \in D_1$, $\overline{D}_1 \subset D_2$, $\overline{D}_2 \neq M$. Then, for $0 < t_1 < t_2 \leqslant t$, the stopping times $\sigma := \inf\{s \geqslant 0\colon X_s(x) \notin D_1\} \wedge t_1$ and $\tau := \inf\{s \geqslant 0\colon X_s(x) \notin D_2\} \wedge t_2$ will do it, by the fact that all coefficients are bounded on $\overline{D}_1$, resp. $\overline{D}_2$.

2. Filtering out noise

From now on we assume in addition that (A, ∇) satisfies the Le Jan–Watanabe condition (see [3]), which in particular means that $\operatorname{tr} \nabla A \otimes A = 0$. In the case of a gradient Brownian system, we may use the Levi-Civita connection. Then from the covariant SDE $DTX = \nabla_{TX} A\,\mathrm{d}Z - \frac{1}{2} R(TX, \mathrm{d}X)\,\mathrm{d}X$ for the derivative process of X, where $D = /\!/_{0,\cdot}\, d\, /\!/_{0,\cdot}^{-1}$ with $/\!/_{0,\cdot}$ parallel transport in TM along $X(x)$, we deduce that

$$DTX \otimes \mathrm{d}X = \mathrm{tr}(\nabla_{TX} A \otimes A)\,\mathrm{d}t = 0 \quad \text{and} \quad \nabla_{TX}\,\mathrm{d}X \otimes \mathrm{d}X = \mathrm{tr}(\nabla_{TX} A \otimes A)\,\mathrm{d}t = 0. \tag{4}$$

An immediate consequence is the vanishing of

$$\mathbb{E}\left[F_\sigma\big(X_\sigma(x)\big)\int_0^\sigma \langle W_r \dot{L}_r, \nabla A(X_{r*}v)\,\mathrm{d}Z_r\rangle\right] = \mathbb{E}\left[\int_0^\sigma \big\langle \mathrm{d}F_r\big(X_r(x)\big) \otimes W_r\dot{L}_r, \mathrm{d}X_r \otimes \nabla_{TX_r v}\,\mathrm{d}X_r\big\rangle\right].$$

The next step is to make the remaining part of (3) intrinsic. To this end we start from the following general commutation formula ([2], Theorem 4.5):

$$D\nabla W = \nabla DW + R\big(\mathrm{d}^\nabla X, TX\big)W + R(\mathrm{d}X, TX)DW + \frac{1}{2}\nabla R(\mathrm{d}X, \mathrm{d}X, TX)W + \frac{1}{2}R(\mathrm{d}X, DTX)W.$$

Eq. (4), along with the fact that $\mathrm{d}X \otimes DW = 0$, cancels out the third and the last term on the r.h.s. To calculate the first term we differentiate $DW(w) = -\frac{1}{2}R(W(w), \mathrm{d}X)\,\mathrm{d}X$ in the direction of v where we use the second part of (4). Substituting in the commutation formula gives

$$\begin{aligned} D\nabla W(v,w) \;=\; & R\big(\mathrm{d}^\nabla X, TX(v)\big)W(w) - \frac{1}{2}\nabla R\big(TX(v), W(w), \mathrm{d}X\big)\,\mathrm{d}X \\ & - \frac{1}{2}R\big(\nabla W(v,w), \mathrm{d}X\big)\,\mathrm{d}X + \frac{1}{2}\nabla R\big(\mathrm{d}X, \mathrm{d}X, TX(v)\big)W(w). \end{aligned} \tag{5}$$

