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Gradient estimates for positive harmonic functions by stochastic analysis

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Abstract

We prove Cheng–Yau type inequalities for positive harmonic functions on Riemannian manifolds by using methods of Stochastic Analysis. Rather than evaluating an exact Bismut formula for the differential of a harmonic function, our method relies on a Bismut type inequality which is derived by an elementary integration by parts argument from an underlying submartingale. It is the monotonicity inherited in this submartingale which allows us to establish the pointwise estimates.

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1. Introduction

The effect of curvature on the behavior of harmonic functions on a Riemannian manifold M is a classical problem. A quantitative measurement of this behavior is encoded most directly in terms of gradient estimates and Harnack inequalities involving constants depending only on a lower bound of the Ricci curvature on M, the dimension of M, and the radius of the ball on which the harmonic function is defined. Such estimates in global form, i.e., for positive harmonic

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functions on Riemannian manifolds, are due to Yau [13]; local versions have been established by Cheng and Yau [3].

The classical proof of gradient estimates for a harmonic function u relies on two ingredients. The first ingredient is a comparison theorem for the Laplacian of the Riemannian distance function, which allows one to bound the mean curvature of geodesic spheres from above in terms of a lower bound on the Ricci curvature. The second ingredient is Bochner's formula which is used to give a lower bound on $\Delta|\mathrm{grad}\, u|^2$ in terms of the lower bound on the Ricci curvature. The gradient estimate itself then relies on a clever use of the maximum principle; see [9].

From a probabilistic point of view, it seems tempting to work with Bismut type representation formulas for the differential of a harmonic function. Theorem 1.1 below is taken from [11], [12] and gives a typical gradient formula; for related and more general results the reader may consult [4]. See [10] for additional background on Bismut formulas.

Notation 1. Throughout this paper, M is an n-dimensional Riemannian manifold with $\langle\cdot,\cdot\rangle$, ∇, $\mathrm{div} = \nabla\cdot$, grad, $\Delta = \nabla\cdot\mathrm{grad}$, and Ric denoting (respectively) the Riemannian metric, the Levi-Civita covariant derivative, the divergence operator, the gradient operator, the Laplacian, and the Ricci tensor on M. For $v \in T_m M$, let $|v| := \langle v, v\rangle^{1/2}$, and for $x, y \in M$, let $d(x, y)$ denote the Riemannian distance between x and y.

For a relatively compact open subset, $D \subset M$, let ∂D denote the boundary of D and for $x \in D$ let

$$r_D(x) := \mathrm{d}(x, \partial D) = \inf_{y\in\partial D} d(x, y)$$

be the distance of x to ∂D. (By convention, if the boundary ∂D is empty we set $r_D(x) = \infty$.) Let $K = K(D) \ge 0$ be the smallest non-negative constant such that $\mathrm{Ric}(v, v) \ge -K|v|^2$ for all $v \in TD \subset TM$ and let $k = k(D) \ge 0$ be defined by the equation $K = (n-1)k^2$.

Theorem 1.1 (*Stochastic Representation of the Gradient*). *Let M be a complete Riemannian manifold, $D \subset M$ be a relatively compact open domain, and $u\colon D \to \mathbb{R}$ be a bounded harmonic function. Then, for any $v \in T_x M$ and $x \in D$,*

$$(\mathrm{d}u)_x v = -\mathbb{E}\left[u(X_\tau)\int_0^\tau \left\langle \Theta_s\, \dot{\ell}_s, \mathrm{d}B_s\right\rangle\right], \tag{1.1}$$

where:

(1) *X is a Brownian motion on M, starting at x, and*

$$\tau = \inf\{t > 0 : X_t \notin D\}$$

its time of first exit from D; the stochastic integral is taken with respect to the Brownian motion B in $T_x M$, related to X by the Stratonovich equation $\mathrm{d}B_t = /\!\!/_t^{-1}\delta X_t$, where $/\!\!/_t\colon T_x M \to T_{X_t} M$ denotes the stochastic parallel transport along X.

(2) *The process Θ takes values in the group of linear automorphisms of $T_x M$ and is defined by the pathwise covariant ordinary differential equation*

$$\mathrm{d}\Theta_t = -\frac{1}{2}\,\mathrm{Ric}_{/\!\!/_t}(\Theta_t)\,\mathrm{d}t, \qquad \Theta_0 = \mathrm{id}_{T_x M},$$

where $\mathrm{Ric}_{/\!\!/_t} = /\!\!/_t^{-1} \circ \mathrm{Ric}^\sharp_{X_t} \circ /\!\!/_t$ (a linear transformation of $T_x M$), and $\langle \mathrm{Ric}^\sharp_z u, w\rangle = \mathrm{Ric}_z(u, w)$ for any $u, w \in T_z M$, $z \in M$.

(3) *Finally, ℓ_t may be any adapted finite energy process taking values in T_xM such that $\ell_0 = v$, $\ell_\tau = 0$ and*

$$\left(\int_0^\tau \mathrm{e}^{Kt}\,|\dot{\ell}_t|^2\,\mathrm{d}t\right)^{1/2} \in L^{1+\varepsilon} \quad \text{for some } \varepsilon > 0,$$

where $K = K(D)$ is as in Notation 1.

By making a clever choice for ℓ_t, it is possible to estimate the right hand side of Eq. (1.1) so as to obtain sharp estimates of the form

$$|\mathrm{grad}\,u|\,(x) \le C\,(r_D\,(x)\,, n, K) \sup_{x\in D} |u\,(x)\,|,$$

where $C\,(r, n, K)$ is a certain function of $r > 0$. See [12, Cor. 5.1] for details.

For positive harmonic functions however, one would like to estimate $|\mathrm{grad}\,u|(x)$ in terms of $u(x)$ only. Such estimates provide elliptic counterparts to the famous parabolic Li–Yau estimates for solutions of the heat equation; see [6]. It is an intriguing problem how to gain such estimates in probabilistic terms from Bismut type formulas.

In this paper we deal with pointwise estimates of $\mathrm{grad}\,u(x)$ in terms of $u(x)$ for positive harmonic functions u. We show that such estimates may indeed be derived by stochastic analysis methods involving certain basic submartingales. In particular, we give a stochastic proof of the following gradient estimate due to Cheng and Yau [3]. In addition, our approach provides an explicit value for the constant $c(n)$ appearing in (1.2); see (4.11) and (4.12).

Theorem 1.2. *Let M be a complete Riemannian manifold of dimension $n \ge 2$ and let $D \subset M$ be a relatively compact domain. Let $u\colon D \to]0, \infty[$ be a strictly positive harmonic function. Then*

$$|\mathrm{grad}\,u(x)| \le c(n)\left(k + \frac{1}{r_D(x)}\right)\cdot u(x), \tag{1.2}$$

where $k = k\,(D)$ and $r_D\,(x)$ are as in Notation 1.

Our method of proof is inspired by the stochastic approach to gradient estimates used in [12] for harmonic functions, and in [1] for harmonic maps, where one represents, as described, the differential by a Bismut type mean value formula which may then be evaluated in explicit terms. However in this paper, we do not use the mean value formula directly. Roughly speaking, Bismut type formulas are derived from certain underlying martingales by taking expectation. We sharpen this approach by constructing analogous submartingales (see Theorem 3.1) which after taking expectation provide Bismut type inequalities (see Eq. (3.6)). From these probabilistic inequalities we are able to establish the pointwise estimates; see Theorem 3.2 and Corollary 3.5. Finally, as in [12], explicit constants depend on a reasonable choice of a finite energy process which is used for integration by parts on path space; see Theorems 4.1–4.3.

2. Some elementary geometric calculations

Let M be a (not necessarily complete) Riemannian manifold of dimension $n \ge 2$, and $u\colon M \to \mathbb{R}$ be a harmonic function. For $x \in M$ let $\varphi(x) = |\mathrm{grad}\,u|(x)$. For $x \in M$ with $\varphi(x) \ne 0$, let $\mathfrak{n}(x) = \varphi(x)^{-1}\,\mathrm{grad}\,u(x)$.

If $f\colon M \to \mathbb{R}$ is a smooth function, then we have the well-known formula

$$\square\mathrm{grad}\,f = \mathrm{grad}\,\Delta f + \mathrm{Ric}^\sharp\mathrm{grad}\,f, \tag{2.1}$$

where $\Box = \operatorname{trace}\nabla^2$ denotes the rough Laplacian on $\Gamma(TM)$, and where by definition,

$$\langle \mathrm{Ric}^\sharp X, Y\rangle = \mathrm{Ric}(X, Y), \quad X, Y \in \Gamma(TM).$$

Eq. (2.1) applied to u gives

$$\Box \operatorname{grad} u = \mathrm{Ric}^\sharp \operatorname{grad} u. \tag{2.2}$$

In the following lemma we calculate $\operatorname{grad}\varphi$ and $\Delta\varphi$ in terms of $\mathfrak{n}$.

