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A stochastic approach to a priori estimates and Liouville theorems for harmonic maps ☆

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To the memory of Paul Malliavin

Abstract

Nonlinear versions of Bismut type formulas for the differential of a harmonic map between Riemannian manifolds are used to establish a priori estimates for harmonic maps. A variety of Liouville type theorems is shown to follow as corollaries from such estimates by exhausting the domain through an increasing sequence of geodesic balls. This probabilistic method is well suited for proving sharp estimates under various curvature conditions. We discuss Liouville theorems for harmonic maps under the following conditions: small image, sublinear growth, non-positively curved targets, generalized bounded dilatation, Liouville manifolds as domains, certain asymptotic behaviour.

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1. Introduction

In 1984 Bismut [4] established a derivative formula for the heat kernel on a Riemannian manifold in terms of an expectation (with respect to the Brownian bridge measure) of explicit curvature terms. Since then Bismut's formula has been extended and applied in various directions. The authors [32] obtained explicit gradient estimates of harmonic functions on regular domains by using the localized version of a Bismut type formula developed in [31].

The purpose of this paper is twofold: first we like to present estimates analogous to [32] in the nonlinear setting of harmonic maps. To this end, we investigate a nonlinear version of the derivative formula, first established by Arnaudon and Thalmaier [3].

The second aim is to demonstrate the new method and to give simple proofs of various Liouville type theorems for harmonic maps by exploiting derivative estimates obtained from the nonlinear Bismut formula. It is not our main concern to prove new results, rather than to uncover the way how probability and geometry work together to enforce triviality for harmonic maps in certain situations. The new probabilistic method is well suited to separate the influence of curvature conditions on the domain and on the target, in establishing sharp estimates in various specific situations.

A Liouville theorem is a statement of the form that under certain curvature conditions on domain and target all harmonic maps u in a certain mapping class are necessarily constant, see for instance [8,23], as well as [10] for estimates. Roughly speaking, Liouville theorem may be classified into two categories: those requiring a restriction on the size of $u(M)$, and those where the condition on u is expressed in terms of the growth rate of the energy of u on geodesic balls.

The traditional probabilistic approach to Liouville type theorems is typically concerned with non-existence of certain types of harmonic maps by verifying that, under the given constraints, such maps would link random processes on the domain and the target in an incompatible way, see for instance [16] for an account in this direction. A stochastic approach along these lines has been applied by S. Stafford [29] to prove S.-Y. Cheng's Liouville theorem [5]. Probabilistic methods relying on coupling of Brownian motions on Riemannian manifolds have been used by F.-Y. Wang [33]. See also J. Picard [22].

Our approach here is quite different: We start from an exact Bismut type expression for the differential of a harmonic map. This formula is exploited under the given curvature conditions to derive precise estimates on certain domains, like on geodesic balls. Since the underlying tool is an exact formula in a general setting, it is not surprising that the method can be adapted to deal with quite a variety of different situations, as will be shown in Section 5; there is also no doubt that this approach can be used to prove many other results in this direction. It can be seen as a unifying approach to a priori estimates for harmonic maps.

The paper is organized as follows. In Section 2 we collect some basic estimates for geodesic transports along semimartingales and in Section 3 we recall the nonlinear derivative formula obtained in [3]. To follow the lines of [32], a variant of the formula is presented in Theorem 3.3. In Section 4 we estimate the first-order differential of harmonic maps via lower bounds of the Ricci curvature on the domain manifold and upper bounds of the sectional curvatures on the target. Finally, in Section 5, Liouville type theorems for harmonic maps are deduced from our derivative estimates in a variety of different situations.

2. Geodesic transports

Definition 2.1 *(Geodesic transport along a semimartingale).* Let (N, h) be a Riemannian manifold and Y be a continuous semimartingale taking values in N. The *geodesic transport* (also called damped or deformed parallel transport [3])

$$\Theta_{0,t} : T_{Y_0}N \to T_{Y_t}N$$

along Y is defined by the following covariant equation along Y:

$$d\big(//_{0,\bullet}^{-1}\Theta_{0,\bullet}\big) = -\frac{1}{2}//_{0,\bullet}^{-1}R^N(\Theta_{0,\bullet}, dY)\,dY \quad \text{with } \Theta_{0,0} = \mathrm{id}, \tag{2.1}$$

where R^N denotes the Riemann curvature tensor on N.

Example 2.2 *(Geodesic transport along a Brownian motion).* Let Y be Brownian motion on (N, h). Then Eq. (2.1) reduces to

$$d\big(//_{0,\bullet}^{-1}\Theta_{0,\bullet}\big) = -\frac{1}{2}\,//_{0,\bullet}^{-1}\,\mathrm{Ric}^N(\Theta_{0,s})\,ds \quad \text{with } \Theta_{0,0} = \mathrm{id},$$

where Ric^N is the Ricci curvature of N. We use the convention $\mathrm{Ric}^N \in \Gamma(\mathrm{End}(TN))$, i.e., $\mathrm{Ric}^N : T_xN \to T_xN$ for each $x \in N$, $\mathrm{Ric}^N(v) \equiv \mathrm{Ric}^N_x(v, \cdot)^\#$.

Definition 2.3 *(Anti-development).* Let Y be a continuous semimartingale taking values in a Riemannian manifold N. The $T_{Y_0}N$-valued processes $\mathscr{A}(Y)$, resp. $\mathscr{A}_{\mathrm{def}}(Y)$, defined by the following Stratonovich integrals,

$$\mathscr{A}(Y) := \int_0^{\bullet} //_{0,s}^{-1}\delta Y_s, \qquad \mathscr{A}_{\mathrm{def}}(Y) := \int_0^{\bullet} \Theta_{0,s}^{-1}\delta Y_s, \tag{2.2}$$

are called the *anti-development* of Y, resp. *deformed anti-development* of Y.

Remarks 2.4. a) The notion of the geodesic transport (Definition 2.1), as well as the notion of anti-development (Definition 2.3), requires a differentiable manifold endowed with a connection; the Riemannian structure is not needed. However, since we only deal with the Levi–Civita connection on a Riemannian manifold, this point will not be important for us. In the sequel ∇ always denotes the Levi–Civita connection.

b) In Eq. (2.2) the Stratonovich integrals with respect to δY_s can be replaced by Ito integrals with respect to $d^\nabla Y_s$ as well, see [3] for details.

c) A continuous semimartingale Y on (N, h) is a ∇-martingale if and only if $\mathscr{A}(Y)$, or equivalently $\mathscr{A}_{\mathrm{def}}(Y)$, is a local martingale in $T_{Y_0}N$.

d) A continuous semimartingale Y on (N, h) is a Brownian motion if and only if $\mathscr{A}(Y)$ is a Brownian motion in $T_{Y_0}N$.

In the following let (N, h) be a Riemannian manifold and Y be a continuous semimartingale taking values in N. Let $Z := \mathscr{A}(Y)$ be the anti-development of Y and $\int h(dY, dY)$ its Riemannian quadratic variation. Then

$$h(dY, dY) = \mathrm{trace}\, d[Z, Z] = \sum_{i=1}^n d\big[Z^i, Z^i\big]$$

with respect to any orthonormal basis of $T_{Y_0}N$. Consider the $\mathrm{End}(T_{Y_0}N)$-valued process Λ given (with respect to an arbitrary orthonormal basis of $T_{Y_0}N$) by the $n \times n$ matrix

$$\Lambda^{jk} := \frac{d[Z^j, Z^k]}{h(dY, dY)}, \quad 1 \leqslant j, k \leqslant n$$

(defined as Radon–Nikodym derivative for $h(dY, dY)$-almost all t, and by 0 else).

Definition 2.5 *(Semimartingales of bounded dilatation).* Let Y be a continuous semimartingale taking values in a Riemannian manifold N. We say that Y is of generalized K-bounded dilatation (e.g. [16]) if $\mathbb{P}$-almost surely,

$$\lambda_1(t) \leqslant K^2\big(\lambda_2(t) + \cdots + \lambda_n(t)\big) \tag{2.3}$$

holds for some $K > 0$ and all $t \in \mathbb{R}_+$, where

$$\lambda_1(t) \geqslant \lambda_2(t) \geqslant \cdots \geqslant \lambda_n(t) \geqslant 0 \tag{2.4}$$

denotes the eigenvalues of Λ_t. The semimartingale Y is said to be K-quasi-conformal, if $\lambda_1(t) \leqslant K^2\lambda_n(t)$ instead of condition (2.3) holds for all $t \geqslant 0$.

The following lemma is an extension of Proposition 2.8 in [2], and gives the basic estimates for the geodesic transport along a semimartingale in terms of the quadratic variation. We use the notation

$$(\sup \mathrm{Sect})(y) = \text{supremum of the sectional curvatures of } N \text{ at } y,$$
$$(\inf \mathrm{Sect})(y) = \text{infimum of the sectional curvatures of } N \text{ at } y$$

for any $y \in N$.

Lemma 2.6. *Let $\Theta_{0,\bullet} : T_{Y_0}N \to T_{Y_\bullet}N$ be the geodesic transport along a continuous semimartingale Y in (N, h) and let*

$$\Theta_{s,t} := \Theta_{0,t} \circ \Theta_{0,s}^{-1} : T_{Y_s}N \to T_{Y_t}N.$$

Then for any $0 \leqslant s \leqslant t$ the following estimates hold:

$$\exp\left(-\frac{1}{2}\int_s^t L_r\, h(dY_r, dY_r)\right) \leqslant |\Theta_{s,t}| \leqslant \exp\left(-\frac{1}{2}\int_s^t \ell_r\, h(dY_r, dY_r)\right) \tag{2.5}$$

where

$$L_r := \begin{cases} (\sup \mathrm{Sect})(Y_r) & \text{if } (\sup \mathrm{Sect})(Y_r) \geqslant 0, \\ [\lambda_2(r) + \cdots + \lambda_n(r)](\sup \mathrm{Sect})(Y_r) & \text{if } (\sup \mathrm{Sect})(Y_r) \leqslant 0, \end{cases} \tag{2.6}$$

respectively,

$$\ell_r := \begin{cases} (\inf \mathrm{Sect})(Y_r) & \text{if } (\inf \mathrm{Sect})(Y_r) \leqslant 0, \\ [\lambda_2(r) + \cdots + \lambda_n(r)](\inf \mathrm{Sect})(Y_r) & \text{if } (\inf \mathrm{Sect})(Y_r) \geqslant 0. \end{cases} \tag{2.7}$$

Here the $\lambda_i(r)$ are as in Definition 2.5.

