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A stochastic approach to the harmonic map heat flow on manifolds with time-dependent Riemannian metric

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Abstract

We first prove stochastic representation formulae for space–time harmonic mappings defined on manifolds with evolving Riemannian metric. We then apply these formulae to derive Liouville type theorems under appropriate curvature conditions. Space–time harmonic mappings which are defined globally in time correspond to ancient solutions to the harmonic map heat flow. As corollaries, we establish triviality of such ancient solutions in a variety of different situations.

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1. Introduction

A smooth mapping $u\colon M \to N$ between Riemannian manifolds (M, g) and (N, h) is said to be *harmonic* if its tension field $\Delta^{g,h} u \equiv \mathrm{trace}\nabla du$ vanishes, see e.g. [9,19]. Since harmonic maps are characterized by the property that they map M-valued Brownian motions to N-valued martingales (see e.g. [11, Satz 7.157(ii)]), it is natural to study them using stochastic methods,

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and this has been done in a number of papers, e.g. [2,8,14,15,29,33]. In particular, stochastic representation formulae for the differential of harmonic maps have turned out to be a powerful tool to prove Liouville theorems, i.e. theorems stating that harmonic maps in a certain class of maps and under certain topological or geometric constraints are necessarily constant [33].

Due to Perelman's proof of the geometrization and hence the Poincaré conjecture using Ricci flow [25,27,26], there is now a strong interest in studying manifolds M with time-dependent geometry. In such a context, the notion of harmonic map turns out to be no longer appropriate; however, it is natural to study *space–time harmonic maps* which by time reversal provide solutions to the *harmonic map heat flow* (or *nonlinear heat equation*), see e.g. [18,23,35].

The behavior of (positive) solutions to the linear heat equation under Ricci flow has been intensively studied during the last decade, e.g. [5,21,22]. It is clear from the static case that in the nonlinear situation under Ricci flow also the geometry of the target space will naturally play a crucial role, see e.g. [19,31].

Building on our previous work on martingales on manifolds with time-dependent connection [10], we establish stochastic representation formulae for space–time harmonic maps and solutions to the harmonic map heat flow defined on a manifold with time-dependent metric. We then apply these formulae to prove Liouville theorems for space–time harmonic maps and ancient solutions to the harmonic map heat flow under appropriate curvature conditions.

2. Stochastic representation formulae

Let M be a differentiable manifold equipped with a smooth family $g(t)$ of Riemannian metrics ($t \in (T_0, T]$ with $T_0 < T$), and let (N, h) be a Riemannian manifold. Let $u\colon (T_0, T] \times M \to N$ be a solution to the harmonic map heat flow

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta^{g(t),h} u. \tag{2.1}$$

Here $\Delta^{g(t),h} u := \operatorname{trace}\nabla du \in \Gamma(u^*TN)$ denotes the tension field of u with respect to $g(t)$ and h. Recall that, for fixed t, we have the differential $du(t,\cdot) \in \Gamma(T^*M, u(t,\cdot)^*TN)$ and the Hessian $(\nabla du)(t,\cdot) \in \Gamma(T^*M \otimes T^*M, u(t,\cdot)^*TN)$ which provides for each (t,x) a bilinear map

$$(\nabla du)(t,x)\colon T_xM \times T_xM \to T_{u(t,x)}N.$$

The trace of this bilinear map gives the tension $(\Delta^{g(t),h}u)(t,x) \in T_{u(t,x)}N$ of u at (t,x). Any solution to Eq. (2.1) with $T_0 = -\infty$ is called an *ancient solution* to the harmonic map heat flow.

Remark 2.1. Let $u\colon (T_0, T] \times M \to N$ be a solution to the harmonic map heat flow (2.1) and let

$$\hat{u}(t,\cdot) = u(T-t,\cdot) \quad \text{and} \quad \hat{g}(t) = g(T-t,\cdot)$$

be defined by time reversal. Then the mapping

$$\hat{u}\colon [0, T-T_0) \times M \to N$$

is space–time harmonic with respect to the family $\hat{g}(t)$ of metrics, i.e.

$$\frac{\partial \hat{u}}{\partial t} + \frac{1}{2}\Delta^{\hat{g}(t),h}\hat{u} = 0.$$

In particular, any ancient solution to the harmonic map heat flow (2.1) gives rise to a space–time harmonic map defined on $\mathbb{R}_+ \times M$, and vice versa. For this reason, in the sequel, we formulate our results for space–time harmonic maps; however the statements immediately apply to solutions of the harmonic map heat flow by time reversal.

From now on, let $(g(t))_{t\geq 0}$ be a smooth family of Riemannian metrics on M and (N,h) be a Riemannian manifold. Let $u\colon [0,\infty)\times M\to N$ be a space–time harmonic map in the sense that

$$\frac{\partial u}{\partial t}+\frac{1}{2}\Delta^{g(t),h}u=0. \tag{2.2}$$

Notation 2.2. Fixing a point $x\in M$, let $(X_t)_{t\geq 0}$ be a $g(t)$-Brownian motion on M [1,7,17,16,24] starting at x, and consider the image process $\tilde{X}_t := u(t,X_t)$ taking values in the target manifold N. As in [10, Theorem 9.3 and Remark 9.4] let $\Theta_{0,t} : T_xM\to T_{X_t}M$ be the damped parallel transport along X, defined by the covariant equation

$$d\left(\left(//^{\mathrm{Riem}}_{0,t}\right)^{-1}\Theta_{0,t}\right)=-\frac{1}{2}\left(//^{\mathrm{Riem}}_{0,t}\right)^{-1}\left(-\frac{\partial g}{\partial t}+\mathrm{Ric}^{g(t)}\right)^{\#}\Theta_{0,t}\,dt,\qquad \Theta_{0,0}=\mathrm{id}_{T_xM},$$

where $//^{\mathrm{Riem}}_{0,t} : T_xM\to T_{X_t}M$ is the Riemann-parallel transport along X, see [10, Definition 3.3]. Similarly, in terms of the Riemann curvature tensor $\tilde{R}$ on N, let

$$\tilde{\Theta}_{0,t}\colon T_{\tilde{X}_0}N\to T_{\tilde{X}_t}N$$

be the damped parallel transport along $\tilde{X}$, defined by the covariant equation

$$\begin{aligned} d\left(//^{-1}_{0,t}\,\tilde{\Theta}_{0,t}\right)&=-\frac{1}{2}//^{-1}_{0,t}\,\tilde{R}(\tilde{\Theta}_{0,t},d\tilde{X}_t)d\tilde{X}_t\\ &=-\frac{1}{2}\sum_{i=1}^{m}//^{-1}_{0,t}\,\tilde{R}\left(\tilde{\Theta}_{0,t},du(t,X_t)\xi^i_t\right)du(t,X_t)\xi^i_t\,dt,\qquad \tilde{\Theta}_{0,0}=\mathrm{id}_{T_{\tilde{X}_0}N}, \end{aligned}$$

where $m=\dim M$ and $(\xi^1_t,\ldots,\xi^m_t)$ is an adapted $g(t)$-orthonormal basis of $T_{X_t}M$:

$$\xi^i_t=//^{\mathrm{Riem}}_{0,t}\,e_i$$

for some fixed orthonormal basis $(e_1,\ldots,e_m)$ of T_xM.