Lemma 2.1. *Let $u\colon M \to \mathbb{R}$ be a harmonic function, $\varphi = |\operatorname{grad} u|$, and $\mathfrak{n} = (\operatorname{grad} u)/\varphi$ where it is defined. Then on M, where φ does not vanish,*

$$\Delta\varphi = \varphi[\mathrm{Ric}(\mathfrak{n}, \mathfrak{n}) + |\nabla\mathfrak{n}|^2_{\mathrm{H.S.}}], \tag{2.3}$$

$$\operatorname{grad}\,\log\varphi = \nabla_{\mathfrak{n}}\mathfrak{n} - (\operatorname{div}\mathfrak{n})\mathfrak{n}, \tag{2.4}$$

and

$$|\operatorname{grad}\,\log\varphi|^2 \le (n-1)\,|\nabla\mathfrak{n}|^2_{\mathrm{H.S.}}. \tag{2.5}$$

Proof. (i) We start by proving Eq. (2.3). To this end, fix $x \in M$ such that $\varphi(x) \neq 0$ and choose an orthonormal frame $(e_i)_{1\le i\le n}$ at x such that $(\nabla_{e_i} e_j)(x) = 0$ for all i, j. Then, we have at x,

$$\Box(\varphi\mathfrak{n}) = (\Delta\varphi)\,\mathfrak{n} + 2\left\langle\nabla_{e_i}\operatorname{grad} u, \mathfrak{n}\right\rangle\nabla_{e_i}\mathfrak{n} + \varphi\,\Box\mathfrak{n}. \tag{2.6}$$

Since $\langle\mathfrak{n}, \mathfrak{n}\rangle = 1$, we have $0 = v\langle\mathfrak{n}, \mathfrak{n}\rangle = 2\langle\nabla_v\mathfrak{n}, \mathfrak{n}\rangle$ for any $v \in T_xM$. Thus, taking scalar product with $\mathfrak{n}$ makes the second term of the right hand side of (2.6) vanish and yields

$$\Delta\varphi = \langle\Box(\varphi\mathfrak{n}), \mathfrak{n}\rangle - \varphi\,\langle\Box\mathfrak{n}, \mathfrak{n}\rangle\,.$$

It is easily seen that

$$\langle\Box\mathfrak{n}, \mathfrak{n}\rangle = -|\nabla\mathfrak{n}|^2_{\mathrm{H.S.}},$$

so that with Eq. (2.2) the claimed equality for $\Delta\varphi$ follows.
(ii) To establish (2.4), note that $\Delta u = 0$ can be written as

$$0 = \operatorname{div}(\varphi\mathfrak{n}) = \mathrm{d}\varphi\,(\mathfrak{n}) + \varphi\operatorname{div}\,\mathfrak{n}.$$

Let $\mathfrak{n}^\flat = \langle\mathfrak{n}, \cdot\rangle = \varphi^{-1}\,\langle\operatorname{grad} u, \cdot\rangle = \varphi^{-1}\mathrm{d}u$. Then on one hand,

$$\begin{aligned}\iota_{\mathfrak{n}}\mathrm{d}\mathfrak{n}^\flat &= \iota_{\mathfrak{n}}(-\varphi^{-2}\mathrm{d}\varphi\wedge\mathrm{d}u) = -\varphi^{-2}(\mathrm{d}\varphi\,(\mathfrak{n})\,\mathrm{d}u - \mathrm{d}u\,(\mathfrak{n})\,\mathrm{d}\varphi)\\ &= -\varphi^{-2}(-\varphi\operatorname{div}\mathfrak{n}\,\mathrm{d}u - \varphi|\mathfrak{n}|^2\mathrm{d}\varphi)\\ &= \operatorname{div}\mathfrak{n}\,\langle\mathfrak{n}, \cdot\rangle + \langle\operatorname{grad}\,\log\varphi, \cdot\rangle\,,\end{aligned}$$

while on the other hand,

$$\iota_{\mathfrak{n}}\mathrm{d}\mathfrak{n}^\flat = \langle\nabla_{\mathfrak{n}}\mathfrak{n}, \cdot\rangle - \langle\nabla_{\cdot}\mathfrak{n}, \mathfrak{n}\rangle = \langle\nabla_{\mathfrak{n}}\mathfrak{n}, \cdot\rangle\,.$$

Comparing these last two equations proves Eq. (2.4).
(iii) Finally, to establish (2.5), note first that, as a consequence of (2.4),

$$|\operatorname{grad}\,\log\varphi|^2 = (\operatorname{div}\mathfrak{n})^2 + |\nabla_{\mathfrak{n}}\mathfrak{n}|^2.$$

Next, fix $x \in M$ such that $\varphi(x) \neq 0$ and choose an orthonormal frame $(e'_i)_{1\le i\le n}$ at x such that $e'_n = \mathfrak{n}$. Then

$$\begin{aligned}|\mathrm{grad}\,\log\varphi|^2 &= \left(\sum_{j=1}^{n-1}\langle\nabla_{e'_j}\mathfrak{n}, e'_j\rangle\right)^2 + |\nabla_{\mathfrak{n}}\mathfrak{n}|^2\\ &\le (n-1)\sum_{j=1}^{n-1}\langle\nabla_{e'_j}\mathfrak{n}, e'_j\rangle^2 + |\nabla_{e'_n}e'_n|^2\\ &\le (n-1)\sum_{j=1}^{n}|\nabla_{e'_j}\mathfrak{n}|^2 = (n-1)\,|\nabla\mathfrak{n}|^2_{\mathrm{H.S.}}. \quad \square\end{aligned}$$

3. Gradient estimates for positive harmonic functions

The following theorem gives the submartingale property which will be crucial for our estimates; see Bakry [2] for related analytic results.

Theorem 3.1. *Let M be a (not necessarily complete) Riemannian manifold of dimension $n \ge 2$. Let X be a Brownian motion on M and let $u: M \to \mathbb{R}$ be a harmonic function. For any $\alpha \ge \frac{n-2}{n-1}$, the process*

$$Y_t := |\mathrm{grad}\,u|^\alpha(X_t)\,\exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\,\mathrm{d}r\right\} \tag{3.1}$$

is a local submartingale, where by convention, $\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(x) := 0$ *at points x where* $\mathrm{grad}\,u(x)$ *vanishes.*

Proof. First assume that $\mathrm{grad}\,u$ does not vanish on M. Then, making use of Eq. (2.3), we have

$$\begin{aligned}\Delta\varphi^\alpha &= \alpha\varphi^{\alpha-1}\Delta\varphi + \alpha(\alpha-1)\varphi^{\alpha-2}|\mathrm{grad}\,\varphi|^2\\ &= \alpha\varphi^\alpha(\mathrm{Ric}(\mathfrak{n},\mathfrak{n}) + |\nabla\mathfrak{n}|^2_{\mathrm{H.S.}} + (\alpha-1)|\mathrm{grad}\,\log\varphi|^2).\end{aligned} \tag{3.2}$$

Since our assumption on α is equivalent to $\alpha - 1 \ge -1/(n-1)$, it now follows from the estimate in Eq. (2.5) that

$$|\nabla\mathfrak{n}|^2_{\mathrm{H.S.}} + (\alpha-1)|\mathrm{grad}\,\log\varphi|^2 \ge |\nabla\mathfrak{n}|^2_{\mathrm{H.S.}} - \frac{1}{n-1}|\mathrm{grad}\,\log\varphi|^2 \ge 0.$$

Combining this estimate with Eq. (3.2) shows

$$\Delta\varphi^\alpha \ge \alpha\varphi^\alpha\mathrm{Ric}(\mathfrak{n},\mathfrak{n}). \tag{3.3}$$

An application of Itô's lemma now implies $Y_t = N_t + A_t$ where

$$\mathrm{d}N_t = \exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\,\mathrm{d}r\right\}\langle /\!\!/_t^{-1}\mathrm{grad}\,\varphi^\alpha(X_t), \mathrm{d}B_t\rangle,$$

and

$$\mathrm{d}A_t = \frac{1}{2}\left(\frac{\Delta\varphi^\alpha}{\varphi^\alpha}(X_t) - \alpha\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_t)\right)Y_t\,\mathrm{d}t.$$

(Here $/\!\!/_t$ is stochastic parallel translation along X and B is the T_xM-valued Brownian motion introduced in Theorem 1.1.) By the inequality (3.3), $\mathrm{d}A_t \ge 0$ and therefore, Y_t is a local submartingale, which completes the proof under the assumption that φ never vanishes on M. This assumption however is easily removed by letting $\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(x) = 0$ in (3.1) at points x where $\operatorname{grad} u(x) = 0$.

