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Characterization of pinched Ricci curvature by functional inequalities

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Abstract In this article, functional inequalities for diffusion semigroups on Riemannian manifolds (possibly with boundary) are established, which are equivalent to pinched Ricci curvature, along with gradient estimates, L^p-inequalities and log-Sobolev inequalities. These results are further extended to differential manifolds carrying geometric flows. As application, it is shown that they can be used in particular to characterize general geometric flow and Ricci flow by functional inequalities.

Keywords Curvature · gradient estimate · log-Sobolev inequality · evolving manifold · Ricci flow

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1 Introduction

Let (M,g) be a d-dimensional Riemannian manifold, possibly with boundary. Let ∇ and Δ be the Levi-Civita connection and the Laplacian associated with the Riemannian metric g, respectively. For a given C^1-vector field Z on M and tangent vectors X,Y on M, let

$$\mathrm{Ric}^Z(X,Y) := \mathrm{Ric}(X,Y) - \langle \nabla_X Z, Y\rangle,$$

where Ric is the Ricci curvature tensor with respect to g and $\langle\cdot,\cdot\rangle := g(\cdot,\cdot)$. We denote by $C(M)$, $C_b(M)$, $C^\infty(M)$ and $C_0^\infty(M)$ the sets of continuous functions, bounded continuous functions, smooth functions, smooth test functions on M, respectively.

Given a C^1-vector field Z on M, we consider the elliptic operator $L := \Delta + Z$. Let X_t^x be a diffusion process starting from $X_0^x = x$ with generator L, called a L-diffusion process.

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We assume that X_t^x is non-explosive for each $x \in M$. Let $B_t = (B_t^1, \ldots, B_t^d)$ be a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. The L-diffusion process X_t^x starting from x solves the Stratonovich equation

$$\mathrm{d}X_t^x = \sqrt{2}u_t^x \circ \mathrm{d}B_t + Z(X_t^x)\,\mathrm{d}t, \quad X_0^x = x, \tag{1.1}$$

where u_t^x is the horizontal process of X_t^x taking values in the orthonormal frame bundle $\mathrm{O}(M)$ over M such that $\pi(u_0^x) = x$. Note that

$$//_{s,t} := u_t^x \circ (u_s^x)^{-1} \colon T_{X_s^x}M \to T_{X_t^x}M, \quad s \leq t,$$

defines parallel transport along the paths $r \mapsto X_r^x$. By convention, an orthonormal frame $u \in \mathrm{O}(M)$ is interpreted as isometry $u \colon \mathbb{R}^d \to T_xM$ where $\pi(u) = x$. Note that parallel transport $//_{s,t}$ is independent of the choice of the initial frame u_0^x above x.

The diffusion process X_t^x gives rise to a Markov semigroup P_t with infinitesimal generator L: for $f \in C_b(M)$, we have

$$P_t f(x) = \mathbb{E}[f(X_t^x)], \quad t \geq 0,$$

where $\mathbb{E}$ stands for expectation with respect to the underlying probability measure $\mathbb{P}$.

The problem of characterizing boundedness of Ric^Z from below in terms of gradient estimates and other functional inequalities for the semigroup P_t, has been thoroughly studied in the literature, e.g. [14, 18, 19]. For instance, it is well-known that the curvature condition

$$\mathrm{Ric}^Z(X,X) \geq \kappa |X|^2, \quad X \in TM,$$

is equivalent to each of the following inequalities:

1) (gradient estimate) for all $f \in C_0^\infty(M)$,

$$|\nabla P_t f|^2 \leq \mathrm{e}^{-2\kappa t}\, P_t |\nabla f|^2;$$

2) (Poincaré inequality) for all $p \in (1,2]$ and $f \in C_0^\infty(M)$,

$$\frac{p}{4(p-1)}\big(P_t f^2 - (P_t f^{2/p})^p\big) \leq \frac{1-\mathrm{e}^{-2\kappa t}}{2\kappa}\, P_t |\nabla f|^2;$$

3) (log-Sobolev inequality) for all $f \in C_0^\infty(M)$,

$$P_t(f^2 \log f^2) - P_t f^2 \log P_t f^2 \leq \frac{2(1-\mathrm{e}^{-2\kappa t})}{\kappa}\, P_t |\nabla f|^2.$$

If P_t has a non-trivial invariant measure μ, such inequalities can be used to obtain the corresponding functional inequalities with respect to μ instead of the heat kernel measure (see [19, Section 2.4]). Typically, one takes $Z = \nabla V$ for some $V \in C^2(M)$, then

$$\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\,\mathrm{vol}(\mathrm{d}x)$$

where vol denotes the Riemannian volume measure on M. Assuming that μ is a probability measure, this allows to recover classical versions of the Poincaré and log-Sobolev inequality.