In [10, Proposition 9.6], we obtained the following theorem which is crucial for all subsequent results.

Theorem 2.3. *For each $v\in T_xM$ the $T_{u(0,x)}N$-valued process*

$$\tilde{\Theta}^{-1}_{0,t}\,du(t,X_t)\,\Theta_{0,t}\,v,\quad t\geq 0,$$

is a local martingale.

The following corollary extends [3, Theorem 5.5] from the case of a fixed metric to the case of evolving metrics.

Corollary 2.4. *Let $T>0$. Assume that there is a constant $\alpha\in\mathbb{R}$ such that*

$$\mathrm{Ric}_{g(t)}-\frac{\partial g}{\partial t}\geq\alpha\quad \text{on } [0,T]\times M,$$

that the sectional curvatures of N are bounded from above and that the differential du of u is uniformly bounded on $[0, T] \times M$. Then, for each $0 \le t \le T$,

$$du(0, x) = \mathbb{E}\left[\tilde{\Theta}_{0,t}^{-1}\, du(t, X_t)\, \Theta_{0,t}\right]. \tag{2.3}$$

Proof. Under the above assumptions the local martingale of Theorem 2.3 is bounded on the time interval $[0, T]$ and hence a true martingale. The claim follows by taking expectations. □

Corollary 2.4 implies a first Liouville theorem under the assumptions that the metric on M evolves under uniformly strict backward super Ricci flow and that the curvature of N is non-positive.

Theorem 2.5 (*Cf. [10, Proposition 9.6]*). *Let M be connected. Suppose that there is a constant $\alpha > 0$ such that*

$$\mathrm{Ric}_{g(t)} - \frac{\partial g}{\partial t} \ge \alpha$$

(uniformly strict backward super Ricci flow), that the sectional curvatures of N are non-positive and that the differential of u is uniformly bounded. Then u is constant.

Proof. By Corollary 2.4

$$du(0, x) = \mathbb{E}\left[\tilde{\Theta}_{0,t}^{-1}\, du(t, X_t)\, \Theta_{0,t}\right]$$

for every $t \ge 0$. The curvature conditions imply that $\|\Theta_{0,t}\| \le e^{-\alpha t/2}$ and $\|\tilde{\Theta}_{0,t}^{-1}\| \le 1$, so that

$$|du(0, x)| \le e^{-\alpha t/2} \sup_{y \in M} |du(t, y)|.$$

The claim now follows from letting $t \to \infty$. □

To prove Liouville theorems under the weaker assumption that the metric on M evolves under backward super Ricci flow (not necessarily uniformly strict backward super Ricci flow) one needs more refined representation formulae which rely on integration by parts arguments.

For X and $\tilde{X}$ as above, let $B = (B_t)_{t \ge 0}$ be the Riemann-anti-development of X into $T_x M$ (hence a $T_x M$-valued Brownian motion, see [10, Remark 8.4]), and

$$\mathcal{A}_{\mathrm{def}}(\tilde{X})_t := \int_0^t \tilde{\Theta}_{0,s}^{-1} \circ d\tilde{X}_s = \int_0^t \tilde{\Theta}_{0,s}^{-1}\, du(s, X_s)\, //_{0,s}^{\mathrm{Riem}}\, dB_s \tag{2.4}$$

the deformed anti-development of $\tilde{X}$ (cf. [3, Eq. (5.32)]).

Theorem 2.6 (*Cf. [3, Theorem 5.6] for the Case of a Fixed Metric*). *Let $\ell = (\ell(t))_{t \ge 0}$ be a $T_x M$-valued process with absolutely continuous trajectories. The following $T_{u(0,x)}N$-valued processes are local martingales:*

$$N_t := \tilde{\Theta}_{0,t}^{-1}\, du(t, X_t)\, \Theta_{0,t}\, \ell(t) - \int_0^t \tilde{\Theta}_{0,s}^{-1}\, du(s, X_s)\, \Theta_{0,s}\, \dot{\ell}(s)\, ds,$$

$$M_t := \tilde{\Theta}_{0,t}^{-1}\, du(t, X_t)\, \Theta_{0,t}\, \ell(t) - \int_0^t \left(//_{0,s}^{\mathrm{Riem}}\right)^{-1} \Theta_{0,s}\, \dot{\ell}(s) \cdot dB_s\, \mathcal{A}_{\mathrm{def}}(\tilde{X})_t.$$

Proof. We have

$$d\left(\tilde{\Theta}_{0,t}^{-1}\, du(t,X_t)\, \Theta_{0,t}\, \ell(t)\right) = \tilde{\Theta}_{0,t}^{-1}\, du(t,X_t)\, \Theta_{0,t}\, \dot{\ell}(t)\, dt + d\left(\tilde{\Theta}_{0,t}^{-1}\, du(t,X_t)\, \Theta_{0,t}\right) \ell(t),$$

so that N is a local martingale by Theorem 2.3. Since the quadratic covariation of $\mathcal{A}_{\text{def}}(\tilde{X})_t$ and $\int_0^t \left(//_{0,s}^{\text{Riem}}\right)^{-1} \Theta_{0,s}\dot{\ell}(s) \cdot dB_s$ equals

$$\left\langle \mathcal{A}_{\text{def}}(\tilde{X}), \int_0^{\cdot} \left(//_{0,s}^{\text{Riem}}\right)^{-1} \Theta_{0,s}\, \dot{\ell}(s) \cdot dB_s \right\rangle_t = \int_0^t \tilde{\Theta}_{0,s}^{-1}\, du(s,X_s)\, \Theta_{0,s}\, \dot{\ell}(s)\, ds,$$

it follows that M is a local martingale as well. □

In general, the processes N and M of Theorem 2.6 are only local martingales, not necessarily true martingales. To obtain stochastic representation formulas by taking expectations, a possible strategy is to stop these processes before the underlying Brownian motion X leaves a relatively compact domain. The T_xM-valued process ℓ may then be chosen appropriately.