Indeed, let $[0, \zeta[$ be the maximal interval on which our Brownian motion X is defined. Fixing $\varepsilon > 0$, we consider the partition

$$0 = \tau_0 \le \sigma_1 \le \tau_1 \le \sigma_2 \le \tau_2 \le \cdots$$

of $[0, \zeta[$ defined by

$$\sigma_i = \inf\{t \ge \tau_{i-1} : Y_t \le \varepsilon/2\} \quad \text{and} \quad \tau_i = \inf\{t \ge \sigma_i : Y_t \ge \varepsilon\}, \quad i \ge 1.$$

Now consider $Y_t^\varepsilon := Y_t \vee \varepsilon$ which is seen to be a local submartingale on each of the subintervals of our partition. Indeed, on $[\sigma_i, \tau_i[$ the process is constant, $Y^\varepsilon \equiv \varepsilon$, while on $[\tau_{i-1}, \sigma_i[$ it is a local submartingale, since there Y itself is a local submartingale by Itô's formula, as shown above, using the fact that $X|[\tau_{i-1}, \sigma_i[$ takes its values in $\{x \in M : \operatorname{grad} u(x) \ne 0\}$. Now since each Y^ε is a local submartingale, $Y_t = \lim_{\varepsilon \downarrow 0} Y_t^\varepsilon$ itself is a local submartingale. □

Remark 2. In (3.1) we adopted the convention that $\mathrm{Ric}(\mathfrak{n},\mathfrak{n})(x) = 0$ at points x where $\operatorname{grad} u(x)$ vanishes. It should be noted that any other convention also gives a local submartingale as well.

Remark 3. In Appendix we provide a generalization of Eq. (3.3), along with a unified proof of the submartingale property of Y_t in (3.1). The argument there directly takes care of the possible vanishing of the gradient of u and does not require the case distinction made in the proof of Theorem 3.1.

Theorem 3.2. *Let M be a Riemannian manifold of dimension $n \ge 2$ and let $u : M \to \mathbb{R}$ be a harmonic function. Further let $\alpha \in [\frac{n-2}{n-1}, 2[$ with $\alpha > 0$, and $p > 1$, $q > 1$ such that $p^{-1} + q^{-1} = 1$. For $x \in M$, let X be a Brownian motion on M starting at x, and denote by τ the time of first exit of X from some relatively compact neighbourhood D of x. Further assume that ρ is a bounded stopping time with $\rho \le \tau$ and that ℓ_t is a real-valued decreasing, adapted process with C^1 paths such that $\ell_0 = 1$, $\ell_\rho = 0$. Then*

$$|\operatorname{grad} u|(x) \le I_1(\alpha, p) \cdot I_2(\alpha, p) \tag{3.4}$$

where

$$I_1(\alpha, p) = \mathbb{E}\left[\left(\int_0^\rho |\operatorname{grad} u|^2(X_s)\,\mathrm{d}s\right)^{\alpha p/2}\right]^{1/\alpha p} \quad \text{and}$$

$$I_2(\alpha, p) = \mathbb{E}\left[\left(\int_0^\rho \exp\left\{\frac{-\alpha}{2-\alpha}\int_0^s \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\,\mathrm{d}r\right\} |\dot{\ell}_s|^{2/(2-\alpha)}\,\mathrm{d}s\right)^{(2-\alpha)q/2}\right]^{1/\alpha q}.$$

Proof. Let Y_t be the process defined in Eq. (3.1). Under our assumptions $\hat{Y}_t := Y_{\rho \wedge t}$ is a bounded non-negative local submartingale which according to Doob and Meyer may be decomposed as $\hat{Y}_t = \hat{Y}_0 + \hat{N}_t + \hat{A}_t$ where $\hat{N}_0 = \hat{A}_0 = 0$; $\hat{N}$ is the local martingale part, and $\hat{A}$ is the drift part. We further assert that $\hat{Y}_t$ is a L^2-submartingale in the sense that

$$\|\hat{N}_\infty\|_2^2 = \mathbb{E}[\langle \hat{N}, \hat{N}\rangle_\infty] < \infty \quad \text{and} \quad \|\hat{A}_\infty\|_2 < \infty. \tag{3.5}$$

To verify the estimates in (3.5), let $(T_n)_{n\geq 0}$ be an increasing sequence of stopping times converging almost surely to $+\infty$, such that each stopped process $\hat{Y}^n := \hat{Y}^{T_n}$ is a submartingale. Writing $\hat{N}^n = \hat{N}^{T_n}$, $\hat{A}^n = \hat{A}^{T_n}$, we have by Itô's formula

$$\langle \hat{N}^n, \hat{N}^n\rangle_t = (\hat{Y}^n_t)^2 - (\hat{Y}^n_0)^2 - 2\int_0^t \hat{Y}^n_s \, \mathrm{d}\hat{Y}^n_s \leq R^2 - 2\int_0^t \hat{Y}^n_s \, \mathrm{d}\hat{N}^n_s$$

where R is an upper bound for $\hat{Y}$. (Here we already used the fact that $\hat{Y} \geq 0$ and that $\hat{A}$ is non-decreasing.) Taking expectation yields $\mathbb{E}[\langle \hat{N}^n, \hat{N}^n\rangle_\infty] \leq R^2$, from which the first estimate in (3.5) follows by monotone convergence. To bound the L^2-norm of the total variation of $\hat{A}$, since $\hat{A}$ is non-decreasing, it suffices to bound $\|\hat{A}_\infty\|_2$. But since $\hat{Y}_\infty := \lim_{t\to\infty} \hat{Y}_t$ exists almost surely, it follows that

$$\|\hat{A}_\infty\|_2 \leq \|\hat{Y}_\infty - \hat{Y}_0\|_2 + \|\hat{N}_\infty\|_2 \leq R + \|\hat{N}_\infty\|_2 < \infty.$$

Let

$$S_t := Y_t \ell_t - \int_0^t Y_s \dot{\ell}_s \, \mathrm{d}s.$$

Since $S_0 = Y_0 = \varphi^\alpha(x)$ and $\mathrm{d}S_t = \ell_t \, \mathrm{d}Y_t$,

$$S_t = \varphi^\alpha(x) + \int_0^t \ell_s \mathrm{d}Y_s$$

is a local submartingale. Moreover, since ℓ is bounded and

$$S_{\rho\wedge t} = \varphi^\alpha(x) + \int_0^t \ell_s \, \mathrm{d}\hat{Y}_s,$$

it is clear that $S_{\rho\wedge t}$ is also a L^2-submartingale. In particular we have

$$\varphi^\alpha(x) = S_0 \leq \mathbb{E}\left[S_\rho\right] = -\mathbb{E}\left[\int_0^\rho Y_s \dot{\ell}_s \, \mathrm{d}s\right] = \mathbb{E}\left[\int_0^\rho Y_s |\dot{\ell}_s| \, \mathrm{d}s\right].$$

Combining this inequality with the definition of Y in Eq. (3.1) implies

$$\varphi^\alpha(x) \leq \mathbb{E}\left[\int_0^\rho \varphi^\alpha(X_s) \, \exp\left\{-\frac{\alpha}{2}\int_0^s \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\, \mathrm{d}r\right\} |\dot{\ell}_s| \, \mathrm{d}s\right]. \tag{3.6}$$

Assuming $\alpha < 2$, an application of Hölder's inequality shows

$$\varphi(x) \leq \mathbb{E}\left[\left(\int_0^\rho \varphi^2(X_s)\, \mathrm{d}s\right)^{\alpha/2} \times \left(\int_0^\rho \exp\left\{-\frac{\alpha}{2-\alpha}\int_0^s \mathrm{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\, \mathrm{d}r\right\} |\dot{\ell}_s|^{2/(2-\alpha)} \, \mathrm{d}s\right)^{(2-\alpha)/2}\right]^{1/\alpha}. \tag{3.7}$$

One more application of Hölder's inequality to Eq. (3.7) then gives Eq. (3.4). $\square$

To estimate $I_1(\alpha, p)$ we use the following lemma.

Lemma 3.3. *For $\beta \in]0,1[$, let*

$$\gamma_\beta = 2^{-1/2}\left(\frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/\beta} = 2^{-1/2}\left(\frac{\Gamma\left(\frac{1-\beta}{2}\right)}{\sqrt{\pi}}\right)^{1/\beta}. \tag{3.8}$$

Then for every positive local martingale Z with infinite lifetime and deterministic starting point $Z_0 = z$, we have

$$\mathbb{E}[\langle Z\rangle_\infty^{\beta/2}]^{1/\beta} \le \gamma_\beta\, z,$$

where $\langle Z\rangle_\infty = \lim_{t\uparrow\infty}\langle Z\rangle_t$ and $\langle Z\rangle_t$ is the quadratic variation process associated with Z.

Proof. Without loss of generality we may assume that $z = 1$. Moreover, applying the Dambis–Dubins–Schwarz Theorem (cf. [7] or [8]), by "enriching" the filtered probability space if necessary, we may assume there exists a Brownian motion B starting at 0 such that

$$Z_t = 1 + B_{\langle Z\rangle_t}.$$

Let $T := \inf\{t \ge 0,\ 1 + B_t = 0\}$. By the reflection principle,

$$\mathbb{P}\{B_t \le -1\} = \frac{1}{2}\mathbb{P}\{T \le t\} \quad \text{for all } t \ge 0,$$

and the scaling property of Brownian motion, we conclude that T has the same law as $1/B_1^2$. Consequently,

$$\mathbb{E}[T^{\beta/2}]^{1/\beta} = \mathbb{E}[|B_1|^{-\beta}]^{1/\beta} = \gamma_\beta.$$

Moreover, we have $\langle Z\rangle_\infty \le T$ a.s., so that

$$\mathbb{E}[\langle Z\rangle_\infty^{\beta/2}]^{1/\beta} \le \mathbb{E}[T^{\beta/2}]^{1/\beta} = \gamma_\beta. \quad \square$$

To exploit estimate (3.4) we now choose $\alpha \in [\frac{n-2}{n-1}, 1[$ if $n \ge 3$, $\alpha \in]0,1[$ if $n = 2$, and $p > 1$ such that $\beta := \alpha p < 1$.