But Itô's product formula provides $W\,\mathrm{d}(W^{-1}\nabla W) = W\,\mathrm{d}(W^{-1}/\!/_{0,\cdot})\ /\!/_{0,\cdot}^{-1}\nabla W + /\!/_{0,\cdot}\,\mathrm{d}(/\!/_{0,\cdot}^{-1}\nabla W) = \frac{1}{2}R(\nabla W, \mathrm{d}X)$ $\mathrm{d}X + D\nabla W$, which combined with Eq. (5) gives

$$W\,\mathrm{d}\big(W^{-1}\nabla W\big) = R\big(\mathrm{d}^\nabla X, TX\big)W - \frac{1}{2}\nabla\,\mathrm{Ric}^\sharp(TX, W) - \frac{1}{2}\,\mathrm{d}^*R(TX)W, \tag{6}$$

where $\mathrm{Ric}^\sharp(w) = \mathrm{Ric}(w,\cdot)^\sharp$ and $\mathrm{d}^*R(w) = -\,\mathrm{tr}\,\nabla.R^\nabla(\cdot, w)$. We write $\widetilde{\nabla W} = /\!/_{0,\cdot}\mathbb{E}[/\!/_{0,\cdot}^{-1}\nabla W \mid \mathscr{F}^{X(x)}]$ for the covariant conditional expectation of ∇W with respect to $\mathscr{F}^{X(x)}$. Then filtering redundant noise from Eq. (6) and integrating with respect to time leads to

$$W_r^{-1}\widetilde{\nabla W}_r = \int_0^r W^{-1}\left[R\big(\mathrm{d}^\nabla X, W\big)W - \frac{1}{2}\nabla\,\mathrm{Ric}^\sharp(W,W)\,\mathrm{d}s - \frac{1}{2}\,\mathrm{d}^*R(W)W\,\mathrm{d}s\right]. \tag{7}$$

Filtering out noise, i.e., taking conditional expectation with respect to $\mathscr{F}^{X(x)}$ on the r.h.s. of (3) and using (7), we obtain the following intrinsic stochastic representation formula for the Hessian of F_0.

Theorem 2.1. *Let ∇ satisfy the Le Jan–Watanabe condition and F be a space–time harmonic function on M in the sense that $F(s, X_s(x))$, $s \geqslant 0$, is a local martingale for any $x \in M$. Writing $F_s(x) = F(s,x)$, and choosing stopping times σ, τ according to Remark* 2*, we have*

$$\begin{aligned} (\nabla\,\mathrm{d}F_0)(v,w) \;=\; & \mathbb{E}\Bigg[F_\sigma \circ X_\sigma(x)\Bigg\{-\int_0^\sigma \Big\langle W_r \int_0^r W_s^{-1}R\big(\mathrm{d}^\nabla X_s, W_s v\big)W_s\,\dot{L}_r, \mathrm{d}^\nabla X_r(x)\Big\rangle \\ & + \frac{1}{2}\int_0^\sigma \Big\langle W_r \int_0^r W_s^{-1}\big(\nabla\,\mathrm{Ric}^\sharp + \mathrm{d}^*R\big)\big(W_s v, W_s \dot{L}_r\big)\,\mathrm{d}s, \mathrm{d}^\nabla X_r(x)\Big\rangle\Bigg\}\Bigg] \\ & + \mathbb{E}\left[F_\tau \circ X_\tau(x)\int_0^\sigma \big\langle W_r\dot{L}_r, \mathrm{d}^\nabla X_r(x)\big\rangle \int_\sigma^\tau \big\langle W_r\dot{K}_r, \mathrm{d}^\nabla X_r(x)\big\rangle\right], \end{aligned} \tag{8}$$

where K, L are $\mathscr{F}^{X(x)}$-adapted L^2-finite energy processes such that $K_s = v$ for $s \leqslant \sigma$ and $K_s = 0$ for $s \geqslant \tau$, *respectively,* $L_0 = w$ *and* $L_s = 0$ *for* $s \geqslant \sigma$.

Remark 3. If $F_s = P_{t-s} f$, where f is bounded measurable, denotes the minimal heat semigroup on M with respect to $\mathscr{L} = \frac{1}{2}\Delta$, the Markov property $P_{t-\tau} f(X_\tau(x)) = \mathbb{E}[F_t(X_t(x))1_{\{t<\zeta(x)\}}|\mathscr{F}_\tau^{X(x)}]$ allows us to replace $F_\tau(X_\tau(x))$ (the same for σ) in the r.h.s. of (8) by $f(X_t(x))1_{\{t<\zeta(x)\}}$.