Theorem 2.7 *(Cf. [33, Theorem 3.1, Remark 3.4 and Theorem 4.1]). Let $v \in T_xM$, $R > 0$,*

$$D_R := \{(t,y) \in \mathbb{R}_+ \times M \mid d_{g(t)}(x,y) < R\}$$

and τ a bounded stopping time satisfying $\tau \le \tau_R$, where

$$\begin{aligned}\tau_R &:= \inf\{t \ge 0 \mid d_{g(t)}(x,X_t) \ge R\}\\ &= \inf\{t \ge 0 \mid (t,X_t) \notin D_R\}.\end{aligned}$$

Suppose that the process ℓ satisfies $\ell(0) = v$, $\ell(\tau) = 0$ and

$$\mathbb{E}\left[\left(\int_0^\tau |\dot{\ell}(s)|^2 ds\right)^{(1+\varepsilon)/2}\right] < \infty \tag{2.5}$$

for some $\varepsilon > 0$. Then the following stochastic representation formulas hold:

$$du(0,x)v = -\mathbb{E}\left[\int_0^\tau \left(//_{0,s}^{\text{Riem}}\right)^{-1} \Theta_{0,s}\, \dot{\ell}(s) \cdot dB_s\; \mathcal{A}_{\text{def}}(\tilde{X})_\tau\right] \tag{2.6}$$

and

$$du(0,x)v = -\mathbb{E}\left[\int_0^\tau \tilde{\Theta}_{0,s}^{-1}\, du(s,X_s)\, \Theta_{0,s}\, \dot{\ell}(s)\, ds\right].$$

Proof. To prove the claimed representation formulas it is sufficient to show that the stopped processes $(M_{t\wedge\tau})_{t\ge0}$ and $(N_{t\wedge\tau})_{t\ge0}$ are true martingales. By Theorem 2.6 we already know that they are local martingales. To show that the process $(M_{t\wedge\tau})_{t\ge0}$ is a true martingale it suffices to show that for each $c \ge 0$ the family $\{M_{\sigma\wedge\tau} \mid \sigma \text{ stopping time}, \sigma \le c\}$ is uniformly integrable, which holds if there is a constant $C < \infty$ such that for every stopping time $\sigma \le c$

$$\mathbb{E}\left[|M_{\sigma\wedge\tau}|^{1+\varepsilon}\right] \le C. \tag{2.7}$$

We first observe that the terms $\tilde{\Theta}_{0,t}^{-1}\, du(t,X_t)\, \Theta_{0,t}\, \dot{\ell}(t)$ and $\mathcal{A}_{\text{def}}(\tilde{X})_t$ are bounded as long as $t \le \tau_R \wedge c$. Moreover, using the Burkholder–Davis–Gundy inequality (see e.g. [13, Theorem

3.3.28]) and the fact that $\Theta_{0,s}$ is bounded for $s \leq \tau_R \wedge c$, we obtain

$$\mathbb{E}\left[\left|\int_0^{\sigma\wedge\tau}\left(//_{0,s}^{\mathrm{Riem}}\right)^{-1}\Theta_{0,s}\dot{\ell}(s)\cdot dB_s\right|^{1+\varepsilon}\right] \leq \tilde{C}\,\mathbb{E}\left[\left(\int_0^{\tau}|\dot{\ell}(s)|^2 ds\right)^{(1+\varepsilon)/2}\right]$$

and hence (2.7), so that the process $(M_{t\wedge\tau})_{t\geq 0}$ is indeed a martingale.

In a similar way the process $(N_{t\wedge\tau})_{t\geq 0}$ is shown to be a martingale as well. □

3. A priori estimates

In this section we prove differential estimates for space–time harmonic maps $u\colon \mathbb{R}_+ \times M \to N$ under the assumption that the metric on M evolves under backward super Ricci flow

$$\frac{\partial g}{\partial t} \leq \mathrm{Ric}_{g(t)}.$$

By time reversal the results apply to ancient solutions to the harmonic map heat flow under forward super Ricci flow. These estimates will then be used in the next section to derive Liouville type results for space–time harmonic mappings, respectively ancient solutions to the harmonic map heat flow. The starting point of our approach is the estimate

$$|du(0,x)v| \leq \mathbb{E}\left[\left|\int_0^{t\wedge\tau_R}\left(//_{0,s}^{\mathrm{Riem}}\right)^{-1}\Theta_{0,s}\,\dot{\ell}(s)\cdot dB_s\right|^p\right]^{1/p}\mathbb{E}\left[|\mathcal{A}_{\mathrm{def}}(\tilde{X})_{t\wedge\tau_R}|^q\right]^{1/q} \quad (3.8)$$

for $p, q > 1$ such that $1/p + 1/q = 1$, which follows immediately from formula (2.6). The process ℓ satisfies $\ell(0) = v$, $\ell(\tau) = 0$ and condition (2.5); otherwise it may be chosen arbitrarily.

Note that in estimate (3.8) geometric information of the evolving manifold M only enters through the first term on the right-hand side, while the second term captures the geometry of the target N. In this sense, estimate (3.8) allows to separate the contributions of the curvatures of M and N to the differential of u.

3.1. Estimation of the first factor

To estimate the first factor on the right-hand side of (3.8), we have to choose the process ℓ in a suitable way. To this end we fix $R > 0$ and, similarly to [32, Proof of Corollary 5.1], define $f : \bar{D}_R \to [0, 1]$ by

$$f(u, y) := \cos\left(\frac{\pi}{2R}\, d_{g(u)}(x, y)\right).$$

For $p \geq 1$ let

$$c_p(R) := \sup_{(u,y)\in\tilde{D}_R}\left\{f^{p+2}\left(\frac{\partial f^{-p}}{\partial u} + \frac{1}{2}\Delta_{g(u)}(f^{-p})\right)\right\},$$

where $\tilde{D}_R := \{(t, y) \in D_R : y \neq x \text{ and } y \notin \mathrm{Cut}_{g(t)}(x)\}$.

Lemma 3.1. *Suppose that*

$$\frac{\partial g}{\partial t} \leq \mathrm{Ric}_{g(t)}$$

and that there exists $r_0 > 0$ *such that*

$$C(x, r_0) := \sup\left\{|\mathrm{Ric}(t, y)| : t \geq 0,\ d_{g(t)}(x, y) \leq r_0\right\} \tag{3.9}$$

is finite. Then $c_p(R)$ *is finite for each* $R > 0$*, and moreover*

$$c_p(R) = O(1/R), \quad \text{as } R \to \infty. \tag{3.10}$$

Remark 3.2. In the case of a fixed metric g with non-negative Ricci curvature, instead of (3.10) one obtains the much better estimate

$$c_p(R) \leq \frac{p\pi^2(p+d+1)}{8R^2}, \tag{3.11}$$

cf. [32, Proof of Corollary 5.1] for the case $p = 2$. This is due to the fact that the estimate for the drift of the radial part of Brownian motion is much better in the case of a fixed metric with non-negative Ricci curvature than in the case of backward super Ricci flow, see Remark 3.3.