Proposition 3.4 (*Gradient Estimate; General Form*). *Let M be a Riemannian manifold of dimension $n \ge 2$ and $D \subset M$ be a relatively compact domain. For $x \in D$, let X be Brownian motion starting at x, τ be the time of first exit of X from D, and ρ be a bounded stopping time such that $\rho \le \tau$. Assume that $u\colon M \to]0,\infty[$ is a positive harmonic function and let $\mathfrak{n} = \operatorname{grad} u/|\operatorname{grad} u|$ when $\operatorname{grad} u \ne 0$ and $\mathfrak{n} = 0$ otherwise. Further, let $\alpha \in [\frac{n-2}{n-1}, 1[\cap]0,1[$ and $q > 1$ be such that $\beta := \frac{q}{q-1}\alpha < 1$. Then, for each $x \in D$, the following estimate holds:*

$$|\operatorname{grad}\log u|(x) \le \gamma_\beta\,\mathbb{E}\left[\left(\int_0^\rho \exp\left\{-\frac{\alpha}{2-\alpha}\int_0^s \operatorname{Ric}(\mathfrak{n},\mathfrak{n})(X_r)\,\mathrm{d}r\right\}|\dot{\ell}_s|^{2/(2-\alpha)}\,\mathrm{d}s\right)^{(2-\alpha)q/2}\right]^{1/\alpha q} \tag{3.9}$$

where γ_β is given by (3.8) and the process ℓ_s is chosen as in Theorem 3.2.

Proof. Lemma 3.3 applied to $Z_t := u(X_t^\rho)$ gives

$$\mathbb{E}\left[\left(\int_0^\rho \varphi^2(X_s)\,\mathrm{d}s\right)^{\beta/2}\right]^{1/\beta} \le \gamma_\beta\, u(x). \tag{3.10}$$

Bounding the term $I_1(\alpha, p)$ in estimate (3.4) by means of (3.10), and dividing by $u(x)$, the claimed inequality follows from (3.4). $\square$

Corollary 3.5. *Keeping the assumptions of Proposition 3.4, if* $u: M \to]0, \infty[$ *is a positive harmonic function, then*

$$|\text{grad}\, \log u|(x) \le \gamma_\beta\, \mathbb{E}\left[\left(\int_0^\rho \exp\left\{\frac{\alpha K s}{2-\alpha}\right\} |\dot{\ell}_s|^{2/(2-\alpha)}\, \mathrm{d}s\right)^{(2-\alpha)q/2}\right]^{1/\alpha q}, \tag{3.11}$$

where $\text{Ric} \ge -K = -K(D)$ *as in Notation 1.*

4. Explicit constants

In order to estimate the right hand sides in Eqs. (3.9) and (3.11), we are going to use the methods developed in [12]; see especially [12, Corollary 5.1]. Throughout this section the assumptions on α and q from Proposition 3.4 will be preserved.

Fix $x \in M$ and let $D \subset M$ be a relatively compact open neighbourhood of x in M with smooth boundary. Let $f \in C^2(\bar{D})$ be a positive function on D which is bounded by 1 and vanishing on ∂D and let X_t be a Brownian motion on M starting at $x \in D$. Define

$$\begin{aligned} &T(s) := \int_0^s f^{-2}(X_t)\, \mathrm{d}t, \quad s < \tau_D, \quad \text{and} \\ &\rho(t) := \inf\{s \ge 0 : T(s) \ge t\}, \end{aligned} \tag{4.1}$$

where τ_D is the time of first exit of X from D. Alternatively we may express ρ as

$$\rho(t) = \begin{cases} T^{-1}(t) & \text{if } t \le T(\tau_D) \\ \infty & \text{if } t > T(\tau_D), \end{cases}$$

from which we see that

$$\dot{\rho}(t) = \frac{1}{T'(\rho(t))} = f^2(X_{\rho(t)}) \quad \text{for } t < T(\tau_D).$$

In particular it follows that $\rho(t) \le t$ for $t < T(\tau_D)$. By [12, Proposition 2.3], the process $X'_t := X_{\rho(t)}$ is a diffusion with generator $L' := \frac{1}{2} f^2 \Delta$, and infinite lifetime and as a consequence, $T(\tau_D) = \infty$ a.s.

The idea now is to use the fact that $T(t) \uparrow \infty$, as $t \uparrow \tau_D$, to construct the required finite energy process ℓ_s meeting the crucial conditions $\ell_0 = 1$ and $\ell_\rho = 0$. More precisely, we fix $t > 0$ ($\rho(t)$ will be our ρ in Corollary 3.5) and let

$$h_0(s) := \int_0^s f^{-2}(X_r)\, 1_{\{r<\rho(t)\}}\, \mathrm{d}r = T(\rho(t) \wedge s).$$

Hence $h_0(s) = h_0(\rho(t)) = T(\rho(t)) = t$ for $s \ge \rho(t)$. Further let $h_1 \in C^1([0, t], \mathbb{R})$ be a function with non-positive derivative such that $h_1(0) = 1$ and $h_1(t) = 0$, and define $\ell_s := (h_1 \circ h_0)(s)$. Since ℓ_s has non-positive derivative, $|\dot{\ell}_s|\, \mathrm{d}s$ is a probability measure on $[0, \rho(t)]$.

Theorem 4.1 (*Gradient Estimate; Specific Form*). *Let M be a Riemannian manifold of dimension $n \ge 2$, and D be a relatively compact open domain in M with smooth boundary*

∂D. Let $f \in C^2(\bar{D})$ be strictly positive on D and vanish on ∂D, and $K = K(D) \geq 0$ be chosen so that Ric $\geq -K$ *as in* Notation 1. *Further suppose that $0 < \alpha \in [\frac{n-2}{n-1}, 1[$ and $q > 1/(1-\alpha)$. Then, letting $p = q/(q-1)$ be the conjugate exponent of q, for any positive harmonic function $u: M \to]0, \infty[$ and any $x \in D$, we have*

$$|\text{grad}\ \log u|(x) \leq \frac{1}{f(x)} \left(\frac{\Gamma\left(\frac{1-\alpha p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^{1/\alpha p} \sqrt{C(\alpha, K, q, f)} \tag{4.2}$$

where

$$C(\alpha, K, q, f) := \sup_{y \in D} \{K f^2(y) - (f \Delta f)(y) + (\alpha q + 1)|\text{grad}\ f|^2(y)\}.$$

Proof. Our assumptions on α imply that $p\alpha < 1$ and $q > 2/(2-\alpha)$. Since the inequality in Eq. (4.2) is invariant under scaling f by a positive constant, we may assume without loss of generality that f is bounded above by 1. We want to estimate $I^{1/\alpha q}$ where

$$I := \mathbb{E}\left[\left(\int_0^{\rho(t)} \exp\left\{ \frac{\alpha K s}{2-\alpha} \right\} |\dot{\ell}_s|^{2/(2-\alpha)}\, \mathrm{d}s \right)^{(2-\alpha)q/2} \right].$$

By means of Jensen's inequality, we get

$$\begin{aligned}
I &= \mathbb{E}\left[\left(\int_0^{\rho(t)} \exp\left\{ \frac{\alpha K}{2-\alpha} s \right\} |\dot{\ell}_s|^{\alpha/(2-\alpha)} |\dot{\ell}_s|\, \mathrm{d}s \right)^{(2-\alpha)q/2} \right] \\
&\leq \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{ \frac{\alpha K q}{2} s \right\} |\dot{\ell}_s|^{q\alpha/2} |\dot{\ell}_s|\, \mathrm{d}s \right] \\
&= \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{ \frac{\alpha K q}{2} s \right\} |\dot{\ell}_s|^{(q\alpha+2)/2}\, \mathrm{d}s \right] \\
&= \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{ \frac{\alpha K q}{2} s \right\} |\dot{h}_1(h_0(s))|^{(q\alpha+2)/2} |\dot{h}_0(s)|^{(q\alpha+2)/2}\, \mathrm{d}s \right] \\
&= \mathbb{E}\left[\int_0^{\rho(t)} \exp\left\{ \frac{\alpha K q}{2} s \right\} |\dot{h}_1(h_0(s))|^{(q\alpha+2)/2} f^{-q\alpha-2}(X_s)\, \mathrm{d}s \right] \\
&= \mathbb{E}\left[\int_0^{t} \exp\left\{ \frac{\alpha K q}{2} \rho(r) \right\} |\dot{h}_1(r)|^{(q\alpha+2)/2} f^{-q\alpha}(X'_r)\, \mathrm{d}r \right] \\
&= \int_0^t |\dot{h}_1(r)|^{(q\alpha+2)/2}\, \mathbb{E}\left[\exp\left\{ \frac{\alpha K q}{2} \rho(r) \right\} f^{-q\alpha}(X'_r) \right] \mathrm{d}r.
\end{aligned} \tag{4.3}$$