Remark 4. If M is Ricci parallel, i.e., $\nabla \operatorname{Ric} \equiv 0$, an easy argument based on the second Bianchi identity shows that also $\mathrm{d}^* R \equiv 0$, which cancels out the second line of (8). Moreover, the damped parallel transport in this particular case reads as $W_s = /\!\!/_{0,s} \exp(-s \operatorname{Ric}_x^\sharp /2)$.

Since $\operatorname{Hess}_x(\log g)(v, w) = \operatorname{Hess}_x g(v, w)/g(x) - \langle \operatorname{grad}_x g, v\rangle\langle \operatorname{grad}_x g, w\rangle/g^2(x)$, desintegrating with respect to the volume measure leads to the following formula for the Hessian of the logarithmic heat kernel:

Corollary 2.2. *Let* $p(t, x, y)$ *be the smooth heat kernel on* $]0, \infty[\times M \times M$ *to the minimal heat semigroup* $P_t f$. *Defining* $I_\sigma^1 = \int_0^\sigma \langle W_r \dot{L}_r, \mathrm{d}^\nabla X_r\rangle$, $I_\tau^2 = \int_\sigma^\tau \langle W_r \dot{K}_r, \mathrm{d}^\nabla X_r\rangle$, *and denoting by* J_σ *the term in curly brackets in the r.h.s. of* (8), *we have for* $y \in M$ *the formula*:

$$\operatorname{Hess}_x\big(\log p(t, \cdot, y)\big)(v, w) = \mathbb{E}\big[J_\sigma + I_\sigma^1 I_\tau^2 | X_t(x) = y\big] - \mathbb{E}\big[I_\sigma^1 | X_t(x) = y\big]\,\mathbb{E}\big[I_\tau^2 | X_t(x) = y\big].$$

3. Hessian estimates

Theorem 3.1. *Let* $n = \dim M \geqslant 2$ *and consider open regular domains* D_1, D_2 *and stopping times* $\sigma,\ \tau$ *as in Remark* 2 (*where* $t_1 = t/2$ *and* $t_2 = t$). *Moreover fix* $f \in C^2(\overline{D}_1)$ *such that* $0 < f|D_1 \leqslant 1/\sqrt{2}$ *and* $f|\partial D_1 = 0$, *as well as* $\tilde f \in C^2(\overline{D}_2)$ *such that* $0 < \tilde f|D_2 \leqslant 1/\sqrt{2}$, *being constant on* D_1 *and* $\tilde f|\partial D_2 = 0$. *For* $t > 0$ *let* $F : [0, t] \times M \to \mathbb{R}$ *be a function which is space–time harmonic on* $[0, t] \times \overline{D}_2$. *We introduce curvature bounds on* D_1 *as there are*

$$C_1 := \sup_{y\in D,\ w\in T_yM,\ |w|=1} \big|\big(\nabla \operatorname{Ric}^\sharp + \mathrm{d}^* R\big)(w, w)\big|; \qquad C_2 := \sup_{y\in D,\ w,w'\in T_yM,\ |w|=|w'|=1} \big|R(w, w')w'\big|;$$

in addition C_4 *will be the upper constant in the Burkholder–Davis–Gundy inequality for* $p = 4$. *Furthermore, let* k *and* K *be constants,* $2k \neq K$, *such that*

$$k \leqslant \inf_{y\in D_2,\ w\in T_yM,\ |w|=1} \operatorname{Ric}(w, w) \leqslant \sup_{y\in D_1,\, w\in T_yM,\, |w|=1} \operatorname{Ric}(w, w) \leqslant K. \tag{9}$$