Proof of Lemma 3.1. We first observe that

$$c_p(R) = \sup_{(u,y)\in \tilde{D}_R} \left\{\frac{p(p+1)}{2}|\nabla f|^2 - pf\left(\frac{\partial f}{\partial u} + \frac{1}{2}\Delta_{g(u)} f\right)\right\}.$$

Let now $\rho(u, y) := d_{g(u)}(x, y)$ and $\bar{f}(\xi) := \cos\left(\frac{\pi\xi}{2R}\right)$, so that $f(u, y) = \bar{f}(\rho(u, y))$ and consequently

$$c_p(R) = \sup_{(u,y)\in \tilde{D}_R} \left\{\frac{p(p+1)}{2}\bar{f}'(\rho)^2|\nabla\rho|^2 - p\bar{f}(\rho)\left(\frac{1}{2}\bar{f}''(\rho)|\nabla\rho|^2 \right.\right.$$
$$\left.\left. + \bar{f}'(\rho)\left(\frac{\partial\rho}{\partial u} + \frac{1}{2}\Delta_{g(u)}\rho\right)\right)\right\}.$$

Since $|\bar{f}| \leq 1$, $|\bar{f}'| \leq \pi/(2R)$, $|\bar{f}''| \leq \pi^2/(4R^2)$ and $|\nabla\rho| \equiv 1$ (on $\tilde{D}_R$), it follows that

$$c_p(R) \leq \frac{(p^2+2p)\pi^2}{8R^2} + \frac{p\pi}{2R}\sup_{(u,y)\in\tilde{D}_R}\left\{\sin\left(\frac{\pi\rho}{2R}\right)\left(\frac{\partial\rho}{\partial u} + \frac{1}{2}\Delta_{g(u)}\rho\right)\right\}.$$

By [17, Proposition 2] we have

$$\frac{\partial\rho}{\partial u} + \frac{1}{2}\Delta_{g(u)}\rho \leq \frac{d-1}{2}\Big(k(r_0)\coth(k(r_0)(\rho(u,y)\wedge r_0)) + k(r_0)^2(\rho(u,y)\wedge r_0)\Big), \tag{3.12}$$

where $k(r_0) := \sqrt{\frac{C(x,r_0)}{d-1}}$. Therefore, using the inequality $\coth\xi \leq 1 + 1/\xi$ valid for $\xi > 0$, we obtain for $\rho(u, y) \leq r_0$,

$$\begin{aligned}\frac{p\pi}{2R}\sin\left(\frac{\pi\rho}{2R}\right)\left(\frac{\partial\rho}{\partial u} + \frac{1}{2}\Delta_{g(u)}\rho\right) &\leq \frac{p\pi}{2R}\sin\left(\frac{\pi\rho}{2R}\right)\frac{d-1}{2}\Big(k(r_0)\coth(k(r_0)\rho) + k(r_0)^2\rho\Big)\\ &\leq \frac{(d-1)p\pi^2}{8R^2}\Big(k(r_0)\rho + 1 + k(r_0)^2\rho^2\Big)\\ &\leq \frac{(d-1)p\pi^2}{8R^2}\Big(k(r_0)r_0 + 1 + k(r_0)^2 r_0^2\Big),\end{aligned}$$

and for $\rho(u, y) \geq r_0$,

$$\begin{aligned}
&\frac{p\pi}{2R} \sin\left(\frac{\pi\rho}{2R}\right)\left(\frac{\partial\rho}{\partial u} + \frac{1}{2}\Delta_{g(u)}\rho\right) \\
&\quad \leq \frac{p\pi}{2R} \sin\left(\frac{\pi\rho}{2R}\right) \frac{d-1}{2}\left(k(r_0)\coth(k(r_0)r_0) + k(r_0)^2 r_0\right) \\
&\quad \leq \frac{(d-1)p\pi}{4R}\left(k(r_0) + \frac{1}{r_0} + k(r_0)^2 r_0\right),
\end{aligned}$$

which completes the proof. □

Remark 3.3. The key ingredient of the proof above is estimate (3.12) for the radial drift of Brownian motion, which should be seen as a parabolic version of the Laplacian comparison theorem for evolving manifolds. In the case of a fixed metric with non-negative Ricci curvature the Laplacian comparison theorem however provides the much better estimate

$$\frac{1}{2}\Delta\rho \leq \frac{d-1}{2\rho}. \tag{3.13}$$

Since in many respects manifolds evolving under backward super Ricci flow behave in a similar way as manifolds with a fixed metric of non-negative Ricci curvature (see e.g. [20] or [34, Section 6.5]), one might expect that an estimate similar to (3.13) also holds under backward super Ricci flow. This, however, is not the case, as the following example shows.

Example 3.4 (*Brownian Motion on Hamilton's Cigar*)**.** Let $M = \mathbb{R}^2$ be equipped with the time-dependent metric

$$g(t, x) := \frac{1}{e^{-2t} + |x|^2}\, g_{\mathrm{eucl}}(x),$$

where g_{eucl} denotes the standard metric on $\mathbb{R}^2$. As shown in [6, Section 4.3], the family $(g(t))_{t\in\mathbb{R}}$ is an eternal solution of the backward Ricci flow, called "Hamilton's cigar" or "Witten's black hole". By elementary calculations one obtains

$$\rho(t, x) = \operatorname{arcsinh}(e^t |x|),$$

and consequently

$$\frac{\partial\rho}{\partial t}(t, x) = \frac{1}{\sqrt{1 + e^{-2t}|x|^{-2}}}$$

and

$$\Delta_{g(t)}\rho(t, x) = \frac{1}{e^{2t}|x|^2\sqrt{1 + e^{-2t}|x|^{-2}}}.$$

It is now easy to see that for each $t \in \mathbb{R}$ the function $|x| \mapsto \left(\frac{\partial\rho}{\partial t} + \frac{1}{2}\Delta_{g(t)}\rho\right)(t, x)$ is decreasing, and hence bounded from below by

$$\lim_{|x|\to\infty}\left(\frac{\partial\rho}{\partial t} + \frac{1}{2}\Delta_{g(t)}\rho\right)(t, x) = 1.$$

Consequently, a parabolic analogue to (3.13) cannot hold under backward super Ricci flow. The drift part of the distance process $\rho(t, X_t)$ of Brownian motion grows at least like t.

We now consider the strictly increasing process $(T(s))_{s\in[0,\tau_R)}$ given by

$$T(s) := \int_0^s \frac{1}{f^2(u, X_u)}\, du,$$

and let the process $(\sigma(r))_{r\geq 0}$ be defined by

$$\sigma(r) := \begin{cases} \inf\{s \in [0, \tau_R) \colon\ T(s) \geq r\} & \text{if such an } s \text{ exists,} \\ \tau_R & \text{otherwise.} \end{cases}$$

Lemma 3.5. *For all stopping times $\tau \leq \tau_R$ we have*

$$\mathbb{E}\left[f^{-p}(\tau, X_\tau)\right] \leq \mathbb{E}\left[e^{c_p(R)T(\tau)}\right].$$

Proof. Applying Itô's formula to the process $Y_r := f^{-p}(\sigma(r), X_{\sigma(r)})$ $(0 \leq r < T(\tau_R))$ we obtain

$$\begin{aligned} dY_r &\overset{\text{m}}{\leq} \left(\frac{\partial f^{-p}}{\partial u} + \frac{1}{2}\Delta_{g(\sigma(r))}(f^{-p})\right)(\sigma(r), X_{\sigma(r)})\dot{\sigma}(r)\, dr \\ &= f^{-p}(\sigma(r), X_{\sigma(r)})\left[f^{p+2}\left(\frac{\partial f^{-p}}{\partial u} + \frac{1}{2}\Delta_{g(\sigma(r))}(f^{-p})\right)\right](\sigma(r), X_{\sigma(r)})\, dr \\ &\leq c_p(R)\, f^{-p}(\sigma(r), X_{\sigma(r)})\, dr \\ &= c_p(R)\, Y_r\, dr, \end{aligned}$$

where the inequality (modulo differentials of local martingales) in the first step is due to the local time at the cut-locus, see [17, Theorem 2]. Since $Y_0 = 1$, it follows that

$$\mathbb{E}\left[f^{-p}(\tau, X_\tau)\right] = \mathbb{E}\left[Y_{T(\tau)}\right] \leq \mathbb{E}\left[e^{c_p(R)T(\tau)}\right]. \quad \square$$

Lemma 3.6. *If $c_p(R)$ is finite, we have*

$$\lim_{s\uparrow\tau_R} T(s) = +\infty$$

almost surely.