Proof. We start with the lower bound in Eq. (2.5). From the defining Eq. (2.1) we deduce

$$\begin{aligned} d\big|//_{0,\bullet}^{-1}\Theta_{0,\bullet}v\big|^2 &= -\big\langle //_{0,\bullet}^{-1}R^N\big(\Theta_{0,\bullet}v, dY\big)\,dY, //_{0,\bullet}^{-1}\Theta_{0,\bullet}v\big\rangle \\ &\geqslant -(\sup \mathrm{Sect})(Y_\bullet)\big(|\Theta_{0,\bullet}v|^2 h(dY,dY) - \big\langle\Theta_{0,\bullet}v, dY\big\rangle^2\big). \end{aligned}$$

Thus on $\{(\sup \mathrm{Sect})(Y_\bullet) \geqslant 0\}$ we have

$$d|\Theta_{0,\bullet}v|^2 \geqslant -(\sup \mathrm{Sect})(Y_\bullet)|\Theta_{0,\bullet}v|^2\, h(dY,dY).$$

On $\{(\sup \mathrm{Sect})(Y_\bullet) \leqslant 0\}$ finally we may use

$$\big\langle\Theta_{0,\bullet}v, dY\big\rangle^2 = \sum_{i,k=1}^{n}\big\langle //_{0,\bullet}^{-1}\Theta_{0,\bullet}v, e^i\big\rangle\big\langle //_{0,\bullet}^{-1}\Theta_{0,\bullet}v, e^k\big\rangle d\big[Z^i, Z^k\big] \leqslant \lambda_1(t)\big|\Theta_{0,\bullet}v\big|^2 h(dY,dY)$$

to get the estimate

$$d|\Theta_{0,\bullet}v|^2 \geqslant -(\sup \mathrm{Sect})(Y_\bullet)\big\{|\Theta_{0,\bullet}v|^2\big[\lambda_2(r)+\cdots+\lambda_n(r)\big]\,h(dY,dY)\big\}.$$

The upper bound in Eq. (2.5) is established in a completely analogous way. □

Remark 2.7. In the special case when Y is a semimartingale of generalized K-bounded dilatation, estimate (2.5) for the geodesic transport $\Theta_{0,\bullet}$ along Y in Lemma 2.6 holds with

$$L_r := \begin{cases} (\sup \mathrm{Sect})(Y_r) & \text{if } (\sup \mathrm{Sect})(Y_r) \geqslant 0, \\ (\sup \mathrm{Sect})(Y_r)\lambda_1(r)/K^2 & \text{if } (\sup \mathrm{Sect})(Y_r) \leqslant 0, \end{cases} \tag{2.8}$$

and

$$\ell_r := \begin{cases} (\inf \mathrm{Sect})(Y_r) & \text{if } (\inf \mathrm{Sect})(Y_r) \leqslant 0, \\ (\inf \mathrm{Sect})(Y_r)\lambda_1(r)/K^2 & \text{if } (\inf \mathrm{Sect})(Y_r) \geqslant 0, \end{cases} \tag{2.9}$$

respectively.

Definition 2.8 *(Maps of bounded dilatation).* Let M, N be Riemannian manifolds of dimensions m and n, respectively. Let $u: M \to N$ be a C^2 map and denote by

$$\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_m(x) \geqslant 0$$

the eigenvalues of $(du_x)^* \circ du_x : T_xM \to T_xM$. The map $u: M \to N$ is said to be of generalized K-bounded dilatation (e.g. [28,3]), if there is a number $K > 0$ such that

$$\lambda_1(x) \leqslant K^2\big[\lambda_2(x)+\cdots+\lambda_m(x)\big] \tag{2.10}$$

for any $x \in M$. The map $u: M \to N$ is said to be K-quasi-conformal, if condition (2.10) is replaced by $\lambda_1(x) \leqslant K^2\lambda_m(x)$. A map with

$$\lambda_1(x) \leqslant K^2\lambda_2(x)$$

is said to be of bounded variation.

Note that the spectrum of $(du_x)^* \circ du_x : T_xM \to T_xM$ coincides with the spectrum of $du_x \circ (du_x)^* : T_{u(x)}N \to T_{u(x)}N$ for each $x \in M$.

Notation 2.9. In the sequel we denote by $\|du\|(x)$ the Hilbert–Schmidt norm of $(du_x)^* \circ du_x$, i.e. $\|du\|(x) = (\sum_i |(du)_x e_i|^2)^{1/2}$, where (e_i) runs through an orthonormal basis of T_xM, and by $|du|(x)$ the usual Euclidean norm, i.e. $|du|(x) = \sup_{|v|\leqslant 1} |(du)_x v|$.

Remark 2.10. For a harmonic map $u : M \to N$ between Riemannian manifolds are equivalent:

(i) u is of generalized K-bounded dilatation (resp. K-quasi-conformal);
(ii) u maps Brownian motions on M to martingales of generalized K-bounded dilatation on N (resp. to K-quasi-conformal martingales on N).

Proof. (See also [16].) Let X be a Brownian motion on M starting from x, and $B := \mathscr{A}(X)$ (by definition a Brownian motion on T_xM). Analogously let $Y = u \circ X$ and $Z := \mathscr{A}(Y)$ (by definition a local martingale on $T_{u(x)}N$). Then

$$dZ_t = \big(//_{0,t}^N\big)^{-1} (du)_{X_t} //_{0,t}^M \, dB_t$$

and hence

$$\begin{aligned} d\big[Z^j, Z^k\big]_t &= \sum_{i=1}^m \big\langle \big(//_{0,t}^N\big)^{-1} (du)_{X_t} //_{0,t}^M e_i, e_j\big\rangle \big\langle \big(//_{0,t}^N\big)^{-1} (du)_{X_t} //_{0,t}^M e_i, e_k\big\rangle \, dt \\ &= h\big((du)_{X_t} (du)_{X_t}^* //_{0,t}^N e_j, //_{0,t}^N e_k\big) \, dt \end{aligned}$$

for any orthonormal basis $(e_1, \dots, e_n)$ of $T_{u(x)}N$. Thus

$$\Lambda_t^{jk} = \frac{h((du)_{X_t} (du)_{X_t}^* //_{0,t}^N e_j, //_{0,t}^N e_k)}{\sum_{i=1}^n h((du)_{X_t} (du)_{X_t}^* //_{0,t}^N e_i, //_{0,t}^N e_i)}. \tag{2.11}$$

Note that we can choose the orthonormal basis (e_i) diagonalizing the process Λ in an adapted way. (We use the fact that one can find a diagonalizing basis for a symmetric matrix which depends measurably on the matrix.) The claims are then easily read off from Eq. (2.11). □

Remark 2.11. Let $Y = u(X_{\bullet}(x))$ where $u : M \to N$ is C^2 (not necessarily harmonic) and $X(x)$ is a Brownian motion on M. Then $h(dY, dY) = \|du\|^2(X_s(x))\, ds$.

Using Remark 2.11 we specialize Lemma 2.6 to the case when Y is the image of a Brownian motion X on M under a mapping $u : M \to N$.

Lemma 2.12. *Let (M, g), (N, h) be Riemannian manifolds, $u : M \to N$ be a C^2 map, and X a Brownian motion on M. Let $\Theta_{s,t} : T_{Y_s}N \to T_{Y_t}N$ be the geodesic transport along the semimartingale $Y = u \circ X$ in N. Then for any $0 \leqslant s \leqslant t$ the following estimates hold*:

$$\exp\left(-\frac{1}{2}\int_s^t L_r \, dr\right) \leqslant |\Theta_{s,t}| \leqslant \exp\left(-\frac{1}{2}\int_s^t \ell_r \, dr\right) \tag{2.12}$$

where

$$L_r = \begin{cases} \|du\|^2(X_r)(\sup \mathrm{Sect}^N)(Y_r) & \textit{if } (\sup \mathrm{Sect}^N)(Y_r) \geqslant 0, \\ [\lambda_2(X_r) + \cdots + \lambda_m(X_r)](\sup \mathrm{Sect}^N)(Y_r) & \textit{if } (\sup \mathrm{Sect}^N)(Y_r) \leqslant 0 \end{cases}$$

and

$$\ell_r = \begin{cases} \|du\|^2(X_r)(\inf \mathrm{Sect}^N)(Y_r) & \textit{if } (\inf \mathrm{Sect}^N)(Y_r) \leqslant 0, \\ [\lambda_2(X_r) + \cdots + \lambda_m(X_r)](\inf \mathrm{Sect}^N)(Y_r) & \textit{if } (\inf \mathrm{Sect}^N)(Y_r) \geqslant 0 \end{cases}$$

and with $\lambda_i(x)$ as in Definition 2.8.

Remark 2.13. In the special case when u is a map of generalized K-bounded dilatation, estimate (2.12) for the geodesic transport $\Theta_{0,\bullet}$ holds with

$$L_r = \begin{cases} \|du\|^2(X_r)(\sup \mathrm{Sect}^N)(Y_r) & \text{if } (\sup \mathrm{Sect}^N)(Y_r) \geqslant 0, \\ (\sup \mathrm{Sect}^N)(Y_r)\lambda_1(X_r)/K^2 & \text{if } (\sup \mathrm{Sect}^N)(Y_r) \leqslant 0 \end{cases}$$

and

$$\ell_r = \begin{cases} \|du\|^2(X_r)(\inf \mathrm{Sect}^N)(Y_r) & \text{if } (\inf \mathrm{Sect}^N)(Y_r) \leqslant 0, \\ (\inf \mathrm{Sect}^N)(Y_r)\lambda_1(X_r)/K^2 & \text{if } (\inf \mathrm{Sect}^N)(Y_r) \geqslant 0. \end{cases}$$

Note that $\lambda_1(x) = |du|^2(x)$.

Remark 2.14. We always have the obvious estimate

$$\exp\left(-\frac{1}{2}\int_s^t \kappa_+(Y_r)\|du\|^2(X_r)\,dr\right) \leqslant |\Theta_{s,t}| \leqslant \exp\left(-\frac{1}{2}\int_s^t \kappa_-(Y_r)\|du\|^2(X_r),dr\right)$$

where $\kappa_+(y) := (\sup \mathrm{Sect}^N)(y) \vee 0$ and $\kappa_-(y) := \inf \mathrm{Sect}^N(y) \wedge 0$.

3. Derivative formulas for harmonic maps

To formulate derivative formulas for harmonic maps, we consider the following setup. Let $u : M \to N$ be a C^2 map between two Riemannian manifolds. Further let $X_\bullet(x)$ be a Brownian motion on M, starting from x, and let $Y := u(X(x))$ be its image process under u on N, starting from $u(x)$. Let $\Theta^M_{0,\bullet}$ denote the geodesic transport on M along $X_\bullet(x)$, and $\Theta^N_{0,\bullet}$ the corresponding transport on N along Y. Finally let $B := \mathscr{A}(X(x)) = \int_0 //_{0,s}^{-1}\delta X_s(x)$ be the anti-development of $X(x)$, by definition a Brownian motion on T_xM, and let $\mathscr{A}_{\mathrm{def}}(Y) = \int_0 (\Theta^N_{0,s})^{-1}\delta Y_s$ be the deformed anti-development of Y.

Theorem 3.1 *(Basic derivative formula for harmonic maps). (Cf. [3], Section 5.) Let (M,g) and (N,h) be Riemannian manifolds and $u \in C^2(M,N)$. Let $x \in M$ and let D be a relatively compact open neighbourhood of x in M. Suppose that u is harmonic on D. Then $Y = u(X_\bullet(x))$ is a ∇-martingale on N up to the first exit time $\tau(x)$ of $X_\bullet(x)$ from D. For any $v \in T_xM$ the following formula holds*:

$$(du)_x v = -\mathbb{E}\left[\mathscr{A}_{\mathrm{def}}(Y)_\tau \int_0^\tau \langle \Theta^M_{0,s}\dot\ell(s), //^M_{0,s}\,dB_s\rangle\right] \tag{3.1}$$

where τ is any bounded stopping time such that $0 < \tau \leqslant \tau(x)$ *and ℓ is any adapted process with finite energy taking values in T_xM such that*

(i) $\ell(0) = v$ *and* $\ell(s) = 0$ *for* $s \geqslant \tau$,
(ii) $(\int_0^\tau |\dot{\ell}(s)|^2\, ds)^{1/2} \in L^{1+\varepsilon}$ *for some* $\varepsilon > 0$.