Let $Z_s = \mathrm{e}^{\alpha K q \rho(s)/2} f^{-\alpha q}(X'_s)$. Writing $\stackrel{\mathrm{m}}{=}$ for equality up to a differential of a local martingale, we have

$$\begin{aligned}
\mathrm{d}Z_s &\stackrel{\mathrm{m}}{=} \frac{1}{2} Z_s (\alpha K q \rho'(s) + f^2(X'_s)(\Delta f^{-\alpha q})(X'_s))\mathrm{d}s \\
&\stackrel{\mathrm{m}}{=} \frac{1}{2} Z_s (\alpha K q f^2(X'_s) - \alpha q (f \Delta f)(X'_s) + \alpha q(\alpha q + 1)|\text{grad}\ f|^2(X'_s))\, \mathrm{d}s
\end{aligned}$$

which implies

$$\mathrm{d}Z_s \le \mathrm{d}M_s + C_1 Z_s\,\mathrm{d}s \tag{4.4}$$

where M is a local martingale and

$$C_1 := \frac{1}{2}\sup_{x\in D}\{\alpha K q f^2(x) - \alpha q(f\Delta f)(x) + \alpha q(\alpha q+1)|\mathrm{grad}\, f|^2(x)\}.$$

Let $D_n \subset D$ be an increasing sequence of relatively compact open subsets of D such that $x \in D_n$ and $\cup D_n = D$. If σ_n is the time of first exit of X' from D_n, then M^{σ_n} is a martingale, $\mathbb{E}[Z_s^{\sigma_n}] < \infty$ for all s, and from Eq. (4.4),

$$\mathbb{E}[Z_s^{\sigma_n}] \le C_1 \int_0^s \mathbb{E}[Z_t^{\sigma_n}]\,\mathrm{d}t.$$

It now follows by an application of Gronwall's lemma along with Fatou's lemma that

$$\mathbb{E}[Z_s] \le \liminf_{n\to\infty} \mathbb{E} Z_s^{\sigma_n} \le Z_0 \mathrm{e}^{C_1 s} = f^{-\alpha q}(x)\mathrm{e}^{C_1 s}.$$

Using this estimate in Eq. (4.3) gives

$$\begin{aligned} I^{1/\alpha q} &\le \left(\int_0^t f^{-\alpha q}(x)\,\mathrm{e}^{C_1 s}\,|\dot h_1(s)|^{(q\alpha+2)/2}\,\mathrm{d}s\right)^{1/\alpha q} \\ &= \frac{1}{f(x)}\left(\int_0^t \mathrm{e}^{C_1 s}\,|\dot h_1(s)|^{(q\alpha+2)/2}\,\mathrm{d}s\right)^{1/\alpha q} =: \frac{1}{f(x)}\bar J(t,h_1). \end{aligned}$$

For $a > 0$, let $J(t,a) := \bar J(t,h_1)$ where $h_1(s) := 1 - \frac{1-\mathrm{e}^{-as}}{1-\mathrm{e}^{-at}}$ for $s \in [0,t]$. It then follows by Corollary 3.5 (with $\rho = \rho(t) \le \tau \wedge t \le \tau$) that

$$|\mathrm{grad}\,\log u|(x) \le \gamma_\beta \frac{1}{f(x)} J(t,a) \tag{4.5}$$

where, by a simple computation,

$$J(t,a) = \left(\frac{a}{1-\mathrm{e}^{-at}}\right)^{(q\alpha+2)/(2\alpha q)} \left(\frac{1-\mathrm{e}^{(C_1-(q\alpha+2)a/2)t}}{(q\alpha+2)a/2 - C_1}\right)^{1/\alpha q}.$$

Now suppose that $a > 2C_1/(q\alpha+2)$. Then

$$\inf_{t>0} J(t,a) \le \lim_{t\to\infty} J(t,a) = \left(\frac{a^{(q\alpha+2)/2}}{(q\alpha+2)a/2 - C_1}\right)^{1/\alpha q}$$

and the minimum in $a > 2C_1/(q\alpha+2)$ of the latter expression is $\sqrt{2C_1/\alpha q} = \sqrt{C(\alpha,K,q,f)}$ which is attained at $a = 2C_1/\alpha q$. Consequently we have shown

$$\inf_{a>0}\inf_{t>0} J(t,a) \le \sqrt{C(\alpha,K,q,f)}$$

which combined with Eq. (4.5) (recall that $\beta = \alpha p$ and γ_β is given as in Eq. (3.8)) gives the estimate in Eq. (4.2). □

Remark 4. Note that $C(\alpha,K,q,f)$ in Theorem 4.1 differs from the constant $c_1(f)/2$ in [12] only by the coefficient, $\alpha q + 1$, which in [12] was the number "3".

Remark 5. Let M be a complete Riemannian manifold, $x \in M$, $R(y) := \mathrm{d}(x, y)$, $\mathrm{Cut}(x) \subset M$ denote the cut-locus of x, and for $r > 0$ let $D = B_r(x)$ (the open ball about x of radius r) and define $f: D \to \mathbb{R}$ by

$$f(y) := \cos\left(\frac{\pi}{2}\frac{r}{R(y)}\right). \tag{4.6}$$

It is shown in the proof of corollary 5.1 in [12] that

$$|\mathrm{grad}\, f| \le \frac{\pi}{2r} \quad \text{and} \quad -\Delta f \le \frac{\pi\sqrt{K(n-1)}}{2r} + \frac{\pi^2 n}{4r^2} \quad \text{on } D \setminus \mathrm{Cut}(x).$$

Hence it follows that on $D \setminus \mathrm{Cut}(x)$,

$$Kf^2 - (f\Delta f) + (\alpha q + 1)|\mathrm{grad}\, f|^2 \le C(\alpha, q, r)$$

where

$$\begin{aligned} C(\alpha, q, r) &= \frac{\pi^2}{4}\frac{n + \alpha q + 1}{r^2} + \frac{\pi}{2}\frac{\sqrt{K(n-1)}}{r} + K \\ &= \frac{\pi^2}{4}\frac{n + \alpha q + 1}{r^2} + \frac{\pi}{2}\frac{k(n-1)}{r} + k^2(n-1). \end{aligned}$$

Here $K = (n-1)k^2$ is as in Notation 1.

Now suppose that $r \in]0, \iota_x[$, where $\iota_x = d(x, \mathrm{Cut}(x))$ is the distance from x to $\mathrm{Cut}(x)$. In this case D is precompact (because M is complete), ∂D is smooth, and $f \in C^2(\bar{D})$ with $f > 0$ on D and $f = 0$ on ∂D. Hence by Theorem 4.1, if u is a positive harmonic function defined on a neighbourhood of $\bar{D}$, then for all $y \in D$,

$$|\mathrm{grad}\, \log u|(y) \le \left(\cos\left(\frac{\pi d(x, y)}{2r}\right)\right)^{-1} \left(\frac{\Gamma\left(\frac{1-\alpha p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/\alpha p} \sqrt{C(\alpha, q, r)},$$

where α, q, and p are as in Theorem 4.1. In particular, taking $y = x$ in this inequality shows

$$|\mathrm{grad}\, \log u|(x) \le \left(\frac{\Gamma\left(\frac{1-\alpha p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{1/\alpha p} \sqrt{C(\alpha, q, r)}. \tag{4.7}$$

When $r \ge \iota_x$, the function $d^2(x, \cdot)$ is no longer smooth on $B_r(x)$ and the boundary of $B_r(x)$ need not be smooth either. Hence, it is not permissible to apply Theorem 4.1 directly to obtain the estimate in (4.7). Nevertheless, using the methods in the proof of [12, Corollary 5.1] the estimate in Eq. (4.7) still holds for arbitrary $r > 0$.