Proof. Let $\tau^n := \inf\{t \geq 0 \colon\ f(t, X_t) \leq 1/n\}$. The previous lemma with $p = 1$ and $\tau = \sigma(t) \wedge \tau^n$ implies that for each $t \geq 0$

$$n\mathbb{P}\left\{\tau^n \leq \sigma(t)\right\} \leq \mathbb{E}\left[f^{-1}(\sigma(t) \wedge \tau^n, X_{\sigma(t)\wedge\tau^n})\right] \leq e^{c_1(R)t}$$

and consequently, since $\tau^n \uparrow \tau_R$,

$$\mathbb{P}\{\sigma(t) = \tau_R\} = \lim_{n\to\infty} \mathbb{P}\left\{\tau^n \leq \sigma(t)\right\} = 0.$$

Since

$$\left\{\lim_{s\uparrow\tau_R} T(s) < \infty\right\} = \bigcup_{0\leq t\in\mathbb{Q}} \{\sigma(t) = \tau_R\},$$

the claim follows. $\square$

Lemma 3.7 (*Cf. [33, Lemma 4.3] for the Case of a Fixed Metric*)**.** *Assume that*

$$\frac{\partial g}{\partial t} \leq \mathrm{Ric}_{g(t)}$$

and that there exists $r_0 < 0$ such that $C(x, r_0)$ defined in (3.9) is finite. Then, for all $t \geq 0$,

$$\mathbb{E}\left[\left|\int_0^{t\wedge\tau_R} \left(//_{0,s}^{\text{Riem}}\right)^{-1} \Theta_{0,s}\, \dot{\ell}(s)\cdot dB_s\right|^p\right] \leq \frac{C_p\left(2c_p(R)/p\right)^{p/2}}{\left(1-\exp(-2c_p(R)t/p)\right)^{p/2}}|v|^p \tag{3.14}$$

where C_p is the constant in the Burkholder–Davis–Gundy inequality with exponent p.

Proof. Fix $t > 0$. As a consequence of the previous lemma we have

$$\sigma(r) = \inf\{s \in [0, \tau_R) : T(s) \geq r\}.$$

Moreover, $\sigma(r) \leq r, \sigma(r) \leq \tau_R, T(\sigma(r)) = r$ and

$$\dot{\sigma}(r) = 1/\dot{T}(\sigma(r)) = f^2(\sigma(r), X_{\sigma(r)})$$

for all $r \geq 0$, and $\sigma(T(s)) = s$ for all $s \in [0, \tau_R)$. Now let

$$h_0(s) := \int_0^{s\wedge\sigma(t)} \frac{1}{f^2(r, X_r)}\, dr = T(s\wedge\sigma(t)) = T(s)\wedge t$$

and

$$h_1(r) = 1 - \frac{1-\exp(-2c_p(R)r/p)}{1-\exp(-2c_p(R)t/p)},$$

and define

$$\ell(s) := h_1(h_0(s))\, v.$$

Note that $h_1(0) = 1, h_1(t) = 0$ and $\dot{h}_1(r) < 0$ for all $r \geq 0$, so that $|\dot{h}_1(r)|\, dr$ is a probability measure on $[0, t]$.

Since $\frac{\partial g}{\partial t} \leq \text{Ric}$ implies that $|\Theta_{0,s}| \leq 1$, we obtain using the Burkholder–Davis–Gundy inequality that

$$\mathbb{E}\left[\left|\int_0^{t\wedge\tau_R} \left(//_{0,s}^{\text{Riem}}\right)^{-1} \Theta_{0,s}\, \dot{\ell}(s)\cdot dB_s\right|^p\right] \leq C_p\,\mathbb{E}\left[\left|\int_0^{t\wedge\tau_R} \left|\dot{\ell}(s)\right|^2 ds\right|^{p/2}\right].$$

Moreover, since

$$\dot{h}_0(s) = \begin{cases} f^{-2}(s, X_s) & \text{if } s < \sigma(t), \\ 0 & \text{if } s > \sigma(t), \end{cases}$$

we have

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^{t\wedge\tau_R} \left|\dot{\ell}(s)\right|^2 ds\right|^{p/2}\right] &= \mathbb{E}\left[\left|\int_0^{t\wedge\tau_R} |\dot{h}_1(h_0(s))|^2\, |\dot{h}_0(s)|^2\, ds\right|^{p/2}\right]|v|^p \\
&= \mathbb{E}\left[\left|\int_0^{\sigma(t)} |\dot{h}_1(h_0(s))|^2\, \frac{1}{f^4(s, X_s)}\, ds\right|^{p/2}\right]|v|^p \\
&= \mathbb{E}\left[\left|\int_0^{t} |\dot{h}_1(h_0(\sigma(r)))|^2\, \frac{1}{f^4(\sigma(r), X_{\sigma(r)})}\dot{\sigma}(r)\, dr\right|^{p/2}\right]|v|^p \\
&= \mathbb{E}\left[\left|\int_0^{t} |\dot{h}_1(r)|^2\, \frac{1}{f^2(\sigma(r), X_{\sigma(r)})}\, dr\right|^{p/2}\right]|v|^p
\end{aligned}$$

$$\begin{aligned}
&\le \mathbb{E}\left[\int_0^t |\dot{h}_1(r)|^{p/2+1} \frac{1}{f^p(\sigma(r), X_{\sigma(r)})}\, dr\right] |v|^p \\
&= \int_0^t |\dot{h}_1(r)|^{p/2+1}\, \mathbb{E}\left[\frac{1}{f^p(\sigma(r), X_{\sigma(r)})}\right] dr\ |v|^p \\
&\le |v|^p \int_0^t |\dot{h}_1(r)|^{p/2+1}\, e^{c_p(R) r}\, dr,
\end{aligned}$$

where in the fifth step we used Jensen's inequality with respect to the probability measure $|\dot{h}_1(r)|\, dr$ on $[0, t]$ (recall that $p \ge 2$).