Then we have

$$\begin{aligned} &|\operatorname{Hess}_x F_0| \equiv \sup_{v\in T_xM,\, |v|=1} \big|\operatorname{Hess}_x F_0(v, v)\big| \leqslant C_{\mathrm{Hess}}\Big(\sup_{D_2} |F_t| \vee \sup_{[0,t]\times\partial D_2} |F|\Big), \quad \textit{with}\\ &C_{\mathrm{Hess}} := \frac{\sqrt{c(f)}}{f(x)\sqrt{1 - \mathrm{e}^{-c(f)t}}}\left(\frac{C_1}{|K - 2k|} + \frac{C_2 C_4^{1/4}\sqrt{n}}{\sqrt{|K - 2k|}} + \frac{(1 \vee \mathrm{e}^{-kt/4})\sqrt{\tilde c(\tilde f)}}{\tilde f(x)\sqrt{1 - \mathrm{e}^{-\tilde c(\tilde f)t}}}\right), \end{aligned} \tag{10}$$

where the constants $c(f)$, $\tilde c(\tilde f)$ *depend on* f (*resp.* $\tilde f$), *its derivatives and* k, K, *and may be chosen as* $c(f) = \sup_{D_1}(-kf^2 + 5|\operatorname{grad} f|^2 - f\Delta f)_+ + (K - 2k)_+ \sup_{D_1} f^2$, *resp.* $\tilde c(\tilde f) = \sup_{D_2}(-k\tilde f^2 + 3|\operatorname{grad} \tilde f|^2 - \tilde f\Delta \tilde f)_+$.

Proof. Since $F \circ X(x)$ is a martingale on $[0, \tau]$, we replace $F_\sigma(X_\sigma)$ in (8) by $F_\tau(X_\tau)$ and estimate $|F_\tau(X_\tau(x))| \leqslant \sup_{D_2} |F_t| \vee \sup_{[0,t]\times\partial D_2} |F|$. Concerning the stochastic integrals in the representation formula, we calculate bounds of their L^2-norms by appropriate choices of L_s and K_s related to f and $\tilde f$.

To this end we construct time changes $T(r) = \int_0^r f^{-2}(X_s(x))\,\mathrm{d}s$ for $r \leqslant$ the first exit time of $X_r(x)$ from D_1, and $\widetilde{T}(r) = \int_{\sigma\wedge r}^r \tilde f^{-2}(X_s(x))\,\mathrm{d}s$ for $r \leqslant$ the first exit time from D_2. We consider the according (right) inverse processes $\tau(r) = \inf\{s \geqslant 0\colon\ T(s) \geqslant r\}$ and $\widetilde{\tau}(r) = \inf\{s \geqslant 0\colon\ \widetilde{T}(s) \geqslant r\}$. Then the upper bound is found by using $L_r := h_1 \circ h_0(r)t^{-1}v$ and $K_r := \tilde{h}_1 \circ \tilde{h}_0(r)t^{-1}v$, where $h_0(r) = \int_0^r f^{-2}(X_s(x))1_{\{s<\tau(t)\}}\,\mathrm{d}s$ and $\tilde{h}_0(r) = \int_{\sigma\wedge r}^r \tilde{f}^{-2}(X_s(x))1_{\{s<\tilde{\tau}(t)\}}\,\mathrm{d}s$, as well as $h_1(r) = t - tc(f)/(1-\mathrm{e}^{-c(f)t})\int_0^r \mathrm{e}^{-c(f)s}\,\mathrm{d}s$ and $\tilde{h}_1(r) = t - t\tilde{c}(\tilde{f})/(1-\mathrm{e}^{-\tilde{c}(\tilde{f})t})\int_0^r \mathrm{e}^{-\tilde{c}(\tilde{f})s}\,\mathrm{d}s$. The proof is then completed following the lines of [7]; details can be found in [6]. □