Sketch of the proof. We briefly recall the idea behind Theorem 3.1. First one shows that

$$\big\{\big(\Theta_{0,s}^N\big)^{-1}(du)_{X_s(x)}\Theta_{0,s}^M\colon\ s < \tau(x)\big\} \tag{3.2}$$

is a local martingale in $T_xM \otimes T_{u(x)}N$, see [3], Section 5. Hence also

$$n_s := \big(\Theta_{0,s}^N\big)^{-1}(du)_{X_s(x)}\Theta_{0,s}^M\ell(s) - \int_0^s \big(\Theta_{0,r}^N\big)^{-1}(du)_{X_r(x)}\Theta_{0,r}^M\dot{\ell}(r)\, dr \tag{3.3}$$

is a local martingale up to the random time $\tau(x)$. Thus taking into account that

$$\mathscr{A}_{\mathrm{def}}(Y) = \int_0^{\cdot} \big(\Theta_{0,s}^N\big)^{-1}(du)_{X_s(x)}//_{0,s}^M\, dB_s$$

we see that $\{m_s\colon\ s < \tau(x)\}$ where

$$m_s := \big(\Theta_{0,s}^N\big)^{-1}(du)_{X_s(x)}\Theta_{0,s}^M\ell(s) - \mathscr{A}_{\mathrm{def}}(Y)_s \int_0^s \big\langle\Theta_{0,r}^M\dot{\ell}(r), //_{0,r}^M\, dB_r\big\rangle \tag{3.4}$$

is a local martingale as well. The proof is completed by verifying that the local martingale $\{m_{s\wedge\tau}\colon s \geqslant 0\}$ is a uniformly integrable martingale under the given assumptions. Formula (3.1) then follows by taking expectations: $\mathbb{E}[m_0] = \mathbb{E}[m_\tau]$. □

Note that formula (3.1) holds true for any stopping time τ and any finite energy process ℓ which meet the given constraints. It will be essential for our applications to deal with specific choices for ℓ. For further reference, we recall a general scheme for constructing appropriate finite energy processes, see [32].

Remark 3.2 *(Construction of finite energy processes).* Let $D \subset M$ be a relatively compact open domain with nonempty smooth boundary, and choose $f \in C^2(\overline{D})$ with $f > 0$ in D and $f|\partial D = 0$. Let

$$\sigma(s) = \inf\left\{r \geqslant 0\colon \int_0^r f^{-2}\big(X_u(x)\big)\, du \geqslant s\right\} \tag{3.5}$$

be the time change as defined at the beginning of Section 4 in [32]. (Note that $\sigma(s)$ is denoted $\tau(s)$ in [32].) We fix $t > 0$ and let

$$h_0(s) = \int_0^s f^{-2}\big(X_r(x)\big)1_{\{r<\sigma(t)\}}\, dr. \tag{3.6}$$

Finally let $h(s) = (h_1 \circ h_0)(s)$ where $h_1 \in C^1[0,t]$ such that $h_1(0) = 1$, $h_1(t) = 0$, $\dot{h}_1 \leqslant 0$, and fix $v \in T_xM$. Then $\ell(s) := h(s)v$ satisfies

(i) $\ell(0) = v$ and $\ell(s) = 0$ for $s \geqslant \sigma(t)$, and
(ii) $(\int_0^{\sigma(t)} |\dot{\ell}(s)|^2\, ds)^{1/2} \in L^p$ for any $1 \leqslant p < \infty$.

Note that $\sigma(t) < \tau(x)$ for any $t > 0$, and finally if we choose $f \leqslant 1$ then $\sigma(t) \leqslant t$ for each $t > 0$.

Proof. See [32] for details. The proof of (ii) given in [32] for $p = 2$ is easily adapted to general p. More precisely, we get for $|v| \leqslant 1$,

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau(x)} \left|\dot{\ell}(s)\right|^2 ds\right]^{p/2} &= \mathbb{E}\left[\int_0^{\sigma(t)} \left|\dot{h}(s)\right|^2 ds\right]^{p/2} \\ &= \mathbb{E}\left[\int_0^{t} \left|\dot{h}_1(s)\right|^2 f^{-2}\big(X_{\sigma(s)}(x)\big)\, ds\right]^{p/2} \\ &\leqslant \int_0^{t} \left|\dot{h}_1(s)\right|^{p/2+1} \mathbb{E}\big[f^{-p}\big(X_{\sigma(s)}(x)\big)\big]\, ds\end{aligned}$$

where the last estimate follows from Jensen's inequality along with the observation that $|\dot{h}_1(s)|\, ds$ defines a probability measure on $[0, t]$.

Applying Ito's formula to $f^{-p}(X_{\sigma(s)})$, we obtain

$$d\big(f^{-p}\big(X_{\sigma(s)}(x)\big)\big) \leqslant dL + cf^{-p}\big(X_{\sigma(s)}(x)\big)\, ds$$

for some positive constant $c = c(p)$, where L is a local martingale. Thus, by the argument right before Theorem 4.1 in [32], it follows that

$$\mathbb{E}\big[f^{-p}\big(X_{\sigma(s)}(x)\big)\big] \leqslant e^{cs} f^{-p}(x) \tag{3.7}$$

which achieves the proof. □

The following theorem gives a slightly different variant of the derivative formula presented in Theorem 3.1.

Theorem 3.3 *(Derivative formula for harmonic maps; alternative version). Let (M, g) and (N, h) be Riemannian manifolds. Suppose that M is complete with Ricci curvature bounded from below and that the sectional curvatures of N are bounded above by a constant $\kappa \geqslant 0$. Let $u : M \to N$ be a harmonic map and let $t > 0$, $p > 1$.*

Assume that either,

$$\mathbb{E}\left[\exp\left(\frac{\kappa q}{2}\int_0^t \|du\|^2\big(X_r(x)\big)\, dr\right)\right] < \infty \quad \text{if } \kappa > 0,\ \text{respectively,} \tag{3.8}$$

$$\mathbb{E}\left[\left(\int_0^t \|du\|^2\big(X_r(x)\big)\, dr\right)^{q/2}\right] < \infty \quad \text{if } \kappa = 0, \tag{3.9}$$

where q is the dual index to p, or that for some $b > 0$,

$$\frac{(\sup \mathrm{Sect}^N)(u(\cdot))}{K^2(\cdot)} \leqslant -b \quad \text{on } M \tag{3.10}$$

where $K^2(x) := \lambda_1(x)/[\lambda_2(x) + \cdots + \lambda_m(x)]$ *if* $\lambda_2(x) + \cdots + \lambda_m(x) > 0$ *and* $K^2(x) := \infty$ *elsewhere, and* $\lambda_i(x)$ *are the eigenvalues of* $(du_x)^* \circ du_x$ *as in Definition* 2.8.

Then

$$(du)_x v = -\mathbb{E}\left[\mathscr{A}_{\mathrm{def}}(Y)_t \int_0^t \langle \Theta_{0,s}^M \dot{\ell}(s)v, //_{0,s}^M\, dB_s\rangle\right] \tag{3.11}$$

where ℓ *may be any bounded non-negative real-valued, adapted, finite energy process such that* $\ell(0) = 1$, $\ell(t) = 0$, *and* $(\int_0^t |\dot{\ell}(s)|^2\, ds)^{1/2} \in L^p$.

Observe that Theorem 3.3 applies in particular for deterministic $\ell \in C^1[0, t]$ with $\ell(0) = 1$ and $\ell(t) = 0$.

Proof of Theorem 3.3. We reduce Theorem 3.3 to Theorem 3.1. Since M is complete with Ricci curvature bounded below, according to Theorem 4 in [34] there is a positive smooth function ψ such that $|\nabla\psi| \leqslant 1$, $\Delta\psi \leqslant 1$ and $\{\psi \leqslant n\}$ is compact for any $n > 0$. Further let $\gamma \in C^\infty(\mathbb{R})$ such that $0 \leqslant \gamma \leqslant 1$, $\gamma|(-\infty, 0] \equiv 1$ and $\gamma|[1, \infty) = 0$. Let $f_n = n\gamma(\psi - n)$ on $D = B_n := \{\psi < n + 1\}$. (Note that $f_n = n$ on B_{n-1} and $f_n = 0$ on $\complement B_n$.) Let ℓ^* be an upper bound for ℓ (i.e. $\ell_s \leqslant \ell^*$). Define

$$\sigma_n(s) = \inf\left\{\rho \geqslant 0: \int_0^\rho f_n^{-2}\big(X_r(x)\big)\, dr \geqslant s\right\},$$

$$h_n(s) = 2\ell^* - \frac{2\ell^*}{t} \int_0^s f_n^{-2}\big(X_r(x)\big) 1_{\{r<\sigma_n(t)\}}\, dr$$

and let $\ell_n(s) = \ell(s) \wedge h_n(s)$, $n \geqslant 2$. Since $\sigma_n(t) \leqslant \tau := \tau_n(x) \wedge t$, where $\tau_n(x)$ is the first exit time of $X(x)$ from B_n, one has $\ell_n(s) = 0$ for $s \geqslant \tau$. Moreover, $\ell_n(0) = \ell(0) = 1$. Then by Theorem 3.1,

$$(du)_x v = -\mathbb{E}\left[\mathscr{A}_{\mathrm{def}}(Y)_\tau \int_0^\tau \langle \Theta_{0,s}^M \dot{\ell}_n(s)v, //_{0,s}^M\, dB_s\rangle\right]. \tag{3.12}$$

It remains to show that formula (3.12) leads to formula (3.11) as $n \to \infty$. First observe that under any of the conditions (3.8), (3.9) or (3.10) (see Lemma 4.5 below)

$$\mathbb{E}\Big[\sup_{r\leqslant t}\big|\mathscr{A}_{\mathrm{def}}(Y)_r\big|^q\Big] < \infty. \tag{3.13}$$

Similarly, for $|v| \leqslant 1$, again invoking Burkholder–Davis–Gundy,

$$\mathbb{E}\left[\sup_{r\leqslant t}\left|\int_0^r \langle \Theta_{0,s}^M \dot{\ell}_n(s)v, //_{0,s}^M\, dB_s\rangle\right|^p\right] \leqslant C\mathbb{E}\left[\left(\int_0^t \big|\dot{\ell}_n(s)\big|^2\, ds\right)^{p/2}\right] \tag{3.14}$$

with a constant C depending on p, t and the lower bound on the Ricci curvature of M. The wanted formula (3.11) follows now easily from Eq. (3.12) and the dominated convergence theorem, as $n \to \infty$. Recall that as above, see Eq. (3.7),

$$\mathbb{E}\big[f_n^{-p}\big(X_{\sigma(s)}(x)\big)\big] \leqslant e^{c(p)s} f_n^{-p}(x)$$

with a constant $c(p)$ depending only on p. $\Box$

Note that the right-hand sides of Eqs. (3.1) and (3.11) do not involve derivatives of u, since $\mathscr{A}_{\mathrm{def}}(Y)$ is well defined for any continuous semimartingale Y.