Since

$$\dot{h}_1(r) = -\frac{2c_p(R)/p}{1 - \exp(-2c_p(R)t/p)}\, e^{-2c_p(R)r/p}$$

and consequently

$$|\dot{h}_1(r)|^{p/2+1}\, e^{c_p(R)r} = \left(\frac{2c_p(R)/p}{1 - \exp(-2c_p(R)t/p)}\right)^{p/2+1} e^{-2c_p(R)r/p},$$

the claim follows. □

3.2. Estimation of the second factor

To estimate the second factor of (3.8) we start by estimating the inverse of the damped parallel transport.

Lemma 3.8 (*Cf. [33, Lemma 2.12]*). *For $y \in N$ let $\kappa(y)$ be the supremum of the sectional curvatures of N at y. Moreover, for $(s, x) \in \mathbb{R}_+ \times M$, let $\lambda_1(s, x) \ge \cdots \ge \lambda_m(s, x) \ge 0$ be the eigenvalues of the map $du(s, x)^* du(s, x) : T_x M \to T_x M$. Then*

$$\|\tilde{\Theta}_{0,t}^{-1}\| \le \exp\left(\frac{1}{2}\int_0^t L_s\, ds\right),$$

where

$$L_s := \begin{cases} |du|^2(s, X_s)\, \kappa(u(s, X_s)) & \text{if } \kappa(u(s, X_s)) \ge 0 \\ \sum_{i=2}^m \lambda_i(s, X_s)\, \kappa(u(s, X_s)) & \text{if } \kappa(u(s, X_s)) \le 0. \end{cases}$$

Proof. Let $w \in T_{\tilde{X}_0} N$. Using the definitions of $\tilde{\Theta}_{0,t}$ and κ we obtain on $\{\kappa(u(t, X_t)) \ge 0\}$

$$\begin{aligned}
\frac{d}{dt}\left|\tilde{\Theta}_{0,t} w\right|^2 &= -\sum_{i=1}^m \left\langle \tilde{R}\big(\tilde{\Theta}_{0,t} w, du(t, X_t)\, \xi_t^i\big) du(t, X_t)\, \xi_t^i,\, \tilde{\Theta}_{0,t} w\right\rangle \\
&\ge -\sum_{i=1}^m \kappa(u(t, X_t)) \left[\left|\tilde{\Theta}_{0,t} w\right|^2 \left|du(t, X_t)\, \xi_t^i\right|^2 - \left\langle \tilde{\Theta}_{0,t} w, du(t, X_t)\, \xi_t^i\right\rangle^2\right] \\
&\ge -\sum_{i=1}^m \kappa(u(t, X_t)) \left|\tilde{\Theta}_{0,t} w\right|^2 \left|du(t, X_t)\, \xi_t^i\right|^2 \\
&= -\kappa(u(t, X_t)) \left|\tilde{\Theta}_{0,t} w\right|^2 |du|^2(t, X_t) \\
&= -L_t \left|\tilde{\Theta}_{0,t} w\right|^2.
\end{aligned}$$

On the set $\{\kappa(u(t, X_t)) \le 0\}$ we moreover use the fact that

$$\sum_{i=1}^{m} \left\langle \tilde{\Theta}_{0,t} w, du(t, X_t)\xi_t^i \right\rangle^2 \le \lambda_1(t, X_t) \, |\tilde{\Theta}_{0,t} w|^2$$

and obtain

$$\begin{aligned} \frac{d}{dt} \left| \tilde{\Theta}_{0,t} w \right|^2 &\ge -\sum_{i=1}^{m} \kappa(u(t, X_t)) \left[\left| \tilde{\Theta}_{0,t} w \right|^2 \left| du(t, X_t)\xi_t^i \right|^2 - \left\langle \tilde{\Theta}_{0,t} w, du(t, X_t)\xi_t^i \right\rangle^2 \right] \\ &\ge -\kappa(u(t, X_t)) \left| \tilde{\Theta}_{0,t} w \right|^2 \sum_{i=2}^{m} \lambda_i(t, X_t) \\ &= -L_t \left| \tilde{\Theta}_{0,t} w \right|^2 \end{aligned}$$

as well. □

Lemma 3.9 (*Cf. [33, Lemma 4.5(1)]*)**.** *For any stopping time τ we have*

$$\mathbb{E}\left[|\mathcal{A}_{\text{def}}(\tilde{X})_\tau|^q \right] \le C_q \, \mathbb{E}\left[\left(\int_0^\tau |du|^2(s, X_s) \times \exp\left(\int_0^s |du|^2(r, X_r) \, \kappa_+(u(r, X_r)) \, dr \right) ds \right)^{q/2} \right],$$

where $\kappa_+(y) := \max(\kappa(y), 0)$.

Proof. By the Burkholder–Davis–Gundy inequality we have

$$\mathbb{E}\left[|\mathcal{A}_{\text{def}}(\tilde{X})_\tau|^q \right] \le C_q \mathbb{E}\left[\left| \int_0^\tau \left| \tilde{\Theta}_{0,s}^{-1} \, du(s, X_s) \right|^2 ds \right|^{q/2} \right],$$

and by the previous lemma

$$\| \tilde{\Theta}_{0,s}^{-1} \|^2 \le \exp\left(\int_0^s |du|^2(r, X_r) \, \kappa_+(u(r, X_r)) \, dr \right). \quad \square$$

Lemma 3.10 (*Cf. [33, Lemma 4.5(2)]*)**.** *If N is simply connected and has non-positive curvature, then for any bounded stopping time τ*

$$\mathbb{E}\left[|\mathcal{A}_{\text{def}}(\tilde{X})_\tau|^2 \right] \le \mathbb{E}\left[\text{dist}_N\big(u(\tau, X_\tau), u(0, x)\big)^2 \right].$$

Proof. Let $\rho(z) := \text{dist}_N(z, u(0, x))$. Using the chain rule for the tension field (see e.g. [12, Lemma 8.7.2]) and the Hessian comparison theorem (see e.g. [11, Satz 7.236]) we obtain

$$\begin{aligned} -\frac{\partial(\rho^2 \circ u)}{\partial t} + \frac{1}{2}\Delta(\rho^2 \circ u) &= -\nabla(\rho^2)\frac{\partial u}{\partial t} + \frac{1}{2}\text{tr}\left(\text{Hess}(\rho^2) \circ (du \otimes du) \right) + \frac{1}{2}\nabla(\rho^2)\Delta u \\ &\ge |du|^2, \end{aligned}$$

and therefore, using the Burkholder–Davis–Gundy inequality and Itô's formula

$$\mathbb{E}\left[|\mathcal{A}_{\text{def}}(\tilde{X})_\tau|^2 \right] \le \mathbb{E}\left[\int_0^\tau \left| \tilde{\Theta}_{0,s}^{-1} du(s, X_s) \right|^2 ds \right]$$

$$\leq \mathbb{E}\left[\int_0^\tau |du|^2(s, X_s)\, ds\right]$$
$$\leq \mathbb{E}\left[\int_0^\tau \left(-\frac{\partial(\rho^2 \circ u)}{\partial t} + \frac{1}{2}\Delta(\rho^2 \circ u)\right)(s, X_s)\, ds\right]$$
$$= \mathbb{E}\left[\rho^2(u(\tau, X_\tau))\right],$$

as claimed. □

Lemma 3.11 (*Cf.* [33, *Lemma 4.6*]). *Assume that N has non-positive curvature and let*

$$K(t,x) := \frac{\lambda_1(t,x)}{\sum_{i=2}^m \lambda_i(t,x)}$$

(with the convention $0/0 := 0$). Then for any stopping time τ

$$\mathbb{E}\left[|\mathcal{A}_{\mathrm{def}}(\tilde{X})_\tau|^q\right] \leq C_q \mathbb{E}\left[\left|\int_0^\tau |du|^2(s, X_s) \times \exp\left(\int_0^s |du|^2(r, X_r)\, \frac{\kappa(u(s, X_s))}{K(s, X_s)}\, dr\right) ds\right|^{q/2}\right].$$