The question how to use functional inequalities for P_t to characterize upper bounds on Ric^Z is much more delicate. When it comes to stochastic analysis on path space, there is a lot of former work based on bounds of Ric^Z, see e.g. [4, 5, 8, 13]. Recently, A. Naber [16] and R. Haslhofer and A. Naber [12] established gradient inequalities on path space which are shown to characterize boundedness of Ric^Z. These results have been extended by F.-Y. Wang and B. Wu [20] to manifolds with boundary, where Ric^Z may also vary along the manifold and may be unbounded, see also [10] and [7] for related results.

Let us briefly describe R. Haslhofer and A. Naber's work. Among other things, they prove that the functional inequality,

$$|\nabla \mathbb{E} F(X_{[0,T]})|^2 \le \mathrm{e}^{\kappa T}\, \mathbb{E}\left[|D_0^{//}F|^2 + \kappa \int_0^T \mathrm{e}^{\kappa(r-T)}\, |D_r^{//}F|^2\, \mathrm{d}r\right], \quad F \in \mathscr{F}C_0^\infty, \tag{1.2}$$

is equivalent to the curvature condition

$$|\,\mathrm{Ric}^Z\,| \le \kappa \tag{1.3}$$

for some non-negative constant κ, where

$$\begin{aligned} &X_{[0,T]} := \{X_t : 0 \le t \le T\},\\ &\mathscr{F}C_0^\infty := \left\{ f(X_{t_1}, \ldots, X_{t_N}) :\ 0 \le t_1 < \ldots < t_N \le T,\ f \in C_0^\infty(M^N) \right\}, \end{aligned}$$

and

$$D_t^{//} F(X_{[0,T]}) := \sum_{i=1}^N \mathbb{1}_{\{t \le t_i\}}\, //_{t,t_i}^{-1} \nabla_i F(X_{[0,T]}), \quad F \in \mathscr{F}C_0^\infty.$$

From the proof in [16] it is clear that it is sufficient to have gradient estimate (1.2) for very special test functionals $F \in \mathscr{F}C_0^\infty$ on path space, namely

- for $F(X_{[0,T]}) = f(X_t)$, and
- for 2-point cylindrical functions of the form

$$F(X_{[0,T]}) = f(x) - \frac{1}{2} f(X_t)$$

where $x = X_0$. In other words, if one replaces the full path space inequality (1.2) by the one for these special test functions, it is still enough to characterize (1.3). From this observation it is easy to see that the subsequent items (i) and (ii) are equivalent:

(i) $|\,\mathrm{Ric}^Z\,| \le \kappa$ for some $\kappa \ge 0$;
(ii) for $f \in C_0^\infty(M)$ and $t > 0$,

$$\begin{aligned} &|\nabla P_t f|^2 \le \mathrm{e}^{2\kappa t}\, P_t |\nabla f|^2 \quad \text{and}\\ &\left|\nabla f - \frac{1}{2}\nabla P_t f\right|^2 \le \mathrm{e}^{\kappa t}\, \mathbb{E}\left[\left|\nabla f - \frac{1}{2} //_{0,t}^{-1} \nabla f(X_t)\right|^2 + \frac{1}{4}\left(\mathrm{e}^{\kappa t} - 1\right) |\nabla f(X_t)|^2\right]. \end{aligned}$$

The two inequalities in (ii) can be combined to the single condition:

$$\begin{aligned} &|\nabla P_t f|^2 - \mathrm{e}^{2\kappa t}\, P_t |\nabla f|^2\\ &\quad \le 4\left((\mathrm{e}^{\kappa t} - 1)|\nabla f|^2 + \langle \nabla f, \nabla P_t f\rangle - \left\langle \nabla f, \mathrm{e}^{\kappa t}\, \mathbb{E}[//_{0,t}^{-1} \nabla f(X_t)]\right\rangle \right) \wedge 0. \end{aligned}$$

The discussion above gives rise to a natural question: *Are there gradient inequalities on M which allow to characterize pinched curvature with arbitrary upper and lower bounds?*

Our paper is organized as follows. In Section 2 we give a positive answer to the question above. In Section 3, we extend these results to characterize simultaneous bounds on Ric^Z

and II on Riemannian manifolds with boundary, where the curvature bounds are not given by constants, but may vary over the manifold. In Section 4 finally, we present gradient and functional inequalities for the time-inhomogeneous semigroup $P_{s,t}$ on manifolds carrying a geometric flow. We show that these inequalities can be used to characterize solutions to some geometric flows, including Ricci flow.