Remark 3.4. (i) Formula (3.1) can be rewritten as

$$(du)_x v = -\mathbb{E}\left[\int_0^{\tau(x)} \big(\Theta_{0,s}^N\big)^{-1} (du)_{X_s(x)} \Theta_{0,s}^M \dot{\ell}(s)\, ds\right] \tag{3.15}$$

under the assumptions of Theorem 3.1. Similarly, in the situation of Theorem 3.3, formula (3.11) can be written as

$$(du)_x v = -\mathbb{E}\left[\int_0^{t} \big(\Theta_{0,s}^N\big)^{-1} (du)_{X_s(x)} \Theta_{0,s}^M \dot{\ell}(s) v\, ds\right]. \tag{3.16}$$

(ii) Let $h \in C^1[0,t]$ with $h(0)=0$ and $h(t)=1$. If M is complete and $u: M \to N$ is harmonic, then a sufficient condition for the validity of the formula

$$(du)_x v = \mathbb{E}\left[\int_0^{t} \big(\Theta_{0,s}^N\big)^{-1} (du)_{X_s(x)} \Theta_{0,s}^M \dot{h}(s) v\, ds\right], \quad t > 0, \tag{3.17}$$

is that the local martingale

$$\big\{\big(\Theta_{0,s}^N\big)^{-1} (du)_{X_s(x)} \Theta_{0,s}^M \colon s \leqslant t\big\} \tag{3.18}$$

in $T_x M \otimes T_{u(x)} N$ is a martingale.

4. A priori estimates for harmonic maps

Derivative formula (3.1), resp. formula (3.15), can be exploited for estimates on du, for instance, by estimating $\mathscr{A}_{\mathrm{def}}(Y)_\tau$ and $\int_0^\tau \langle \Theta_{0,s}^M \dot{\ell}(s), //_{0,s}^M\, dB_s\rangle$ in formula (3.1) in appropriate norms. Similar arguments apply to the formulas (3.11) and (3.16).

Theorem 4.1 *(Derivative estimates for harmonic maps). Keeping notations and assumptions of Theorem* 3.1, *the following estimates hold*:

$$\big|(du)_x v\big| \leqslant \big\|\mathscr{A}_{\mathrm{def}}(Y)_\tau\big\|_q \left\| \int_0^\tau \big\langle \Theta_{0,s}^M \dot{\ell}(s), //_{0,s}^M\, dB_s\big\rangle \right\|_p, \tag{4.1}$$

respectively,

$$\left|(du)_x v\right| \leqslant \left(\mathbb{E}\left[\int_0^\tau \left|\left(\Theta_{0,s}^N\right)^{-1} du\right|^q \left(X_s(x)\right) ds\right]\right)^{1/q} \left(\mathbb{E}\left[\int_0^\tau \left|\Theta_{0,s}^M \dot{\ell}(s)\right|^p ds\right]\right)^{1/p} \tag{4.2}$$

where $1 \leqslant p < \infty$ *and* $1/p + 1/q = 1$.

Proof. Estimate (4.1) follows from Eq. (3.1) by Hölder's inequality. Estimate (4.2) follows from Eq. (3.15) by applying Hölder's inequality with respect to $\mathbb{P} \otimes ds$. $\square$

Although the inequalities (4.1) and (4.2) are rather similar in nature, they may lead to slightly different estimates in some situations. Note that the second term in the r.h.s. of (4.1), resp. of (4.2), involves only the local geometry on M at the point x and can be estimated by the method developed in [32] for the case of harmonic functions. This part of the problem is identical for harmonic maps and harmonic functions. The geometry of N (and the nonlinearity of u) enters via the inverse deformed parallel transport Θ^N on N along $u(X)$ and concerns only the first term in the r.h.s. of (4.1), resp. (4.2).

It is well known that one cannot expect universal bounds on the derivatives of all harmonic maps belonging to a given homotopy class. From a probabilistic point of view the reason for this difficulty is the appearance of the deformed parallel transport along the image martingale $u(X)$ in our derivative formulas. Reasonable estimates can only be given for certain classes of harmonic maps under various curvature conditions.

We want to illustrate here how (4.1) and (4.2) can be used for explicit estimates under specific assumptions. We first introduce some notations.

Notation 4.2. Let $\Theta_{0,\bullet}^M$ be the geodesic transport on M along $X_\bullet(x)$ and define $Q_s^M := (//_{0,s}^M)^{-1}\Theta_{0,s}^M$. Then Q_s^M takes values in $\mathrm{End}(T_x M)$ and is determined by the pathwise linear differential equation,

$$\frac{d}{ds} Q_s^M = -\frac{1}{2}\mathrm{Ric}_{//_{0,s}}^M\left(Q_s^M\right), \qquad Q_0^M = \mathrm{id}_{T_x M} \tag{4.3}$$

where $\mathrm{Ric}_{//_{0,s}}^M := (//_{0,s}^M)^{-1} \circ \mathrm{Ric}_{X_s(x)}^M \circ //_{0,s}^M : T_x M \to T_x M$.

Note that, by definition of Q_s^M,

$$\int_0 \left\langle \Theta_{0,s}^M \dot{\ell}(s), //_{0,s}^M dB_s \right\rangle = \int_0 \left\langle Q_s^M \dot{\ell}(s), dB_s \right\rangle. \tag{4.4}$$

To derive bounds on $|du|$ we need to estimate the norms involved in the r.h.s. of (4.1) and (4.2) respectively. We focus first on estimates of the local term on M containing the deformed transport $\Theta_{0,s}^M \dot{\ell}_s$.

Lemma 4.3. *We keep the notations and assumptions of Theorem* 3.1. *Assume that* ∂D *is smooth and let* $p \in [2, \infty)$. *For any* $f \in C^2(\overline{D})$ *with* $f > 0$ *in* D *and* $f|\partial D = 0$, *define*

$$\begin{aligned} c_p(f) &:= \sup_D \left\{ \frac{p}{2}\alpha^+ f^2 + \frac{1}{2} f^{2+p} \Delta f^{-p} \right\} \\ &= \frac{p}{2} \sup_D \left\{ \alpha^+ f^2 - f\Delta f + (p+1)|\nabla f|^2 \right\} \end{aligned} \tag{4.5}$$

where $-\alpha$ is the lower bound of Ric^M *on* D *and* $\alpha^+ = \alpha \vee 0$. *For a proper choice of* $\ell(s)$ *we have*

$$\left\| \sup_{0\leqslant r\leqslant\tau(x)} \left| \int_0^r \langle \Theta_{0,s}^M \dot\ell(s), //_{0,s}^M\, dB_s\rangle \right| \right\|_p \leqslant \frac{c(p)|v|}{f(x)} \left(\frac{2c_p(f)}{p}\right)^{1/2} \tag{4.6}$$

where $c(p) > 0$ *is a constant depending only on* p *with* $c(2) = 1$. *In particular, if* $\mathrm{Ric}^M \geqslant 0$ *on* D *then there is* $c = c(m, p) > 0$ *such that for a proper choice of* f *the estimate in* (4.6) *is dominated by* $c/\operatorname{dist}_M(x, \partial D)$*, where* $m = \dim M$ *denotes the dimension of* M.

Proof. By Eq. (4.4) and Burkholder–Davis–Gundy inequality we have

$$\begin{aligned} \left\| \sup_{0\leqslant r\leqslant\tau(x)} \left| \int_0^r \langle \Theta_{0,s}^M \dot\ell(s), //_{0,s}^M\, dB_s\rangle \right| \right\|_p &= \left\| \sup_{0\leqslant r\leqslant\tau(x)} \left| \int_0^r \langle Q_s^M \dot\ell(s), dB_s\rangle \right| \right\|_p \\ &\leqslant c(p) \left(\mathbb{E}\left[\int_0^{\tau(x)} \left|Q_s^M \dot\ell(s)\right|^2 ds \right]^{p/2} \right)^{1/p}. \end{aligned} \tag{4.7}$$

Assume that $|v| \leqslant 1$. To obtain the desired estimate, we go back to the notation and argument in Remark 3.2. For $f \in C^2(\overline{D})$ with $f > 0$ in D and $f|\partial D = 0$, we define $\ell(s) = (h_1 \circ h_0)(s)v$ as in Remark 3.2, where for fixed $t > 0$,

$$h_0(s) = \int_0^s f^{-2}\big(X_r(x)\big) 1_{\{r<\sigma(t)\}}\, dr$$

and $h_1 \in C^1[0, t]$ such that $h_1(0) = 1$, $h_1(t) = 0$, and $\dot h_1 \leqslant 0$. Finally let

$$\sigma(s) = \inf\left\{ r \geqslant 0\colon \int_0^r f^{-2}\big(X_u(x)\big)\, du \geqslant s \right\} \tag{4.8}$$

be the corresponding time change. Then

$$\begin{aligned} \mathbb{E}\left[\int_0^{\tau(x)} \left|Q_s^M \dot\ell(s)\right|^2 ds \right]^{p/2} &\leqslant \mathbb{E}\left[\int_0^{\sigma(t)} \left|\dot\ell(s)\right|^2 \exp(\alpha s)\, ds \right]^{p/2} \\ &= \mathbb{E}\left[\int_0^t \left|\dot h_1(s)\right|^2 f^{-2}(X_{\sigma(s)}) \exp\big(\alpha\sigma(s)\big)\, ds \right]^{p/2} \\ &\leqslant \int_0^t \left|\dot h_1(s)\right|^{p/2+1} \mathbb{E}\left[f^{-p}(X_{\sigma(s)}) \exp\left(p\alpha \frac{\sigma(s)}{2}\right) \right] ds. \end{aligned}$$

Applying Ito's formula to $f^{-p}(X_{\sigma(s)}) \exp(p\alpha\sigma(s)/2)$, we obtain

$$d\left(f^{-p}(X_{\sigma(s)}) \exp\left(p\alpha \frac{\sigma(s)}{2}\right) \right) \leqslant dL_s + \exp\left(p\alpha\frac{\sigma(s)}{2}\right) f^{-p}(X_{\sigma(s)}) c_p(f)\, ds$$

where L is a local martingale. Again by the argument right before Theorem 4.1 in [32], it then follows that

$$\mathbb{E}\big[f^{-p}(X_{\sigma(s)})\exp\big(p\alpha\sigma(s)/2\big)\big]\leqslant \exp\big(c_p(f)s\big)f^{-p}(x).$$

Taking

$$h_1(s)=1-\frac{2c_p(f)}{p(1-\exp(-2c_p(f)t/p))}\int_0^s \exp\big(-2c_p(f)r/p\big)\,dr,$$

we arrive at the estimate

$$\mathbb{E}\Bigg[\int_0^{\tau(x)}\big|Q_s^M\dot{\ell}(s)\big|^2\,ds\Bigg]^{p/2}\leqslant f^{-p}(x)\bigg(\frac{2c_p(f)}{p}\bigg)^{p/2}\big(1-\exp\big(-2c_p(f)t/p\big)\big)^{-p/2}. \tag{4.9}$$

(Note that $\ell(s)=$ constant for $s\geqslant\sigma(t)$ by construction.) Combining this with (4.7) and letting t tend to ∞, we obtain the desired estimate. Moreover, as shown in the proof of Corollary 5.1 in [32], if Ric^M on D is non-negative, we may conclude that the last term of (4.6) is dominated by $c/\operatorname{dist}_M(x,\partial D)$ for some constant c depending only on m and p. $\square$

Example 4.4. Let $D=B_M(x_0,R)$ be a geodesic ball in M of radius $R>0$ about some point x_0 and let $p=2$ for simplicity. Then, by a proper choice for f, estimate (4.9) can be worked out in more explicit forms, e.g.

$$\mathbb{E}\Bigg[\int_0^{\tau(x)}\big|Q_s^M\dot{\ell}(s)\big|^2\,ds\Bigg]\leqslant\frac{C(R)}{2\sin(\pi\delta_x/(2R))} \tag{4.10}$$

where $\delta_x:=\operatorname{dist}_M(x,\partial D)$ and

$$C(r):=\sqrt{\pi^2(m+3)r^{-2}+2\pi\sqrt{\alpha^+(m-1)}r^{-1}+4\alpha^+}; \tag{4.11}$$

here $-\alpha$ is a lower bound on the Ricci curvature, i.e. $\mathrm{Ric}\geqslant-\alpha$ on D, and $\alpha^+=\alpha\vee 0$.