Proof. This follows immediately from the Burkholder–Davis–Gundy inequality, Lemma 3.8 and the definition of K. □

Corollary 3.12. *Suppose that*

$$\frac{\kappa(u(s,x))}{K(s,x)} \leq -b < 0$$

for all $s \geq 0$ and all $x \in M$. Then we have

$$\mathbb{E}\left[|\mathcal{A}_{\mathrm{def}}(\tilde{X})_\tau|^2\right] \leq \frac{1}{b}\,\mathbb{E}\left[\left(1 - \exp\left(-b\int_0^\tau |du|^2(r, X_r)\, dr\right)\right)\right] \leq \frac{1}{b}.$$

Proof. Since

$$\frac{d}{ds}\exp\left(-b\int_0^s |du|^2(r, X_r)\, dr\right) = -b\, |du|^2(s, X_s) \exp\left(-b\int_0^s |du|^2(r, X_r)\, dr\right)$$

we have

$$\int_0^\tau |du|^2(s, X_s) \exp\left(\int_0^s |du|^2(r, X_r) \frac{\kappa(u(s, X_s))}{K(s, X_s)}\, dr\right) ds$$
$$\leq \frac{1}{b} - \frac{1}{b}\exp\left(-b\int_0^\tau |du|^2(r, X_r)\, dr\right). \quad \square$$

4. Liouville theorems

In this section we derive Liouville type results for space–time harmonic mappings, respectively ancient solutions to the harmonic map heat flow. We work out details in three typical cases: Mappings of sub-square-root growth, of bounded dilatation, and of small image. From now on we suppose that M is connected.

4.1. Space–time harmonic maps of sub-square-root growth

We say that a function $u : \mathbb{R}_+ \times M \to N$ is of *sub-square-root growth* if for each $x \in M$ there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\varphi(r)/\sqrt{r} \to 0 \tag{4.15}$$

as $r \to \infty$ such that for all $t \geq 0$ and all $z \in M$

$$\mathrm{dist}_N\,(u(t,z), u(0,x)) \leq \varphi(d_{g(t)}(z,x)). \tag{4.16}$$

Theorem 4.1. *Suppose that M is connected,*

$$\frac{\partial g}{\partial t} \leq \mathrm{Ric}_{g(t)}$$

(backward super Ricci flow), that for each $x \in M$ there exists $r_0 > 0$ such that the constant $C(x, r_0)$ defined in (3.9) is finite, and that N is simply-connected and has non-positive sectional curvatures. Then every space–time harmonic mapping $u\colon \mathbb{R}_+ \times M \to N$ of sub-square-root growth is constant.

Proof. By Lemmas 3.7 and 3.10 and (4.16) we have for each $R > 0$

$$\begin{aligned} |du(0,x)v|^2 &\leq C_2\, c_2(R)\, |v|^2\, \mathbb{E}\Big[\mathrm{dist}_N(u(\tau_R, X_{\tau_R}), u(0,x))^2\Big] \\ &\leq C_2\, c_2(R)\, |v|^2\, \mathbb{E}\Big[\varphi^2(d_{g(\tau_R)}(X_{\tau_R}, x))\Big] \\ &= C_2\, c_2(R)\, |v|^2\, \varphi(R)^2. \end{aligned}$$

The claim now follows by letting $R \to \infty$, taking into account Lemma 3.1 and (4.15). □

Analogously, a mapping $u\colon (-\infty, T] \times M \to N$ is said to be of sub-square-root growth if (4.16) holds for all $(t, z) \in (-\infty, T] \times M$.

Corollary 4.2. *Suppose that M is connected and that*

$$\frac{\partial g}{\partial t} \geq -\mathrm{Ric}_{g(t)} \quad \text{on } (-\infty, T] \times M$$

(forward super Ricci flow). Assume that for each $x \in M$ there exists $r_0 > 0$ such that $\sup\big\{|\mathrm{Ric}(t,y)| :\ t \in (-\infty, T],\ d_{g(t)}(x,y) \leq r_0\big\}$ *(the analogue of the constant $C(x, r_0)$ defined in (3.9)) is finite, and that N is simply-connected and has non-positive sectional curvatures. Then any ancient solution of sub-square-root growth $u\colon (-\infty, T] \times M \to N$ to the harmonic map heat flow is constant.*

Remark 4.3. Theorem 4.1 should be compared with S.-Y. Cheng's Liouville theorem [4] which gives an analogous statement for harmonic maps of sublinear growth when M is equipped with a fixed metric of non-negative Ricci curvature, see [33, Corollary 5.10] and [30] for stochastic proofs. All these proofs depend crucially on the Laplacian comparison theorem. In the case of backward super Ricci flow the stronger assumption of sub-square-root growth is needed because estimate (3.10) is weaker than estimate (3.11) which holds in the case of a fixed metric with non-negative Ricci curvature, see the discussion in Remark 3.2.

4.2. Space–time harmonic maps of bounded dilatation

Let $u\colon \mathbb{R}_+ \times M \to N$ be a space–time harmonic map. We say that u is of *bounded dilatation* if there is a real constant C such that

$$\lambda_1(t,x) \le C \sum_{i=2}^{m} \lambda_i(t,x) \tag{4.17}$$

for all $(t,x) \in \mathbb{R}_+ \times M$. Similarly, an ancient solution $u\colon (-\infty, T] \times M \to N$ to the harmonic map heat flow is said to be of bounded dilatation if (4.17) holds for all $(t,x) \in (-\infty, T] \times M$.

Theorem 4.4 (*Cf. [33, Corollary 5.15] for Harmonic Maps in the Case of a Fixed Metric*)**.** *Suppose that M is connected,*

$$\frac{\partial g}{\partial t} \le \mathrm{Ric}_{g(t)}$$

(backward super Ricci flow), that for each $x \in M$ there exists $r_0 > 0$ such that $C(x, r_0)$ is finite, and that N has uniformly strictly negative sectional curvatures. Then any space–time harmonic map $u\colon \mathbb{R}_+ \times M \to N$ of bounded dilatation is constant.