After finishing our paper we learned of work in progress of Bo Wu [21] aiming at results on path space in a similar direction.

2 Characterizations for Ricci curvature

We start the section by introducing our main results.

Theorem 2.1 *Let (M,g) be a complete Riemannian manifold. Let k_1,k_2 be two real constants such that $k_1 \le k_2$. The following conditions are equivalent:*

(i) $k_1 \le \mathrm{Ric}^Z \le k_2$;

(ii) *for $f \in C_0^\infty(M)$ and $t>0$,*

$$\begin{aligned}&|\nabla P_t f|^2 - \mathrm{e}^{-2k_1 t} P_t |\nabla f|^2\\ &\quad\le 4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f|^2 + \langle \nabla f, \nabla P_t f\rangle - \mathrm{e}^{-k_1 t}\,\mathbb{E}\big\langle \nabla f, //_{0,t}^{-1}\nabla f(X_t)\big\rangle\right]\wedge 0;\end{aligned}$$

(ii') *for $f \in C_0^\infty(M)$ and $t>0$,*

$$|\nabla P_t f|^2 - \mathrm{e}^{-2k_1 t} P_t |\nabla f|^2 \le 4\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}|\nabla P_t f|^2 - \mathrm{e}^{-k_1 t}\,\mathbb{E}\big\langle \nabla P_t f, //_{0,t}^{-1}\nabla f(X_t)\big\rangle\right)\wedge 0;$$

(iii) *for $f \in C_0^\infty(M)$, $p\in(1,2]$ and $t>0$,*

$$\begin{aligned}&\frac{p(P_t f^2-(P_t f^{2/p})^p)}{4(p-1)} - \frac{1-\mathrm{e}^{-2k_1 t}}{2k_1}P_t|\nabla f|^2\\ &\quad\le 4\int_0^t\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}(t-r)}-1\right)P_r|\nabla f|^2\right.\\ &\qquad\left.+\mathbb{E}\big\langle \nabla f(X_r), \nabla P_{t-r}f(X_r) - \mathrm{e}^{-k_1(t-r)}\,//_{r,t}^{-1}\nabla f(X_t)\big\rangle\right]\mathrm{d}r\wedge 0;\end{aligned}$$

(iii') *for $f \in C_0^\infty(M)$, $p\in(1,2]$ and $t>0$,*

$$\begin{aligned}&\frac{p(P_t f^2-(P_t f^{2/p})^p)}{4(p-1)} - \frac{1-\mathrm{e}^{-2k_1 t}}{2k_1}P_t|\nabla f|^2\\ &\quad\le 4\int_0^t\left[\mathrm{e}^{\frac{k_2-k_1}{2}(t-r)}P_r|\nabla P_{t-r}f|^2 - \mathrm{e}^{-k_1(t-r)}\,\mathbb{E}\left\langle \nabla f(X_r), //_{r,t}^{-1}\nabla f(X_t)\right\rangle\right]\mathrm{d}r\wedge 0;\end{aligned}$$

(iv) *for $f \in C_0^\infty(M)$ and $t>0$,*

$$\begin{aligned}&\frac14\left(P_t(f^2\log f^2) - P_t f^2\log P_t f^2\right) - \frac{1-\mathrm{e}^{-2k_1 t}}{2k_1}P_t|\nabla f|^2\\ &\quad\le 4\int_0^t\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}(t-r)}-1\right)P_r|\nabla f|^2\right.\\ &\qquad\left.+\mathbb{E}\big\langle \nabla f(X_r), \nabla P_{t-r}f(X_r) - \mathrm{e}^{-k_1(t-r)}\,//_{r,t}^{-1}\nabla f(X_t)\big\rangle\right]\mathrm{d}r\wedge 0;\end{aligned}$$

(iv') *for* $f \in C_0^\infty(M)$ *and* $t > 0$,

$$\frac{1}{4}\left(P_t(f^2\log f^2) - P_t f^2 \log P_t f^2\right) - \frac{1-\mathrm{e}^{-2k_1 t}}{2k_1} P_t|\nabla f|^2$$
$$\leq 4\int_0^t \left[\mathrm{e}^{\frac{k_2-k_1}{2}(t-r)} P_r|\nabla P_{t-r} f|^2 - \mathrm{e}^{-k_1(t-r)}\,\mathbb{E}\left\langle \nabla f(X_r), //_{r,t}^{-1}\nabla f(X_t)\right\rangle\right] \mathrm{d}r \wedge 0.$$

Remark 2.2 The inequalities in (iv) and (iv') can be understood as limits of the inequalities (iii) and (iii') as $p \downarrow 1$ respectively.