Corollary 3.2. *Let $D \subset M$ be an open regular domain, $n = \dim M \geqslant 2$, and C_1, C_2, k and K from above will be defined with respect to D instead of D_1, D_2. We write $r \equiv r(x) := \mathrm{dist}(x, \partial D)$. Then we have $|\mathrm{Hess}_x F_0| \leqslant C_{\mathrm{Hess}}(\sup_D |F_t| \vee \sup_{[0,t]\times\partial D} |F|)$, where in definition* (10) *of C_{Hess} we may replace $c(f)$ by $C(r)$ and $\tilde{c}(\tilde{f})$ by $\widetilde{C}(r)$ as follows*:

$$\begin{aligned} C(r) &:= \sqrt{\left(6(n+16)/r^2 + 3\sqrt{(n-1)(-k)_+}/r + (-k)_+/2\right)_+ + (K-2k)_+/2}\,, \\ \widetilde{C}(r) &:= \sqrt{\left(6(n+10)/r^2 + 3\sqrt{(n-1)(-k)_+}/r + (-k)_+/2\right)_+}\,. \end{aligned} \tag{11}$$

Proof. We apply Theorem 3.1 in the case of $D_1 = B_{r(x)/2}$ and $D_2 = B_{r(x)}$. The constants in (11) correspond to the choices $f(p) = \varphi(\mathrm{dist}(x,p))/\sqrt{2}$ and $\tilde{f}(p) = \psi(\mathrm{dist}(x,p))/\sqrt{2}$, where by definition $\varphi(\varrho) = 1 - (2\varrho/r(x))^3$ for $0 \leqslant \varrho \leqslant r(x)/2$, resp., $\psi(\varrho) = 1 \wedge [1 - (2\varrho/r(x) - 1)^3]$ for $0 \leqslant \varrho \leqslant r(x)$. For details cf. [7], Section 5, as well as [6]. □

In our last application we focus on the case of harmonic functions on rotationally symmetric manifolds.

Corollary 3.3. *Keeping the assumptions of Corollary* 3.2, *let M be rotationally symmetric with respect to $x \in M$ and u be a harmonic function on M. Then, at the center of symmetry, we have the estimates*

$$|\mathrm{grad}_x u| \leqslant C_{\mathrm{grad}}(r)u(x) \quad \textit{and} \quad |\mathrm{Hess}_x u| \leqslant C_{\mathrm{Hess}}(r)u(x), \tag{12}$$

where $C_{\mathrm{Hess}} \equiv C_{\mathrm{Hess}}(r)$ is given as in Corollary 3.2 *above* (*for arbitrary $t > 0$*) *and*

$$C_{\mathrm{grad}}(r) := \frac{1}{2}\sqrt{\pi^2(n+3)/r^2 + 2\pi\left(\sqrt{(-k)_+(n-1)}\right)/r + 4(-k)_+}. \tag{13}$$

Proof. We extend the probability space underlying $X_.(x)$ by tensoring with the Lie group G of isometries on M fixing x, equipped with the Haar measure invariant under the left action of the group. We may replace $X_.(x)$ by the Brownian motion Y on the product space defined by $Y_r(\omega,\theta) = \theta(X_r(x))(\omega)$. The intrinsic gradient formula then can be written $\langle \mathrm{grad}_x u, v\rangle = \mathbb{E}[u(Y_\tau)S_\tau(v)]$ with $S_\tau(v) = \int_0^\tau \langle W_r \dot{K}_r(v), \mathrm{d}^\nabla Y_r\rangle$. The crucial observation is that S_τ obeys the rotational invariance property $S_\tau(v)(\omega,\theta) = S_\tau((\mathrm{d}\theta)_x^{-1}v)(\omega, \mathrm{id}_M)$; hence $\|S_\tau\|(\theta,\omega) = \sup_{v\in T_xM,\ |v|=1} |S_\tau(\omega,\theta)(v)|$ is independent of θ. Writing the gradient formula as iterated integral with respect to the initial probability and the Haar measure, and exploiting the mean value property of u, we obtain $|\mathrm{grad}_x u| \leqslant u(x)\,\mathbb{E}\|S_\tau\|$, where the last expectation can be bounded as in [7], Section 5. The method immediately transfers to the Hessian case. □

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