Proof. Estimate (4.10) on a geodesic ball follows from the arguments used in the proof of Corollaries 5.2 and 5.3 in [32]. $\square$

Next we are going to estimate the L^q norm of $\mathscr{A}_{\mathrm{def}}(Y)$. Lemma 4.5 below gives general estimates relying only on upper bounds of the sectional curvature Sect^N of N. The subsequent Lemma 4.6 then also takes the dilatation of the map u into account (assuming that $\mathrm{Sect}^N\leqslant 0$).

Lemma 4.5. *Let $Y=u(X(x))$ and $1<q<\infty$. Then, for any stopping time τ, the following estimate holds*:

$$\begin{aligned}&\Big\|\sup_{0\leqslant r\leqslant\tau}\big|\mathscr{A}_{\mathrm{def}}(Y)_r\big|\Big\|_q\\&\quad\leqslant c(q)\Bigg(\mathbb{E}\Bigg[\int_0^\tau\|du\|^2\big(X_s(x)\big)\exp\Bigg\{\int_0^s\kappa_+(Y_r)\|du\|^2\big(X_r(x)\big)\,dr\Bigg\}\,ds\Bigg]^{q/2}\Bigg)^{1/q}\end{aligned} \tag{4.12}$$

where $\kappa_+(y)=(\sup\mathrm{Sect}^N)(y)\vee 0$ and $c(q)>0$ is a constant with $c(2)=1$.

In particular:

1) *If* $(\sup \mathrm{Sect}^N)(Y_r) \leqslant \kappa$ *for some constant* $\kappa > 0$, *then*

$$\Big\| \sup_{0\leqslant r\leqslant\tau} \big|\mathscr{A}_{\mathrm{def}}(Y)_r\big| \Big\|_q \leqslant c(q)\left(\mathbb{E}\left[\frac{1}{\kappa}\left(\exp\left\{\kappa\int_0^\tau \|du\|^2\big(X_r(x)\big)\,dr\right\}-1\right)\right]^{q/2}\right)^{1/q} \tag{4.13}$$

where the r.h.s. is finite if

$$\mathbb{E}\left[\exp\left(\frac{\kappa q}{2}\int_0^\tau \|du\|^2\big(X_r(x)\big)\,dr\right)\right] < \infty. \tag{4.14}$$

2) *If* $(\sup \mathrm{Sect}^N)(Y_r) \leqslant 0$ *and* N *is simply connected, then*

$$\Big\| \sup_{0\leqslant r\leqslant\tau} \big|\mathscr{A}_{\mathrm{def}}(Y)_r\big| \Big\|_2 \leqslant \mathbb{E}\big[\mathrm{dist}_N\big(u\big(X_\tau(x)\big), u(x)\big)\big]^2. \tag{4.15}$$

Proof. First note that for $Y = u(X_\bullet(x))$

$$\mathscr{A}_{\mathrm{def}}(Y) = \int_0^{\cdot} \big(\Theta_{0,s}^N\big)^{-1}(du)_{X_s(x)}\, //_{0,s}^M\, dB_s.$$

Let $\rho(s) := \|du\|^2(X_s(x))$. Invoking Burkholder–Davis–Gundy, we get then for any stopping time τ, by means of Lemma 2.12, resp. Remark 2.14,

$$\begin{aligned}\Big\| \sup_{0\leqslant r\leqslant\tau} \big|\mathscr{A}_{\mathrm{def}}(Y)_r\big| \Big\|_q &\leqslant c(q)\left(\mathbb{E}\left[\int_0^\tau \big|\big(\Theta_{0,s}^N\big)^{-1}\big|^2\rho(s)\,ds\right]^{q/2}\right)^{1/q}\\ &\leqslant c(q)\left(\mathbb{E}\left[\int_0^\tau \rho(s)\exp\left\{\int_0^s \kappa_+(Y_r)\rho(r)\,dr\right\}ds\right]^{q/2}\right)^{1/q}\end{aligned} \tag{4.16}$$

which gives the first part of the claim. If $\kappa_+(Y_r) \leqslant \kappa$ for some constant $\kappa > 0$, then estimate (4.13) follows from (4.12) by noting that

$$\rho(s)\exp\left\{\kappa\int_0^s \rho(r)\,dr\right\} = \frac{1}{\kappa}\frac{d}{ds}\exp\left\{\kappa\int_0^s \rho(r)\,dr\right\}.$$

Finally, if $\kappa \leqslant 0$, we may conclude from Lemma 2.6 that $|(\Theta_{0,s}^N)^{-1}| \leqslant 1$. Thus, letting $\phi(z) = \mathrm{dist}_N(z, u(x))$, we get

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leqslant r\leqslant\tau} \big|\mathscr{A}_{\mathrm{def}}(Y)_r\big|^2\Big] &\leqslant \mathbb{E}\left[\int_0^\tau \|du\|^2\big(X_s(x)\big)\,ds\right]\\ &\leqslant \mathbb{E}\left[\int_0^\tau \frac{1}{2}\Delta\big(\phi^2\circ u\big)\big(X_s(x)\big)\,ds\right] = \mathbb{E}\big[\big(\phi^2\circ u\big)\big(X_\tau(x)\big)\big]\end{aligned}$$

where we used for the second inequality that $\Delta(\phi^2 \circ u) \geqslant 2\|du\|^2$ under the assumption $\mathrm{Sect}^N \leqslant 0$. □

Lemma 4.6. *As in Lemma* 4.5 *let* $Y = u(X(x))$ *and assume that* $\sup \mathrm{Sect}^N \leqslant 0$. *Then, for any stopping time* τ,

$$\Big\| \sup_{0\leqslant r\leqslant\tau} \big|\mathscr{A}_{\mathrm{def}}(Y)_r\big| \Big\|_q \leqslant c(q)\left(\mathbb{E}\left[\int_0^\tau \|du\|^2\big(X_s(x)\big)\exp\left\{\int_0^s \frac{(\sup \mathrm{Sect}^N)(Y_r)|du|^2(X_r(x))}{K^2(X_r(x))}\,dr\right\}ds\right]^{q/2}\right)^{1/q}$$

where

$$K^2(x) = \begin{cases} \frac{\lambda_1(x)}{\lambda_2(x)+\cdots+\lambda_m(x)}, & \text{if } \lambda_2(x)+\cdots+\lambda_m(x) > 0,\\ \infty, & \text{if } \lambda_2(x)+\cdots+\lambda_m(x) = 0 \end{cases} \tag{4.17}$$

with $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_m(x) \geqslant 0$ *the eigenvalues of* $(du_x)^* \circ du_x : T_xM \to T_xM$.

In particular, if for some constant $b > 0$,

$$\frac{(\sup \mathrm{Sect}^N)(u(\cdot))}{K^2(\cdot)} \leqslant -b \quad \text{on } M, \tag{4.18}$$

then

$$\Big\| \sup_{0\leqslant r\leqslant\tau} \big|\mathscr{A}_{\mathrm{def}}(Y)_r\big| \Big\|_q \leqslant c(m,q)\left(\mathbb{E}\left[\frac{1}{b}\left(1-\exp\left\{-b\int_0^\tau |du|^2\big(X_r(x)\big)\,dr\right\}\right)\right]^{q/2}\right)^{1/q}. \tag{4.19}$$

Proof. Lemma 4.6 is established in the same way as Lemma 4.5. Note that in (4.19) the constant $c(q)$ changed to $c(m,q) = m^2 c(q)$ where $m = \dim M$, as a consequence of the trivial inequality $\|du\|^2 \leqslant m|du|^2$. □

Recall that (4.18) is in particular satisfied if u is of generalized K-bounded dilatation and Sect^N is bounded above by some negative constant.

Remark 4.7. Under assumption (4.18) we have

$$\int_0^\tau \big|\big(\Theta^N_{0,s}\big)^{-1}\big|^2 |du|^2\big(X_s(x)\big)\,ds \leqslant \frac{1}{b}\left(1-\exp\left(-b\int_0^\tau |du|^2\big(X_r(x)\big)\,dr\right)\right). \tag{4.20}$$

Combining Theorem 4.1 and Lemmas 4.3, 4.5 and 4.6, along with Remark 4.7, we are able to present explicit derivative estimates for harmonic maps $u : D \to N$ where $D \subset M$ is a connected and relatively compact open domain in M with smooth boundary. For simplicity we confine ourselves here to the case $p = q = 2$.

Theorem 4.8. *Let $m = \dim M$ and $-\alpha$ be a lower bound of the Ricci curvature on M, and let $\kappa \geqslant 0$ be an upper bound of the sectional curvatures of N. Let $u : D \to N$ be harmonic. For any $f \in C^2(\overline{D})$ with $f > 0$ in D and $f|\partial D = 0$, one has*

$$\left|(du)_x\right| \leqslant \frac{\sqrt{c_2(f)}}{f(x)} \times \begin{cases} (\frac{1}{\kappa}\mathbb{E}[\exp(\kappa \int_0^{\tau(x)} \|du\|^2(X_s(x))\,ds) - 1])^{1/2}, & \text{if } \kappa > 0, \\ (\mathbb{E}[\int_0^{\tau(x)} \|du\|^2(X_s(x))\,ds])^{1/2}, & \text{if } \kappa = 0, \end{cases}$$

where $c_2(f)$ is defined in (4.5). *In particular, if $D = B_M(x_0, R) := \{\mathrm{dist}_M(\cdot, x_0) < R\}$, then for any $x \in D$,*

$$\left|(du)_x\right| \leqslant \frac{C(R)}{2\sin(\pi\delta_x/(2R))} \times \begin{cases} (\frac{1}{\kappa}\mathbb{E}[\exp(\kappa \int_0^{\tau(x)} \|du\|^2(X_s(x))\,ds) - 1])^{1/2}, & \text{if } \kappa > 0, \\ (\mathbb{E}[\int_0^{\tau(x)} \|du\|^2(X_s(x))\,ds])^{1/2}, & \text{if } \kappa = 0, \end{cases}$$

where $\delta_x := \mathrm{dist}_M(x, \partial D)$ and $C(r)$ is defined by Eq. (4.11).