Theorem 4.2. *Let α, q, and p be as in Theorem 4.1 and continue with the notation used in Remark 5. Then for any $r > 0$ and positive harmonic function $u: B_r(x) \to]0, \infty[$, the estimate in Eq. (4.7) holds.*

Proof. Fix $x \in M$ and let $R(y) := d(x, y)$. By Kendall [5, Corollary 1.2], there exists a continuous adapted increasing process, L_t, such that

$$\mathrm{d}(R(X_t)) = \langle \mathrm{grad}\, R(X_t), /\!\!/_t\, \mathrm{d}B_t\rangle + \frac{1}{2}\Delta R(X_t)\,\mathrm{d}t - \mathrm{d}L_t \tag{4.8}$$

where $/\!\!/_t$ and B_t are as in Theorem 1.1, and $\operatorname{grad} R(y)$ and $\Delta R(y)$ are interpreted as being zero at points $y \in M$ where R fails to be smooth. In particular, if $g \in C^2(\mathbb{R})$, then by Eq. (4.8) and Itô's formula we have

$$\mathrm{d}(g(R(X_t))) = g'(R(X_t))\left(\langle \operatorname{grad} R(X_t), /\!\!/_t \mathrm{d}B_t\rangle + \frac{1}{2}\Delta R(X_t)\,\mathrm{d}t - \mathrm{d}L_t\right) + \frac{1}{2} g''(R(X_t))\,|\operatorname{grad} R(X_t)|^2 \mathrm{d}t. \tag{4.9}$$

Now at points where R is smooth,

$$\Delta(g(R)) = \nabla\cdot(g'(R)\nabla R) = g'(R)\,\Delta R + g''(R)\,|\operatorname{grad} R|^2.$$

Hence Eq. (4.9) may be written as

$$\mathrm{d}(g(R(X_t))) = \langle \operatorname{grad}(g\circ R)(X_t), /\!\!/_t \mathrm{d}B_t\rangle + \frac{1}{2}\Delta(g\circ R)(X_t)\,\mathrm{d}t - g'(R(X_t))\,\mathrm{d}L_t$$

with the convention that $\Delta(g\circ R) = 0$ and $\operatorname{grad}(g\circ R) = 0$ at points where R is not smooth.

Let f be defined as in Eq. (4.6), $\rho(s)$ be as in Eq. (4.1), and $X'_s = X_{\rho(s)}$ be as above, then

$$\mathrm{d}(g(R(X'_s))) \overset{\mathrm{m}}{=} \frac{1}{2} f^2(X'_s)\Delta(g(R))(X_{\rho(s)})\mathrm{d}s - g'(R(X'_s))\mathrm{d}L_{\rho(s)}. \tag{4.10}$$

Let $g(\tau) := (\cos(\frac{\pi}{2r}\tau))^{-\alpha q}$ and, as in the proof of Theorem 4.1, let

$$Z_s = \mathrm{e}^{\alpha K q\rho(s)/2} f^{-\alpha q}(X'_s) = \mathrm{e}^{\alpha K q\rho(s)/2} g(X'_s).$$

Then, using Eq. (4.10) and the convention that $\Delta f(y) = 0$ and $\operatorname{grad} f(y) = 0$ at points $y \in D$ where R is not smooth, we have

$$\begin{aligned}\mathrm{d}Z_s &\overset{\mathrm{m}}{=} \frac{1}{2} Z_s(\alpha K q\rho'(s) + f^2(X'_s)(\Delta f^{-\alpha q})(X'_s))\mathrm{d}s - \mathrm{e}^{\alpha K q\rho(s)/2} g'(R(X'_s))\mathrm{d}L_{\rho(s)}\\ &\overset{\mathrm{m}}{\le} \frac{1}{2} Z_s(\alpha K q f^2(X'_s) - \alpha q\,(f\Delta f)(X'_s) + \alpha q\,(\alpha q+1)\,|\operatorname{grad} f(X'_s)|^2)\mathrm{d}s,\end{aligned}$$

where we have used $-\mathrm{e}^{\alpha K q\rho(s)/2} g'(R(X'_s)) \ge 0$ and $L_{\rho(s)}$ is increasing in s. With this observation, the remainder of the proof of Theorem 4.1 starting with Eq. (4.4) goes through without any further modification. Hence the statement of Theorem 4.1 holds for $D = B_r(x)$ and $f = \cos(\pi R/(2r))$. Therefore the argument used to arrive at Eq. (4.7) is still valid provided u is a positive harmonic function defined on a neighbourhood of $\bar{B}_r(x)$. A simple limiting argument shows that Eq. (4.7) is still valid even when u is a positive harmonic function defined only on $B_r(x)$. □

Theorem 4.3 (*Gradient Estimate*). *Let M^n be a complete Riemannian manifold, $n \ge 2$, and let $D \subset M$ be a relatively compact domain, $k = k(D) \ge 0$ be chosen so that* $\mathrm{Ric} \ge -(n-1)k^2$, *and $r_D(x) = \mathrm{dist}(x, \partial D)$ as in Notation 1. If $u\colon D \to]0,\infty[$ is a strictly positive harmonic function, then*

$$|\operatorname{grad} u(x)| \le c(n)\left(k + \frac{1}{r_D(x)}\right)u(x) \quad \textit{for all } x \in D,$$

where

$$c(2) := \frac{\pi}{2}\sqrt{\frac{3}{2}}\exp\left\{-\Gamma'\left(\frac{1}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right)\right\}, \tag{4.11}$$

and for $n \geq 3$,

$$c\,(n) := \frac{\pi}{2}\left(\frac{\Gamma\left(\frac{1}{2(2n-3)}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{(n-3/2)/(n-2)} \sqrt{3\,(n-1)\,/2}. \tag{4.12}$$

As already mentioned, Theorem 4.3 is originally due to Cheng and Yau [3]. Our approach however gives an explicit value for the constant $c(n)$.

Proof. Let $r = r_D\,(x)$. If $n = 2$, let $\alpha \downarrow 0$ in Eq. (4.7) to conclude that

$$|\mathrm{grad}\ \log u|(x) \leq \frac{1}{\sqrt{2}}\ \exp\left\{-\Gamma'\left(\frac{1}{2}\right)\Big/\Gamma\left(\frac{1}{2}\right)\right\}\sqrt{\frac{\pi^2}{4}\frac{3}{r^2}+\frac{\pi}{2}\frac{k}{r}+k^2}.$$

If $n \geq 3$, let $\alpha = \frac{n-2}{n-1}$, and $q = \frac{2}{1-\alpha} = 2\,(n-1)$. Then again from Eq. (4.7) we find

$$|\mathrm{grad}\ \log u|(x) \leq \gamma_{\alpha p}\sqrt{(n-1)\left(\frac{3\pi^2}{4}\frac{1}{r^2}+\frac{\pi}{2}\frac{1}{r}k+k^2\right)}$$

where

$$\gamma_{\alpha p} = \frac{1}{\sqrt{2}}\left(\frac{\Gamma\left(\frac{1}{2(2n-3)}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{(n-3/2)/(n-2)}.$$

This completes the proof of the theorem in view of the following simple estimate:

$$\frac{3\pi^2}{4}\frac{1}{r^2}+\frac{\pi}{2}\frac{k}{r}+k^2 \leq \frac{3\pi^2}{4}\left(\frac{1}{r}+k\right)^2. \quad \square$$

Corollary 4.4 (*Gradient Estimate on Geodesic Balls*). *Let M be a complete Riemannian manifold with* Ric $\geq -(n-1)k^2$, $k \geq 0$. *If u is a positive harmonic function on a geodesic ball $B_r(x) \subset M$, then*

$$\sup_{B_{r/2}(x)} \frac{|\mathrm{grad}\,u|}{u} \leq c(n)\left(k+\frac{2}{r}\right).$$

In particular, if Ric ≥ 0 *then any positive harmonic function u on M is constant.*

Corollary 4.5 (*Elliptic Harnack Inequality*). *Let M be a complete Riemannian manifold with* Ric $\geq -(n-1)k^2$. *Suppose that u is a positive harmonic function on a geodesic ball $B_r(x) \subset M$. Then*

$$\sup_{B_{r/2}(x)} u \leq \exp\left(c(n)\,(2+kr)\right)\inf_{B_{r/2}(x)} u. \tag{4.13}$$

Proof. By the gradient estimate, we have $\sup_{B_{r/2}(x)} |\mathrm{grad}\,u|/u \leq c(n)\,(k+2/r)$. Let $x_1, x_2 \in \bar{B}_{r/2}(x)$ be such that $\sup_{B_{r/2}(x)} u = u(x_1)$ and $\inf_{B_{r/2}(x)} u = u(x_2)$. Then let $\sigma\colon [0,1] \to$

$\bar{B}_{r/2}(x)$ be a parametrization of the path following a length minimizing geodesic from x_2 to x concatenated with a length minimizing geodesic joining x to x_1. Then

$$\begin{aligned}\log \frac{u(x_1)}{u(x_2)} &= \left|\int_0^1 \frac{\mathrm{d}\log u(\sigma(s))}{\mathrm{d}s}\,\mathrm{d}s\right| \\ &\le \int_0^1 |\mathrm{grad}\ \log u|(\sigma\ (s))|\sigma'\ (s)\ |\ \mathrm{d}s \\ &\le c(n)\left(k+\frac{2}{r}\right)\int_0^1 |\sigma'\ (s)\ |\ \mathrm{d}s \\ &\le c(n)\left(k+\frac{2}{r}\right)\cdot r\end{aligned}$$

from which Eq. (4.13) follows. □

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Appendix. Expansion on the submartingale proof

We start by generalizing inequality (3.3).