Proof. The assumptions on the curvature of N and the dilatation of u imply that there is a constant $b > 0$ such that

$$\frac{\kappa(u(t,x))}{K^2(t,x)} \le -b < 0$$

for all $t \ge 0$ and all $x \in M$. Lemma 3.7 and Corollary 3.12 then imply that

$$|du(0,x)v|^2 \le \frac{C_2 c(R)|v|^2}{b(1-\exp(-c(R)t))^2}.$$

Letting first $t \to \infty$ and then $R \to \infty$, one obtains that $du(0,x) = 0$. □

Corollary 4.5. *Suppose that M is connected,*

$$\frac{\partial g}{\partial t} \ge -\mathrm{Ric}_{g(t)} \quad \text{on } (-\infty, T] \times M$$

(forward super Ricci flow), that for each $x \in M$ there exists $r_0 > 0$ such that $\sup\big\{|\mathrm{Ric}(t,y)| : t \in (-\infty, T],\ d_{g(t)}(x,y) \le r_0\big\}$ is finite, and that N has uniformly strictly negative sectional curvatures. Then any ancient solution of bounded dilatation $u\colon (-\infty, T] \times M \to N$ to the harmonic map heat flow is constant.

4.3. Space–time harmonic maps of small image

Let (N, h) be a Riemannian manifold and $\lambda > 0$. Recall that an N-valued martingale Y is said to have *exponential moments of order* λ if

$$\mathbb{E}\left[\exp\Big(\lambda \int_0^\infty h(dY_s, dY_s)\Big)\right] < \infty.$$

Remark 4.6. Let (N, h) be a Riemannian manifold, $B \subset N$ an open subset and $\lambda > 0$. Suppose that there is a real-valued C^2 function f on B satisfying $c_1 \le f \le c_2$ for some positive constants

c_1, c_2 such that

$$\nabla df + 2\lambda f \le 0. \tag{4.18}$$

Then every N-valued martingale taking its values in B has exponential moments of order λ, see [28, Proposition 2.1.2], [33, Remark 5.2].

Definition 4.7. Let (N, h) be a Riemannian manifold, $y \in N$ a point, and $B = B(y, r)$ an open geodesic ball about y of radius r. Such a geodesic ball is said to be *regular* if it does not meet the cut locus of its center y and if $\kappa < (\pi/2r)^2$ where κ denotes an upper bound of the sectional curvatures of N on $B(y, r)$.

Example 4.8. Suppose that $B(y, r)$ is a relatively compact regular geodesic ball in N such that $r < \pi/(2\sqrt{\kappa})$ where $\kappa > 0$ is an upper bound of the sectional curvatures of N on $B(y, r)$. Let $f = \cos\big(\sqrt{\kappa q}\, d(y, \cdot)\big)$ where $q > 1$ is chosen in such a way that $0 < c_1 \le f$ holds on $B(y, r)$ for some $c_1 > 0$. Then

$$\nabla df + \kappa q\, f \le 0,$$

which by Remark 4.6 means that any $B(y, r)$-valued martingale has exponential moments of order $\kappa q/2$.

Theorem 4.9 (*Cf. [33, Corollary 5.5] for Harmonic Maps in the Case of a Fixed Metric*)**.** *Suppose that M is connected and*

$$\frac{\partial g}{\partial t} \le \mathrm{Ric}_{g(t)}$$

(backward super Ricci flow) and that for each $x \in M$ there exists $r_0 > 0$ such that $C(x, r_0)$ is finite. Let B be a relatively compact regular geodesic ball in N of radius r such that $r < \pi/(2\sqrt{\kappa})$ where $\kappa > 0$ is an upper bound of the sectional curvatures of N on B. Then any space–time harmonic map $u : \mathbb{R}_+ \times M \to N$ taking its values in B is constant.

Proof. Let $q \in (1, 2]$ be as in Example 4.8 and p be such that $1/p + 1/q = 1$. Since by Lemma 3.7 the L^p-norm term on the right-hand side of (3.8) tends to 0 as $t \to \infty$ and then $R \to \infty$, it is sufficient to show that $\mathbb{E}\big[|\mathcal{A}_{\mathrm{def}}(\tilde{X})_{t\wedge\tau_R}|^q\big]$ is bounded uniformly in t and R. By Lemma 3.9 we have

$$\mathbb{E}\left[|\mathcal{A}_{\mathrm{def}}(\tilde{X})_{t\wedge\tau_R}|^q\right] \le C_q\, \mathbb{E}\left[\left(\int_0^{t\wedge\tau_R} \rho(s) \exp\left(\int_0^s \rho(r)\, \kappa_+(\tilde{X}_r)\, dr\right) ds\right)^{q/2}\right],$$

where $\rho(s) := |du|^2(s, X_s)$. Since by assumption $\kappa_+(\tilde{X}_r) \le \kappa$ for a constant $\kappa > 0$, and using

$$\rho(s) \exp\left\{\kappa \int_0^s \rho(r)\, dr\right\} = \frac{1}{\kappa}\frac{d}{ds} \exp\left\{\kappa \int_0^s \rho(r)\, dr\right\},$$

we finally obtain

$$\begin{aligned}\mathbb{E}\left[|\mathcal{A}_{\mathrm{def}}(\tilde{X})_{t\wedge\tau_R}|^q\right] &\le C_q \mathbb{E}\left[\left\{\frac{1}{\kappa}\left(\exp\left\{\kappa \int_0^{t\wedge\tau_R} |du|^2(r, X_r)\, dr\right\} - 1\right)\right\}^{q/2}\right] \\ &\le \frac{C_q}{\kappa^{q/2}} \mathbb{E}\left[\left(\exp\left\{\kappa \int_0^{\infty} |du|^2(r, X_r)\, dr\right\} - 1\right)^{q/2}\right].\end{aligned}$$

The last term is finite since by Example 4.8 the martingale $\tilde{X}_t = u(t, X_t)$ has exponential moments of order $\kappa q/2$, which is equivalent to say that

$$\mathbb{E}\left[\exp\left(\frac{\kappa q}{2}\int_0^\infty |du|^2(r, X_r)\,dr\right)\right] < \infty. \quad \square$$

Corollary 4.10. *Suppose that M is connected and*

$$\frac{\partial g}{\partial t} \geq -\mathrm{Ric}_{g(t)} \quad \text{on } (-\infty, T] \times M$$

(forward super Ricci flow) and that for each $x \in M$ there exists $r_0 > 0$ such that $\sup\left\{|\mathrm{Ric}(t, y)| : t \in (-\infty, T],\ d_{g(t)}(x, y) \leq r_0\right\}$ *is finite. Let B be a relatively compact regular geodesic ball in N of radius r such that $r < \pi/(2\sqrt{\kappa})$ where $\kappa > 0$ is an upper bound of the sectional curvatures of N on B. Then any ancient solution $u: (-\infty, T] \times M \to N$ to the harmonic map heat flow taking values in B is constant.*

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