Proof. Taking $p = q = 2$, we obtain the first formula from Theorem 4.1, Lemma 4.3 (letting $t \to \infty$) and Lemma 4.5. The specific estimates on geodesic balls follow then from estimate (4.10) of Example 4.4. □

Corollary 4.9. *We keep the notation of Theorem* 4.8. *Let $u : D \to N$ be a harmonic map defined on a relatively compact open domain $D \subset M$ with smooth boundary. Then, for any $x \in D$,*

$$\left|(du)_x\right| \leqslant \frac{C(\delta_x)}{2} \times \begin{cases} (\frac{1}{\kappa}\mathbb{E}[\exp(\kappa \int_0^{\tau(x)} \|du\|^2(X_s(x))\,ds) - 1])^{1/2}, & \text{if } \kappa > 0, \\ (\mathbb{E}[\int_0^{\tau(x)} \|du\|^2(X_s(x))\,ds])^{1/2}, & \text{if } \kappa = 0. \end{cases}$$

Moreover, let ι_x denote the injectivity radius at x, we have

$$\left|(du)_x\right| \leqslant \frac{C(\delta_x(s))}{2} \times \begin{cases} \frac{1-\cos(\sqrt{2\kappa}s)}{\kappa\cos(\sqrt{2\kappa}s)}, & \text{if } \kappa > 0,\ s \in]0, \pi/(2\sqrt{2\kappa})[, \\ s, & \text{if } \kappa = 0,\ s > 0, \end{cases}$$

where

$$\delta_x(s) := \delta_x \wedge \iota_x \wedge \sup\bigl\{r > 0\colon\ u\bigl(B_M(x,r)\bigr) \subset B_N\bigl(u(x), s\bigr)\bigr\}.$$

Proof. The desired results are immediate consequences of Theorem 4.8 by replacing D with either $B_M(x, \delta_x)$ or $B_M(x, \delta_x(s))$, just note that in the second situation if $\kappa = 0$ one has

$$\mathbb{E}\left[\int_0^\tau \|du\|^2\bigl(X_s(x)\bigr)\,ds\right] \leqslant \mathbb{E}\bigl[\mathrm{dist}_N\bigl(u\bigl(x_\tau(x)\bigr), u(x)\bigr)^2\bigr],$$

while if $\kappa > 0$ we use the exponential estimate (5.3) appearing in the proof of Remark 5.2 below (see also Example 5.4). □

Theorem 4.10. *The notations are again as in Theorem* 4.8 *and Corollary* 4.9. *Let $u : D \to N$ be a harmonic map defined on a relatively compact open domain $D \subset M$ with smooth boundary. We assume that $\mathrm{Ric}^M \geqslant -\alpha$ for some constant α and that*

$$\frac{(\sup \mathrm{Sect}^N)(u(\cdot))}{K^2(\cdot)} \leqslant -b \tag{4.21}$$

for some $b > 0$*. Then for any* $x \in D$*,*

$$\left|(du)_x\right| \leqslant \frac{\sqrt{c_2(f)}}{f(x)}\left(\frac{1}{b}\mathbb{E}\left[1-\exp\left(-b\int_0^{\tau(x)}|du|^2\big(X_s(x)\big)\,ds\right)\right]\right)^{1/2},$$

respectively,

$$\left|(du)_x\right| \leqslant \frac{C(\delta_x)}{2}\left(\frac{1}{b}\mathbb{E}\left[1-\exp\left(-b\int_0^{\tau(x)}|du|^2\big(X_s(x)\big)\,ds\right)\right]\right)^{1/2}$$

where $\delta_x = \operatorname{dist}_M(x, \partial D)$ *and* $C(r)$ *is given by Eq.* (4.11).

If $D = B_M(x_0, R) = \{\operatorname{dist}_M(\cdot, x_0) < R\}$*, then*

$$\left|(du)_x\right| \leqslant \frac{C(R)}{2\sin(\pi\delta_x/(2R))}\left(\frac{1}{b}\mathbb{E}\left[1-\exp\left(-b\int_0^{\tau(x)}|du|^2\big(X_s(x)\big)\,ds\right)\right]\right)^{1/2}.$$

Proof. Theorem 4.10 is derived in the same way as Theorem 4.8 and Corollary 4.9. This time however, instead of (4.1), estimate (4.2) is used along with Remark 4.7. □

Note that assumption (4.21) is in particular satisfied with $b = \beta/K^2$ if u is of generalized K-bounded dilatation and $\operatorname{Sect}^N \leqslant -\beta$ for some $\beta > 0$.

5. Liouville type theorems

The basic strategy to Liouville type theorems based on Theorem 4.1 and the estimates of Section 4, is to show that the r.h.s. of (4.1) or (4.2) tends to 0 when the domain D exhausts the manifold M. Since the r.h.s. is a product of two terms combining the geometry of M and N, there are naturally two types of theorems: one specifying lower Ricci curvature bounds of the manifold M (sufficient for the second term to tend to 0) and one relying on upper sectional curvature bounds of the target N (forcing the first term to tend to 0). As Theorem 4.10 above shows, bounds on the sectional curvature of N may sometimes be combined with properties of the mapping u, like bounds on the dilatation of u.

In this section we prove Liouville theorems for harmonic maps with one of the following features: (1) small image; (2) sublinear growth; (3) non-positively curved target; (4) bounded dilatation; (5) Liouville manifolds as domains; (6) certain asymptotic behaviour. Certain Liouville theorems for harmonic maps of bounded dilatation have already been deduced from the nonlinear derivative formula in [3].

There is an alternative efficient approach of R. Schoen and K. Uhlenbeck to a priori estimates in case of small energy using monotonicity formulas. For a typical version see Theorem 2.2 in [27]. Compare [30] and [1], for a discussion of monotonicity formulas from a stochastic point of view.

5.1. Harmonic maps of small image

Definition 5.1. Let (N, h) be a Riemannian manifold. For a non-negative measurable function λ defined on N, let $\mathscr{E}_\lambda$ denote the set of N-valued martingales Y satisfying

$$\mathbb{E}\left[\exp\left(\int_0^\infty \lambda(Y_s)h(dY_s, dY_s)\right)\right] < \infty. \tag{5.1}$$

Martingales Y in $\mathscr{E}_\lambda$ are said to have exponential moments of order λ.

Remark 5.2. Let $B \subset N$ be an open subset. Suppose that we are given on B a non-negative locally bounded function λ and a C^2 function f satisfying $c_1 \leqslant f \leqslant c_2$ for some positive constants c_1, c_2 and such that

$$\nabla df + 2\lambda f \leqslant 0. \tag{5.2}$$

(This means that $(\nabla df)_x(v, v) + 2\lambda(x) f(x) h_x(v, v) \leqslant 0$ for any $v \in T_xN$, $x \in B$.) Then any B-valued martingale belongs to $\mathscr{E}_\lambda$ (see Picard [20], Proposition 2.1.2, as well as [21]).

Proof. Indeed, Itô's formula along with Eq. (5.2) shows that

$$S_t := (f \circ Y_t) \exp\left(\int_0^t \lambda(Y_s)h(dY_s, dY_s)\right)$$

is a (non-negative) local supermartingale, thus $\mathbb{E}[S_\infty] \leqslant \mathbb{E}[S_0]$ by Fatou's lemma. Using the lower and upper bounds on f, this implies

$$\mathbb{E}\left[\exp\left(\int_0^\infty \lambda(Y_s)h(dY_s, dY_s)\right)\right] \leqslant \frac{c_2}{c_1}. \qquad \square \tag{5.3}$$

Definition 5.3. An open geodesic ball $B_N(y_0, R)$ in a Riemannian manifold N is said to be *regular* if it does not meet the cut locus of its centre y_0 and if $\kappa < (\pi/2R)^2$ where κ denotes an upper bound of the sectional curvature of N on $B_N(y_0, R)$.

Example 5.4. Let $B_N(y_0, R)$ be a regular geodesic ball in N such that $R < \pi/(2\sqrt{\kappa})$ where $\kappa > 0$ is an upper bound of the sectional curvatures of N on $B_N(y_0, R)$. Take $f(y) = \cos(\sqrt{\kappa q}\, d(y_0, y))$ where $q > 1$ is chosen such that $0 < c_1 \leqslant f$ holds on $B_N(y_0, R)$ for some $c_1 > 0$. Then

$$\nabla df + \kappa q f \leqslant 0$$

which means by Remark 5.2 that any $B_N(y_0, R)$-valued ∇-martingale has exponential moments of order $\kappa q/2$.

Corollary 5.5. *(See Hildebrandt, Jost, and Widman [12,11].) Let $u : M \to N$ be a harmonic map between connected complete Riemannian manifolds such that*

$$u(M) \subset B_N(y_0, R)$$

where $B_N(y_0, R)$ is a regular geodesic ball in N about y_0 of radius R. Suppose that M has non-negative Ricci curvature and that $R < \pi/(2\sqrt{\kappa})$ where κ is a positive upper bound for the sectional curvatures of N on $B_N(y_0, R)$. Then u is a constant.

Proof. Exhaust M by a sequence of geodesic balls and let $\tau_n(x)$ be the corresponding sequence of first exit times of $X_\bullet(x)$. Let $q > 1$ be as in Example 5.4 and p be the dual index to q. We apply estimate (4.1) with $\tau(x) = \tau_n(x)$. By Example 5.4 $Y = u(X_\bullet(x))$ has exponential moments of order $\kappa q/2$, which implies according to the estimates (4.13) and (4.14) that the L^q norms of $\mathscr{A}_{\mathrm{def}}(Y)_{\tau_n(x)}$ are bounded, while according to the last assertion in Lemma 4.3, the L^p norm term of the r.h.s. tends to 0 as $n \to \infty$. □

Remark 5.6. Corollary 5.5 is sharp in the sense that it applies to the inclusion map $\iota : S^{n-1} \to S^n$ as the equator which lies in the closed ball of radius $\pi/(2\sqrt{\kappa})$ centred at the North Pole.

However the assumption $\mathrm{Ric}^M \geqslant 0$ in Corollary 5.5 can be weakened to condition (5.4) below. See also [6].

Corollary 5.7. *Let $u : M \to N$ be a harmonic map between connected complete Riemannian manifolds such that*

$$u(M) \subset B_N(y_0, R)$$

where $B_N(y_0, R)$ is a regular geodesic ball in N about y_0 of radius R. Suppose that $R < \pi/2\sqrt{\kappa q}$ for some $q > 1$, where κ is a positive upper bound of the sectional curvatures of N on $B_N(y_0, R)$. Let $p = [q/(q-1)] \vee 2$, and assume that $\mathrm{Ric}^M(x) \geqslant -k_M(x)$ for all $x \in M$ and some measurable function k_M satisfying

$$\liminf_{t\to\infty} \frac{1}{t^{p/2+1}} \int_0^t \mathbb{E} \exp\left[\frac{p}{2}\int_0^s k_M(X_r)\,dr\right] ds = 0. \tag{5.4}$$

Then u is a constant map.