Lemma A.1. *Let $\varepsilon \ge 0$, u: $M \to \mathbb{R}$ be a harmonic function,*

$$M' = \{x \in M : \mathrm{grad}\, u\ (x) \neq 0\}, \qquad \varphi_\varepsilon = \sqrt{|\mathrm{grad}\, u|^2 + \varepsilon^2},$$

and

$$\mathfrak{n}_\varepsilon\ (x) := \varphi_\varepsilon^{-1} \mathrm{grad}\, u = (|\mathrm{grad}\, u|^2 + \varepsilon^2)^{-1/2} \mathrm{grad}\, u,$$

with the convention that $\mathfrak{n}_0\ (x) := 0$ if $x \notin M'$. Then for $\alpha \in [\frac{n-2}{n-1}, 1]$,

$$\Delta\varphi_\varepsilon^\alpha \ge \alpha\varphi_\varepsilon^\alpha\ \mathrm{Ric}\ (\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon) \tag{A.1}$$

where (A.1) holds for all $x \in M$ if $\varepsilon > 0$ and for all $x \in M'$ if $\varepsilon = 0$.

Proof. Suppose either $\varepsilon > 0$ or $\varepsilon = 0$ and $\mathrm{grad}\, u\ (x) \neq 0$. We begin by observing that

$$\begin{aligned}\Delta\varphi_\varepsilon^\alpha &= \alpha\varphi_\varepsilon^{\alpha-1}\Delta\varphi_\varepsilon + \alpha\ (\alpha-1)\ \varphi_\varepsilon^{\alpha-2}|\mathrm{grad}\, \varphi_\varepsilon|^2 \\ &= \alpha\varphi_\varepsilon^\alpha \left\{\frac{\Delta\varphi_\varepsilon}{\varphi_\varepsilon} + (\alpha-1)\ |\mathrm{grad}\ \log\varphi_\varepsilon|^2\right\}.\end{aligned} \tag{A.2}$$

Set $f_\varepsilon\ (s) = (\varepsilon^2 + s)^{1/2}$, so that $\varphi_\varepsilon\ (x) = f_\varepsilon\left(|\mathrm{grad}\, u\ (x)\ |^2\right)$. Then

$$f'_\varepsilon\ (s) = \frac{1}{2}(\varepsilon^2+s)^{-1/2} \quad \text{and} \quad f''_\varepsilon\ (s) = -\frac{1}{4}(\varepsilon^2+s)^{-3/2}$$

and for $v \in TM$,

$$\begin{aligned} v\varphi_\varepsilon &= 2f_\varepsilon'(|\mathrm{grad}\, u|^2)\langle \nabla_v \mathrm{grad}\, u, \mathrm{grad}\, u\rangle \\ &= \varphi_\varepsilon^{-1}\langle \nabla_v \mathrm{grad}\, u, \mathrm{grad}\, u\rangle \\ &= \langle \nabla_v \mathrm{grad}\, u, \mathfrak{n}_\varepsilon\rangle = \langle \nabla_{\mathfrak{n}_\varepsilon} \mathrm{grad}\, u, v\rangle \end{aligned} \tag{A.3}$$

where in the last equality we have used the fact that ∇ has zero torsion. From this equation it follows that

$$\mathrm{grad}\, \varphi_\varepsilon = \nabla_{\mathfrak{n}_\varepsilon} \mathrm{grad}\, u = \nabla_{\mathfrak{n}_\varepsilon}(\varphi_\varepsilon \mathfrak{n}_\varepsilon) = \mathfrak{n}_\varepsilon \varphi_\varepsilon \cdot \mathfrak{n}_\varepsilon + \varphi_\varepsilon \nabla_{\mathfrak{n}_\varepsilon} \mathfrak{n}_\varepsilon, \tag{A.4}$$

and in particular that

$$\mathrm{grad}\, \log \varphi_\varepsilon = \mathfrak{n}_\varepsilon \log \varphi_\varepsilon \cdot \mathfrak{n}_\varepsilon + \nabla_{\mathfrak{n}_\varepsilon} \mathfrak{n}_\varepsilon. \tag{A.5}$$

Since

$$\begin{aligned} \mathrm{div}\, \mathfrak{n}_\varepsilon &= \mathrm{div}\,(\varphi_\varepsilon^{-1}\mathrm{grad}\, u) = -\varphi_\varepsilon^{-2}\langle \mathrm{grad}\, \varphi_\varepsilon, \mathrm{grad}\, u\rangle + \varphi_\varepsilon^{-1}\Delta u \\ &= -\langle \mathrm{grad}\, \log \varphi_\varepsilon, \mathfrak{n}_\varepsilon\rangle = -\mathfrak{n}_\varepsilon \log \varphi_\varepsilon, \end{aligned}$$

Eq. (A.5) may be written as

$$\mathrm{grad}\, \log \varphi_\varepsilon = \nabla_{\mathfrak{n}_\varepsilon} \mathfrak{n}_\varepsilon - (\mathrm{div}\, \mathfrak{n}_\varepsilon)\, \mathfrak{n}_\varepsilon. \tag{A.6}$$

From Eq. (A.3) we also have

$$\begin{aligned} \nabla^2_{v\otimes v}\varphi_\varepsilon &= \langle \nabla^2_{v\otimes v}\mathrm{grad}\, u, \mathfrak{n}_\varepsilon\rangle + \langle \nabla_v \mathrm{grad}\, u, \nabla_v \mathfrak{n}_\varepsilon\rangle \\ &= \langle \nabla^2_{v\otimes v}\mathrm{grad}\, u, \mathfrak{n}_\varepsilon\rangle + \langle \nabla_v(\varphi_\varepsilon \mathfrak{n}_\varepsilon), \nabla_v \mathfrak{n}_\varepsilon\rangle \\ &= \langle \nabla^2_{v\otimes v}\mathrm{grad}\, u, \mathfrak{n}_\varepsilon\rangle + v\varphi_\varepsilon \cdot \langle \mathfrak{n}_\varepsilon, \nabla_v \mathfrak{n}_\varepsilon\rangle + \varphi_\varepsilon \langle \nabla_v \mathfrak{n}_\varepsilon, \nabla_v \mathfrak{n}_\varepsilon\rangle \end{aligned}$$

which upon summing on v running through an orthonormal frame shows

$$\begin{aligned} \Delta\varphi_\varepsilon &= \langle \Box \mathrm{grad}\, u, \mathfrak{n}_\varepsilon\rangle + \langle \mathfrak{n}_\varepsilon, \nabla_{\mathrm{grad}\, \varphi_\varepsilon} \mathfrak{n}_\varepsilon\rangle + \varphi_\varepsilon |\nabla \mathfrak{n}_\varepsilon|^2_{\mathrm{H.S.}} \\ &= \langle \mathrm{grad}\, \Delta u + \mathrm{Ric}^\#\mathrm{grad}\, u, \mathfrak{n}_\varepsilon\rangle + \langle \mathfrak{n}_\varepsilon, \nabla_{\mathrm{grad}\, \varphi_\varepsilon} \mathfrak{n}_\varepsilon\rangle + \varphi_\varepsilon |\nabla \mathfrak{n}_\varepsilon|^2_{\mathrm{H.S.}} \\ &= \langle \mathrm{Ric}^\# \nabla u, \mathfrak{n}_\varepsilon\rangle + \langle \mathfrak{n}_\varepsilon, \nabla_{\mathrm{grad}\, \varphi_\varepsilon} \mathfrak{n}_\varepsilon\rangle + \varphi_\varepsilon |\nabla \mathfrak{n}_\varepsilon|^2_{\mathrm{H.S.}} \\ &= \varphi_\varepsilon\{\mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon) + \langle \mathfrak{n}_\varepsilon, \nabla_{\mathrm{grad}\, \log \varphi_\varepsilon} \mathfrak{n}_\varepsilon\rangle + |\nabla \mathfrak{n}_\varepsilon|^2_{\mathrm{H.S.}}\}. \end{aligned} \tag{A.7}$$

When $\varepsilon \neq 0$ and $\mathrm{grad}\, u\,(x) = 0$, we see from Eq. (A.7) that

$$\Delta\varphi_\varepsilon(x) = \varphi_\varepsilon(x)\; |\nabla \mathfrak{n}_\varepsilon|^2_{\mathrm{H.S.}}(x)$$

and from Eq. (A.4) that $(\mathrm{grad}\, \log \varphi_\varepsilon)(x) = 0$. Combining these identities with Eq. (A.2) gives

$$(\Delta\varphi_\varepsilon^\alpha)(x) = \alpha\varphi_\varepsilon^\alpha(x)\, |\nabla \mathfrak{n}_\varepsilon|^2_{\mathrm{H.S.}}(x) \geq \alpha\varphi_\varepsilon^\alpha(x)\, \mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)(x) = 0. \tag{A.8}$$

This shows that Eq. (A.1) is valid for $x \notin M'$. To finish the proof it suffices to show that Eq. (A.1) is valid for all $x \in M'$.