Remark 2.3 As application, Theorem 2.1 can be used to characterize Einstein manifolds where Ric is a multiple of the metric g (constant Ricci curvature). The case $\mathrm{Ric} = \nabla Z$ can be characterized by all/some of the inequalities in (ii)-(iv) and (ii')-(iv') for $k_1 = k_2 = 0$, where the inequalities in (iii), (iii'), (iv) and (iv') may be understood as $k_2 = k_1$ and $k_1 \to 0$.

Proof (Proof of Theorem 2.1.) We divide the proof into two parts. In Part I, we will derive the functional inequalities from the curvature condition; in Part II, we will prove the reverse.

Part I. We already know that the curvature condition $\mathrm{Ric}^Z \leq k_1$ is equivalent to each of the following functional inequalities (see e.g. [19, Theorem 2.3.1]):

1) for all $f \in C_0^\infty(M)$,

$$|\nabla P_t f|^2 \leq \mathrm{e}^{-2k_1 t} P_t|\nabla f|^2;$$

2) for all $p \in (1,2]$ and $f \in C_0^\infty(M)$,

$$\frac{p}{4(p-1)}\left(P_t f^2 - (P_t f^{2/p})^p\right) \leq \frac{1-\mathrm{e}^{-2k_1 t}}{2k_1} P_t|\nabla f|^2;$$

3) for all $f \in C_0^\infty(M)$,

$$P_t(f^2\log f^2) - P_t f^2 \log P_t f^2 \leq \frac{2(1-\mathrm{e}^{-2k_1 t})}{k_1} P_t|\nabla f|^2.$$

Now, we prove that under the curvature condition (i) in Theorem 2.1, the remaining bounds in (ii)-(iv) and (ii')-(iv') hold true.

(a) (i) $\Rightarrow$ (ii) and (ii'): We start with well-known stochastic representation formulas for diffusion semigroups. By Bismut's formula (see [3,9]), we have

$$(\nabla P_t f)(x) = \mathbb{E}[Q_t //_{0,t}^{-1} \nabla f(X_t^x)].$$

Here Q_t is the $\mathrm{Aut}(T_xM)$-valued process defined by the linear pathwise differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t = -Q_t \,\mathrm{Ric}^Z_{//_{0,t}}, \quad Q_0 = \mathrm{id}_{T_xM}, \tag{2.1}$$

where

$$\mathrm{Ric}^Z_{//_{0,t}} := //_{0,t}^{-1} \circ \mathrm{Ric}^Z_{X_t} \circ //_{0,t} \in \mathrm{End}(T_xM) \tag{2.2}$$

and $//_{0,t}$ is parallel transport in TM along X_t. As usual, Ric^Z_x operates as a linear homomorphism on T_xM via $\mathrm{Ric}^Z_x v = \mathrm{Ric}^Z(\cdot, v)^\sharp$, $v \in T_xM$.

Let a and b be two constants such that $a+b=1$. We first observe that

$$\begin{aligned}
&2a\nabla f+2b\nabla P_t f-Q_t //_{0,t}^{-1}\nabla f(X_t)\\
&=2a\nabla f+2b\nabla P_t f-\mathrm{e}^{-\frac{k_2+k_1}{2}t} //_{0,t}^{-1}\nabla f(X_t)+\mathrm{e}^{-\frac{k_2+k_1}{2}t}\left(\mathrm{id}-\mathrm{e}^{\frac{k_2+k_1}{2}t}Q_t\right) //_{0,t}^{-1}\nabla f(X_t)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left|2(a\nabla f+b\nabla P_t f)-Q_t //_{0,t}^{-1}\nabla f(X_t)\right|\\
&\quad\le\left|2(a\nabla f+b\nabla P_t f)-\mathrm{e}^{-\frac{k_2+k_1}{2}t} //_{0,t}^{-1}\nabla f(X_t)\right|\\
&\qquad+\left|\mathrm{e}^{-\frac{k_2+k_1}{2}t}\left(\mathrm{id}-\mathrm{e}^{\frac{k_2+k_1}{2}t}Q_t\right) //_{0,t}^{-1}\nabla f(X_t)\right|. \qquad (2.3)
\end{aligned}$$