Proof. Since $|\Theta^M_{0,s}|^2 \leqslant \exp[\int_0^s k_M(X_r)\,dr]$, the proof is completed by applying Hölder's inequality to (3.11), choosing $\ell(s) = 1 - s/t$, invoking Burkholder–Davis–Gundy's inequality and the fact that $\sup_{t>0} \|\mathscr{A}_{\mathrm{def}}(Y)_t\|_q < \infty$ under the given assumptions, as already used in the proof to Corollary 5.5. □

5.2. Harmonic maps of sublinear growth

Definition 5.8. A map $u : M \to N$ between complete Riemannian manifolds is said to be of sublinear growth if for some point $x \in M$ and some positive function φ with $\varphi(r)/r \to 0$ as $r \to \infty$

$$\mathrm{dist}_N\big(u(z), u(x)\big) \leqslant \varphi \circ \mathrm{dist}_M(z, x) \quad \text{for all } z \in M. \tag{5.5}$$

Remark 5.9. Let $u : M \to N$ be a harmonic map between complete Riemannian manifolds, and suppose that the sectional curvatures of N are all non-positive, and that N is simply connected. To estimate for $Y = u(X_\bullet(x))$ the L^2 norm of

$$\mathscr{A}_{\mathrm{def}}(Y)_{\tau_r(x)} = \int_0^{\tau_r(x)} \big(\Theta^N_{0,s}\big)^{-1} (du)_{X_s(x)}\, dB_s$$

where $\tau_r(x)$ denotes the first exit time of $X_\bullet(x)$ from the geodesic ball $B_M(x,r)$ about x of radius r, by Eq. (4.15) of Lemma 4.5 we have

$$\mathbb{E}\big|\mathscr{A}_{\mathrm{def}}(Y)_{\tau_r(x)}\big|^2 \leqslant \mathbb{E}\big[\mathrm{dist}_N\big(u\big(X_{\tau_r(x)}(x)\big), u(x)\big)^2\big]. \tag{5.6}$$

Observe that if in addition u is of sublinear growth, then $\|\mathscr{A}_{\mathrm{def}}(Y)_{\tau_r(x)}\|_2$ may be estimated by $c_r r$ with $c_r := \varphi(r)/r \to 0$ as $r \to \infty$.

Corollary 5.10. *(See Cheng [5].) Let $u : M \to N$ be a harmonic map between connected complete Riemannian manifolds. Suppose that $\mathrm{Ric}^M \geqslant 0$, $\mathrm{Sect}^N \leqslant 0$, and N is simply connected. If u is of sublinear growth, then u must be a constant map.*

Proof. Indeed by Eq. (4.1) we have

$$\big|(du)_x v\big| \leqslant \big\|\mathscr{A}_{\mathrm{def}}(Y)_{\tau_r(x)}\big\|_2 \cdot \left\| \int_0^{\tau_r(x)} \big\langle \Theta^M_{0,s}\dot\ell(s), //^M_{0,s}\, dB_s\big\rangle \right\|_2. \tag{5.7}$$

Applying Remark 5.9 to the first and Lemma 4.3 to the second term of the r.h.s. of (5.7), we see that the r.h.s. can be estimated by $C\varphi(r)/r$ for some constant $C > 0$ and thus tends to 0 as $r \to \infty$. □

5.3. Non-positively curved targets

One observes that there is an interior supremum bound for harmonic maps into manifolds of non-positive curvature. The following theorem goes back to the work of Eells and Sampson [9].

Theorem 5.11. *Let (M, g) be a Riemannian manifold and let $D \subset M$, $D \neq M$ be an open region with compact closure. For any compact set $K \subset D$ there is a constant c depending only on K, D and the metric g such that for any Riemannian manifold N of non-positive curvature and any harmonic map $u : D \to N$ we have the bound*

$$\sup_K \|du\|^2 \leqslant c E_D(u) \tag{5.8}$$

where $E_D(u) = \int_D \|du\|^2\, d\,\mathrm{vol}$.

Proof. There exist $r_0 > 0$, two open relatively compact subsets U_0, U_1 and a function $f \in C^2(D)$, $0 \leqslant f \leqslant 1$, such that

$$\bigcup_{x\in K} B_M(x, r_0) \subset U_0, \qquad \overline{U}_0 \subset U_1, \qquad \overline{U}_1 \subset D,$$

and $f \equiv 1$ on U_0 and $\mathrm{supp}\, f \subset \overline{U}_1$. Analogously to (3.5), for $x \in K$, let

$$\sigma(s) = \inf\left\{ r \geqslant 0: \int_0^r f^{-2}\big(X_u(x)\big)\, du \geqslant s \right\}.$$

Then $\sigma(s) \leqslant s$, and $X'_s(x) := X_{\sigma(s)}(x)$ is a diffusion on U_1 with generator $f^2\,\Delta$. Since the sectional curvatures of N are non-positive and the Ricci curvature on D is bounded below, by

Bochner's formula one obtains $\Delta\|du\|^2 \geqslant -C_1\|du\|^2$ for some $C_1 > 0$ and all harmonic maps u defined on D. Therefore

$$\|du\|^2(x) \leqslant e^{C_1}\mathbb{E}\|du\|^2(X_{\sigma(1)}) \leqslant e^{C_1}C_2\,E_D(u)$$

with

$$C_2 := \sup_{x\in K,\, y\in\overline{U}_1} p'(1,x,y)$$

where $p'(s,x,y)$ denotes the smooth heat kernel to X', i.e.

$$p'(s,x,y)\,\mathrm{vol}(dy) = \mathbb{P}\big\{X'_s(x)\in dy\big\}. \qquad \square$$

A notable feature of Theorem 5.11 is that the constant c appearing in (5.8) is independent of the target manifold N. It is straight-forward to adapt the above method to give explicit expressions for the constants involved.

The arguments of Section 5.2 already revealed the crucial role played by assumptions of non-positive curvature of the target: they enter into the formulas through the fact that $|(\Theta^N_{0,s})^{-1}| \leqslant 1$, implying for instance that for any stopping time τ,

$$\mathbb{E}\big|\mathscr{A}_{\mathrm{def}}(Y)_\tau\big|^2 \leqslant \mathbb{E}\Bigg[\int_0^\tau \|du\|^2\big(X_s(x)\big)\,ds\Bigg] \tag{5.9}$$

where $Y = u(X_\bullet(x))$ is the image of a Brownian motion $X_\bullet(x)$ under the harmonic map $u : M \to N$.

For a general C^1 map $u : M \to N$ and a geodesic ball $B_M(x,r)$ in M about x of radius r, let

$$E_{B_M(x,r)}(u) := \int_{B_M(x,r)} \|du\|^2\,d\,\mathrm{vol}$$

denote the energy of u on $B_M(x,r)$. (Here $B_M(x,r)$ is assumed to be relatively compact and $B_M(x,r) \neq M$.) Finally, u is said to have finite energy if

$$E(u) := \int_M \|du\|^2\,d\,\mathrm{vol} < \infty.$$

Corollary 5.12. *(See Schoen and Yau [25,26].) A harmonic map $u : M \to N$ of finite energy, from a connected complete Riemannian manifold M with* $\mathrm{Ric}^M \geqslant 0$ *to a Riemannian manifold N with* $\mathrm{Sect}^N \leqslant 0$*, must be a constant map.*

Proof. For $x \in M$ and $r > 0$, let $D = B_M(x,r)$ and consider

$$f = \cos\big(\pi\,\mathrm{dist}_M(\cdot,x)/2r\big)$$

on D. The corresponding time change $\sigma(s)$ defined by Eq. (4.8) then satisfies $\sigma(s) \leqslant s$. We may estimate as follows:

$$\big|(du)_x v\big|^2 \leqslant \mathbb{E}\big|\mathscr{A}_{\mathrm{def}}(Y)_{\sigma(t)}\big|^2 \cdot \mathbb{E}\Bigg[\int_0^{\sigma(t)} \big|Q^M_s\,\dot\ell(s)\big|^2\,ds\Bigg]. \tag{5.10}$$

First note that estimate (4.9) holds with $\sigma(t)$ replacing $\tau(x)$. We let $p=2$, $t=1$, and evaluate the r.h.s. of (4.9) as in the proof of Corollary 5.1 in [32]. It follows along with (5.9) that

$$|du|^2(x) \leqslant \frac{C}{r^2}\int_0^1 \mathbb{E}\|du\|^2\big(X_s(x)\big)\,ds$$

for some constant $C>0$ independent of r and x. (Here we used that $\sigma(1)\leqslant 1$.) Therefore

$$E(u) \leqslant \frac{C}{r^2}\int_0^1 ds\int_M \mathbb{E}\|du\|^2\big(X_s(x)\big)\,\mathrm{vol}(dx) \leqslant \frac{C}{r^2}E(u),$$

since the heat semigroup on M is a contraction in $L^1(d\,\mathrm{vol})$. Letting $r\to\infty$ we obtain $E(u)=0$ and hence u is a constant. □

For related results see also Okayasu [19] and Rigoli and Setti [24].

5.4. Harmonic maps of bounded dilatation

In this subsection we deal with Liouville theorems for harmonic maps of bounded dilatation. Such theorems typically require non-negative Ricci curvature on the initial manifold and negative sectional curvature bounded away from zero on the target, e.g. [28]. A notable feature of our approach is that the bounded dilatation condition on the mapping can be relaxed on domains with sufficiently enough negative curvature, see (5.11) and (5.13) below for the precise conditions.

Theorem 5.13. *Let (M,g), (N,h) be Riemannian manifolds where M is complete with its Ricci curvature bounded below by a non-positive constant, say $\mathrm{Ric}^M \geqslant -\alpha$ for some real α. Let $u: D\to N$ be a harmonic map where $D\subset M$, defined on some relatively compact open domain $D\neq M$, and suppose that for some constant $b>0$*

$$\sup_D\left[\frac{(\sup\mathrm{Sect}^N)(u(\cdot))}{K^2(\cdot)}\right] \leqslant -b \tag{5.11}$$

where $K^2(\cdot)$ is defined by Eq. (4.17). Then for any $x\in D$,

$$\big|(du)_x\big| \leqslant \frac{C(\mathrm{dist}_M(x,\partial D))}{2\sqrt{b}}, \tag{5.12}$$

respectively, if $D=B_M(x_0,R)$, then

$$\big|(du)_x\big| \leqslant \frac{C(R)}{2\sqrt{b}\sin(\pi\,\mathrm{dist}_M(x,\partial D)/(2R))}$$

where

$$C(r)=\sqrt{\pi^2(m+3)r^{-2}+2\pi\sqrt{\alpha^+(m-1)}r^{-1}+4\alpha^+}.$$

Proof. This is an immediate consequence of Theorem 4.10. □

Corollary 5.14. *Let* (M,g), (N,h) *be Riemannian manifolds where* M *is complete with* $\mathrm{Ric}^M \geqslant -\alpha$ *for some* $\alpha \geqslant 0$. *Let* $u: M \to N$ *be a harmonic map such that for some constant* $b > 0$,

$$\sup_N \left[\frac{(\sup \mathrm{Sect}^N)(u(\cdot))}{K^2(\cdot)} \right] \leqslant -b. \tag{5.13}$$

Then

$$u^* h \leqslant \frac{\alpha}{b} g.$$

Proof. The claim follows from (5.12) by letting $D \nearrow M$. $\square$

Corollary 5.14 includes the following Liouville theorem due to C.L. Shen as a special case.