So for the rest of the proof we will assume that $x \in M'$. Since $\langle \mathfrak{n}_0, \mathfrak{n}_0\rangle = 1$ on M',

$$0 = v1 = 2\langle \nabla_v \mathfrak{n}_0, \mathfrak{n}_0\rangle \quad \text{for all } v \in T_xM \quad \text{and} \quad x \in M'. \tag{A.9}$$

Taking $\varepsilon = 0$ in Eq. (A.7) gives

$$\Delta\varphi_0 = \varphi_0(\mathrm{Ric}\,(\mathfrak{n}_0, \mathfrak{n}_0) + |\nabla \mathfrak{n}_0|^2_{\mathrm{H.S.}}) \tag{A.10}$$

and from Eqs. (A.6) and (A.9) we find

$$|\mathrm{grad}\ \log\varphi_0|^2 = |\nabla_{\mathfrak{n}_0}\mathfrak{n}_0|^2 + (\mathrm{div}\,\mathfrak{n}_0)^2 \tag{A.11}$$

on M'. Let $F_\varepsilon(s) := (s^2+\varepsilon^2)^{\alpha/2}$ so that $\varphi_\varepsilon^\alpha = F_\varepsilon(\varphi_0)$ and by elementary calculus,

$$F_\varepsilon'(s) = \alpha s(s^2+\varepsilon^2)^{-1}F_\varepsilon(s) \quad \text{and}$$
$$F_\varepsilon''(s) = \alpha(s^2+\varepsilon^2)^{-2}(\varepsilon^2-(1-\alpha)s^2)F_\varepsilon(s).$$

Therefore we find

$$\begin{aligned}
\Delta\varphi_\varepsilon^\alpha &= \Delta F_\varepsilon(\varphi_0) = \mathrm{div}\,(F_\varepsilon'(\varphi_0)\,\mathrm{grad}\,\varphi_0) = F_\varepsilon'(\varphi_0)\,\Delta\varphi_0 + F_\varepsilon''(\varphi_0)\,|\mathrm{grad}\,\varphi_0|^2 \\
&= \alpha\varphi_0(\varphi_0^2+\varepsilon^2)^{-1}F_\varepsilon(\varphi_0)\,\Delta\varphi_0 + \alpha(\varphi_0^2+\varepsilon^2)^{-2}(\varepsilon^2-(1-\alpha)\,\varphi_0^2)F_\varepsilon(\varphi_0)\,|\mathrm{grad}\,\varphi_0|^2 \\
&\geq \alpha\varphi_\varepsilon^\alpha\{\varphi_0(\varphi_0^2+\varepsilon^2)^{-1}\Delta\varphi_0 - (\varphi_0^2+\varepsilon^2)^{-2}\,(1-\alpha)\,\varphi_0^2|\mathrm{grad}\,\varphi_0|^2\} \\
&= \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\left\{\frac{\Delta\varphi_0}{\varphi_0} - (1-\alpha)\,\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}|\mathrm{grad}\ \log\varphi_0|^2\right\} \\
&\geq \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\left\{\frac{\Delta\varphi_0}{\varphi_0} - (1-\alpha)\,|\mathrm{grad}\ \log\varphi_0|^2\right\} \\
&= \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\{\mathrm{Ric}\,(\mathfrak{n}_0,\mathfrak{n}_0) + |\nabla\mathfrak{n}_0|^2_{\mathrm{H.S.}} - (1-\alpha)\,|\mathrm{grad}\ \log\varphi_0|^2\}
\end{aligned} \tag{A.12}$$

where in the last equality we have used Eq. (A.10).

Letting $\{e_i\}_{i=1}^n$ be an orthonormal frame such that $e_n = \mathfrak{n}_0$ shows

$$\begin{aligned}
(\mathrm{div}\,\mathfrak{n}_0)^2 &= \left(\sum_{i=1}^n\langle\nabla_{e_i}\mathfrak{n}_0, e_i\rangle\right)^2 = \left(\sum_{i=1}^{n-1}\langle\nabla_{e_i}\mathfrak{n}_0, e_i\rangle\right)^2 \\
&\leq (n-1)\sum_{i=1}^{n-1}\langle\nabla_{e_i}\mathfrak{n}_0, e_i\rangle^2 \leq (n-1)\sum_{i=1}^{n-1}|\nabla_{e_i}\mathfrak{n}_0|^2
\end{aligned}$$

and therefore, using Eq. (A.11),

$$\begin{aligned}
|\mathrm{grad}\ \log\varphi_0|^2 &= |\nabla_{\mathfrak{n}_0}\mathfrak{n}_0|^2 + (\mathrm{div}\,\mathfrak{n}_0)^2 \\
&\leq (n-1)\sum_{i=1}^{n-1}|\nabla_{e_i}\mathfrak{n}_0|^2 + |\nabla_{e_n}\mathfrak{n}_0|^2 \leq (n-1)\,|\nabla\mathfrak{n}_0|^2_{\mathrm{H.S.}}.
\end{aligned}$$

Combining this estimate with Eq. (A.12) shows

$$\Delta\varphi_\varepsilon^\alpha \geq \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\{\mathrm{Ric}\,(\mathfrak{n}_0,\mathfrak{n}_0) + (1-(1-\alpha)\,(n-1))\,|\nabla\mathfrak{n}_0|^2_{\mathrm{H.S.}}\}. \tag{A.13}$$

Taking $\alpha \geq \frac{n-2}{n-1}$ (which is equivalent to $1-(1-\alpha)\,(n-1) \geq 0$) in Eq. (A.13) implies

$$\Delta\varphi_\varepsilon^\alpha \geq \alpha\varphi_\varepsilon^\alpha\frac{\varphi_0^2}{\varphi_0^2+\varepsilon^2}\mathrm{Ric}\,(\mathfrak{n}_0,\mathfrak{n}_0) = \alpha\varphi_\varepsilon^\alpha\mathrm{Ric}\,(\mathfrak{n}_\varepsilon,\mathfrak{n}_\varepsilon)\,.$$

Combining this estimate with that in Eq. (A.8) proves inequality (A.1). □

Theorem A.2. *We keep the notation and the assumptions of Theorem 3.1. For all* $\alpha \in [\frac{n-2}{n-1}, 1]$,

$$Y_t := |\mathrm{grad}\, u\,(X_t)\,|^{\alpha} \exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}\,(\mathfrak{n}, \mathfrak{n})\,(X_s)\,\mathrm{d}s\right\}$$

is a local submartingale, where

$$\mathfrak{n}\,(x) := \mathfrak{n}_0\,(x) = \begin{cases} |\mathrm{grad}\, u\,(x)\,|^{-1}\mathrm{grad}\, u\,(x) & \text{if } \mathrm{grad}\, u\,(x) \neq 0 \\ 0 & \text{if } \mathrm{grad}\, u\,(x) = 0. \end{cases}$$

Proof. Let $\varepsilon > 0$; then by Itô's formula along with Lemma A.1,

$$\begin{aligned} \mathrm{d}\varphi_\varepsilon^\alpha\,(X_t) &= \langle(\mathrm{grad}\,\varphi_\varepsilon^\alpha)\,(X_t)\,,\, /\!\!/_t\, \mathrm{d}B_t\rangle + \frac{1}{2}(\Delta\varphi_\varepsilon^\alpha)\,(X_t)\,\mathrm{d}t \\ &= \mathrm{d}M_t^\varepsilon + \frac{\alpha}{2}\varphi_\varepsilon^\alpha\,(X_t)\,\mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)\,(X_t)\,\mathrm{d}t + \rho_t^\varepsilon\,\mathrm{d}t \end{aligned}$$

where M^ε denotes the local martingale part and ρ_t^ε is a non-negative process. In particular this implies that

$$\begin{aligned} &\mathrm{d}\left(\exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)\,(X_s)\,\mathrm{d}s\right\}\varphi_\varepsilon^\alpha\,(X_t)\right) \\ &\quad = \exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)\,(X_s)\,\mathrm{d}s\right\}\mathrm{d}M_t^\varepsilon + \exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)\,(X_s)\,\mathrm{d}s\right\}\rho_t^\varepsilon\,\mathrm{d}t. \end{aligned}$$

So if $\varepsilon > 0$ and $\alpha \in [\frac{n-2}{n-1}, 1]$, then

$$Y_t\,(\varepsilon) := (|\mathrm{grad}\, u\,(X_t)\,|^2 + \varepsilon^2)^{\alpha/2} \exp\left\{-\frac{\alpha}{2}\int_0^t \mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)\,(X_s)\,\mathrm{d}s\right\}$$

is a local submartingale. If τ is the time of first exit of X_t from a precompact open subset of M, $Y_t^\tau\,(\varepsilon)$ is an honest submartingale. If G is a bounded non-negative $\mathscr{F}_s$-measurable function, then

$$\begin{aligned} &\mathbb{E}[G\left(Y_t^\tau\,(\varepsilon) - Y_s^\tau\,(\varepsilon)\right)] \\ &\quad = \mathbb{E}\left[G\left(\int_{s\wedge\tau}^{t\wedge\tau} \exp\left\{-\frac{\alpha}{2}\int_0^r \mathrm{Ric}\,(\mathfrak{n}_\varepsilon, \mathfrak{n}_\varepsilon)\,(X_s)\,\mathrm{d}s\right\}\rho_r^\varepsilon\,\mathrm{d}r\right)\right] \geq 0. \end{aligned}$$

Using the dominated convergence theorem, we may let $\varepsilon \downarrow 0$ in the above inequality to conclude

$$\mathbb{E}[G\left(Y_t^\tau - Y_s^\tau\right)] \geq 0$$

which completes the proof. □

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