Corollary 5.15. *(See Shen [28].) Let* (M,g), (N,h) *be Riemannian manifolds where* M *is complete. Let* $\mathrm{Ric}^M \geqslant 0$ *and* $\mathrm{Sect}^N \leqslant -\beta$ *for some constant* $\beta > 0$. *Then any harmonic map* $u: M \to N$ *of generalized* K*-bounded dilatation is constant.*

5.5. Harmonic maps defined on Liouville manifolds

Liouville theorems for harmonic maps defined on Liouville manifolds have been studied in probabilistic terms by W.S. Kendall [15,16]; see also H. Donnelly [7]. One typically considers situations where the martingale $Y = u \circ X$ has a.s. a limit as $t \to \infty$, which is then constant by the Liouville property.

Lemma 5.16. *For any martingale* Y *taking values in a Riemannian manifold* N *the following are equivalent*:

(i) Y *converges a.s. as* $t \to \infty$,
(ii) *The anti-development* $\mathscr{A}(Y)$ *of* Y *converges a.s. as* $t \to \infty$.

Under these conditions, if in addition $\mathrm{Sect}^N \leqslant 0$, *the deformed anti-development* $\mathscr{A}_{\mathrm{def}}(Y)$ *converges a.s. as* $t \to \infty$ *as well.*

Proof. Indeed, by the martingale convergence theorem, up to a set of measure 0, Y converges exactly on the set where the Riemannian quadratic variation of Y stays finite up to the lifetime of Y. The Riemannian quadratic variation of Y however coincides with the quadratic variation of the anti-development $\mathscr{A}(Y)$ of Y. If $\mathrm{Sect}^N \leqslant 0$ the quadratic variation of $\mathscr{A}_{\mathrm{def}}(Y)$ can be estimated against the quadratic variation of $\mathscr{A}(Y)$:

$$\begin{aligned}
\left[\mathscr{A}_{\mathrm{def}}(Y), \mathscr{A}_{\mathrm{def}}(Y)\right]_t &= \int_0^t \left\| \left(\Theta_{0,s}^N\right)^{-1} du \right\|^2 \big(X_s(x)\big)\, ds \\
&\leqslant \int_0^t \|du\|^2 \big(X_s(x)\big)\, ds = \left[\mathscr{A}(Y), \mathscr{A}(Y)\right]_t
\end{aligned}$$

which gives the additional claim, see Remark 2.4, c). $\square$

Theorem 5.17. *Let $u : M \to N$ be a bounded harmonic map into a complete Riemannian manifold N with $\mathrm{Sect}^N \leqslant 0$. Suppose that a.s.*

$$u \circ X_t(x) \to u_\infty, \quad \text{as } t \to \infty, \tag{5.14}$$

for some deterministic point $u_\infty \in N$. Then u is a constant mapping.

Proof. Indeed, we may assume that N is simply connected (otherwise we lift $u \circ X$ to the universal cover of N). Let τ_r be the first exit time of $X(x)$ from the geodesic ball of radius r about x. Then

$$\begin{aligned}
0 &\leqslant \mathbb{E}\big[\big|\mathscr{A}_{\mathrm{def}}(Y)_{\tau_r}\big|^2\big] \\
&\leqslant \mathbb{E}\left[\int_0^{\tau_r} \|du\|^2\big(X_s(x)\big)\, ds\right] \\
&\leqslant \frac{1}{2}\mathbb{E}\left[\int_0^{\tau_r} \Delta\, \mathrm{dist}_N^2\big(u(\cdot), u_\infty\big)\big(X_s(x)\big)\, ds\right] \\
&= \mathbb{E}\big[\mathrm{dist}_N^2\big(u\big(X_{\tau_r}(x)\big), u_\infty\big)\big] - \mathrm{dist}_N^2\big(u(x), u_\infty\big)
\end{aligned}$$

which converges to $-\mathrm{dist}_N^2(u(x), u_\infty)$ as $r \to \infty$ according to the dominated convergence theorem. Thus $u(x) = u_\infty$ for any $x \in M$. □

Corollary 5.18. *(See Kendall [16].) Let (M, g) and (N, h) be Riemannian manifolds. Suppose that (M, g) supports no non-constant bounded harmonic functions. Assume that (N, h) is complete and simply connected with non-positive sectional curvature, i.e. $\mathrm{Sect}^N \leqslant 0$. Then any bounded harmonic map $u : M \to N$ is constant.*

Proof. The N-valued martingale $u(X_t(x))$ converges almost surely to a limit L as $t \to \infty$, see Remark 5.2 above. Since M does not carry non-trivial bounded harmonic functions, the limit L must be constant. The claim then follows from Theorem 5.17. □

A similar type Liouville theorem has been first proved by Kendall [15] for harmonic maps $u : M \to N$ of bounded dilatation with negatively curved targets N, see also [17].

Remark 5.19. (See Kendall [15].) Let (M, g) and (N, h) be Riemannian manifolds. Suppose that (M, g) supports no non-constant bounded harmonic functions and that (N, h) is complete and simply connected with sectional curvatures bounded between two strictly negative constants. Then any harmonic map $u : M \to N$ of bounded dilatation is constant.

Here, under the given conditions, convergence of the martingale $u(X_t(x))$ on N takes place at infinity. As in Corollary 5.18 the limit is then constant. It should be noted that the pinched curvature assumption can be weakened, see [18] for details. It is also enough to assume that u is K-quasi-conformal. Analytic proofs have been worked out by H. Donnelly [7].

5.6. Harmonic maps of certain asymptotic behaviour

Besides the various properties discussed so far which imply Liouville theorems for harmonic maps (e.g. small image, bounded dilatation, finite energy, certain growth behaviour, etc.), there seems to be yet another class of conditions: Instead of assuming for instance global "smallness" of the image, one may only prescribe the asymptotic behaviour at infinity for the harmonic maps, e.g. Zhiren Jin [14].

Theorem 5.20. *(See Jin [14].) Let $\mathbb{R}^m$, $m \geqslant 3$, be equipped with the standard Euclidean metric, and let (N, h) be a Riemannian manifold.*

(A) *Let $u : \mathbb{R}^m \to (N, h)$ be a harmonic map. If $u(x) \to y_0 \in N$ as $|x| \to \infty$, then u is a constant map.*
(B) *Suppose that Sect^N is bounded from above. Then for any $y \in N$, there is a (nonempty) open neighbourhood $U_y \subset N$ such that the family $\{U_y\}_{y\in N}$ has the following property: If $u : \mathbb{R}^m \to (N, h)$ is harmonic, and $u(x) \in U_y$ as $|x| \to \infty$ for some $y \in N$, then u is a constant map.*

Recall that $u(x) \in U_y$ as $|x| \to \infty$ means that there exists $r_0 > 0$ such that

$$u\big(\{x\colon |x| \geqslant r_0\}\big) \subset U_y.$$

Obviously Theorem 5.20(B) is stronger than Theorem 5.20(A), however Z.R. Jin [14] proved part (A) in a slightly more general setting by allowing also a certain conformal change of the Euclidean metric on $\mathbb{R}^m$.

Theorem 5.20 can be strengthened by the stochastic method in a straightforward way. The Euclidean domain $\mathbb{R}^m$ may be replaced by a non-compact, complete, connected Riemannian manifold (M, g) of non-negative Ricci curvature. Also the open neighbourhoods U_y can be specified explicitly.

Theorem 5.21. *Let (M, g) and (N, h) be two Riemannian manifolds. Suppose that M is non-compact, connected, complete with Ric^M non-negative, and that all sectional curvatures of N are bounded above, $\mathrm{Sect}^N \leqslant \kappa$ for some positive κ. Let $u : M \to N$ be a harmonic map such that*

$$u\big(\{x \in M\colon \mathrm{dist}_M(x, x_0) \geqslant r_0\}\big) \subset B_N(y_0, R)$$

for some $x_0 \in M$, $y_0 \in N$ and $r_0 > 0$, where $B_N(y_0, R)$ is a regular geodesic ball about y_0 of radius $R < \pi/(2\sqrt{\kappa})$. Then u is a constant map.

Proof. Fix $x \in M$ with $\mathrm{dist}_M(x, x_0) \geqslant r_0 + 1$ and let $D = B_M(x, 1)$. Then, by assumption, $u(D) \subset B_N(y_0, R)$. As in the proof of Corollary 5.5, we get $|du|(x) \leqslant C$ for some constant C independent of x. Thus $|du|$ is bounded. Next, let $D = B_M(x, r)$ for arbitrary $x \in M$ and $r > 0$. We may apply again estimate (5.10), choosing $t = 1$ as in the proof of Corollary 5.12. Since $|du|$ is bounded and $\sigma(1) \leqslant 1$, we can now exploit estimate (4.13). Proceeding as in the proof of Corollary 5.12 shows that $|du|(x) \leqslant \tilde{C}r^{-1}$ for some constant $\tilde{C} > 0$. Therefore, since x, r are arbitrary, the map u must be a constant. □

Harmonic maps having small images and limiting value at infinity are necessarily constant, regardless of the geometry of the domain manifold and the rate of convergence at infinity.

Theorem 5.22 *(Convergent harmonic maps). (See [23], Proposition 6.9.) Let $u : M \to N$ be a harmonic map between non-compact Riemannian manifolds. Suppose that N is complete and that $\overline{u(M)}$ is contained in a regular geodesic ball $B_N(y_0, R)$ in N of radius R. Suppose that*

$$\lim_{x\to\infty} u(x) = y_0 \in N,$$

then u is a constant.

Proof. By assumption, we have $R\sqrt{\kappa} < \pi/2$ where $\kappa \geqslant 0$ denotes an upper bound of the sectional curvature of N on $B_N(y_0, R)$. We may restrict ourselves to the case $\kappa > 0$ (the case $\kappa = 0$ is already covered by Theorem 5.17). On the geodesic ball $B_N(y_0, R)$ we consider the function

$$\phi(y) := \frac{1 - \cos(\sqrt{\kappa}\,\mathrm{dist}_N(y, y_0))}{\kappa}.$$

It is elementary to check that [13]

$$\Delta_N \phi \geqslant \cos\bigl(\sqrt{\kappa}\,\mathrm{dist}_N(\cdot, y_0)\bigr),$$

and hence $\Delta_N \phi \geqslant \varepsilon$ on $B_N(y_0, R)$ for some strictly positive constant ε. This gives the inequality $\Delta_M(\phi \circ u) \geqslant \varepsilon \|du\|^2$ which allows to conclude as in the proof of Theorem 5.17,

$$\begin{aligned}
0 &\leqslant \mathbb{E}\bigl[\bigl|\mathscr{A}_{\mathrm{def}}(Y)_{\tau_r}\bigr|^2\bigr] \leqslant \mathbb{E}\Biggl[\int_0^{\tau_r} \|du\|^2\bigl(X_s(x)\bigr)\,ds\Biggr] \\
&\leqslant \frac{1}{\varepsilon}\mathbb{E}\Biggl[\int_0^{\tau_r} \Delta_M(\phi \circ u)\bigl(X_s(x)\bigr)\,ds\Biggr] \\
&= \frac{1}{\varepsilon}\mathbb{E}\bigl[\phi\,\mathrm{dist}_N^2\bigl(u\bigl(X_{\tau_r}(x)\bigr)\bigr)\bigr] - \frac{1}{\varepsilon}\phi\bigl(u(x)\bigr) \\
&\to -\frac{1}{\varepsilon}\phi\bigl(u(x)\bigr), \quad \text{as } r \to \infty. \qquad \square
\end{aligned}$$

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