We now turn to estimate the last term on the right-hand side above,

$$\left|\left(\mathrm{id}-\mathrm{e}^{\frac{k_2+k_1}{2}t}Q_t\right) //_{0,t}^{-1}\nabla f(X_t)\right|\le\left\|\mathrm{id}-\mathrm{e}^{\frac{k_2+k_1}{2}t}Q_t\right\|\,|\nabla f(X_t)|.$$

To estimate $\|\mathrm{id}-\mathrm{e}^{\frac{k_2+k_1}{2}t}Q_t\|$, we rewrite the involved operator as

$$\mathrm{id}-\mathrm{e}^{\frac{k_2+k_1}{2}t}Q_t=\int_0^t \mathrm{e}^{\frac{k_2+k_1}{2}s}Q_s\left(\mathrm{Ric}^Z_{//_{0,s}}-\frac{k_1+k_2}{2}\,\mathrm{id}\right)\mathrm{d}s.$$

Hence, by the curvature condition (i), we have

$$\begin{aligned}
\left\|\mathrm{id}-\mathrm{e}^{\frac{k_2+k_1}{2}t}Q_t\right\|&\le\int_0^t \mathrm{e}^{\frac{k_2+k_1}{2}s}\|Q_s\|\left|\mathrm{Ric}^Z_{//_{0,s}}-\frac{k_1+k_2}{2}\,\mathrm{id}\right|\mathrm{d}s\\
&\le\int_0^t \mathrm{e}^{\frac{k_2+k_1}{2}s}\mathrm{e}^{-k_1 s}\,\frac{k_2-k_1}{2}\,\mathrm{d}s=\mathrm{e}^{\frac{(k_2-k_1)t}{2}}-1
\end{aligned}$$

which implies

$$\left|\mathrm{e}^{-\frac{k_2+k_1}{2}t}\left(\mathrm{id}-\mathrm{e}^{\frac{k_2-k_1}{2}t}Q_t\right) //_{0,t}^{-1}\nabla f(X_t)\right|\le\mathrm{e}^{-\frac{k_1+k_2}{2}t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f|(X_t).$$

By this and Eq. (2.3), we have

$$\begin{aligned}
&\left|2(a\nabla f+b\nabla P_t f)-Q_t //_{0,t}^{-1}\nabla f(X_t)\right|^2\\
&\quad\le\left[\left|2(a\nabla f+b\nabla P_t f)-\mathrm{e}^{-\frac{k_1+k_2}{2}t} //_{0,t}^{-1}\nabla f(X_t)\right|+\mathrm{e}^{-\frac{k_1+k_2}{2}t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f|(X_t)\right]^2\\
&\quad=\left|2(a\nabla f+b\nabla P_t f)-\mathrm{e}^{-\frac{k_1+k_2}{2}t} //_{0,t}^{-1}\nabla f(X_t)\right|^2\\
&\qquad+2\,\mathrm{e}^{-\frac{k_1+k_2}{2}t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\left|2(a\nabla f+b\nabla P_t f)-\mathrm{e}^{-\frac{k_1+k_2}{2}t} //_{0,t}^{-1}\nabla f(X_t)\right|\,|\nabla f|(X_t)\\
&\qquad+\mathrm{e}^{-(k_1+k_2)t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)^2|\nabla f|^2(X_t). \qquad (2.4)
\end{aligned}$$

By Cauchy's inequality, we have

$$\begin{aligned}
&2\,\mathrm{e}^{-\frac{k_1+k_2}{2}t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\left|2(a\nabla f+b\nabla P_tf)-\mathrm{e}^{-\frac{k_1+k_2}{2}t}\,//_{0,t}^{-1}\nabla f(X_t)\right|\,|\nabla f|(X_t)\\
&=2\sqrt{\mathrm{e}^{\frac{k_2-k_1}{2}t}-1}\,\left|2(a\nabla f+b\nabla P_tf)-\mathrm{e}^{-\frac{k_1+k_2}{2}t}\,//_{0,t}^{-1}\nabla f(X_t)\right|\mathrm{e}^{-\frac{k_1+k_2}{2}t}\sqrt{\mathrm{e}^{\frac{k_2-k_1}{2}t}-1}\,|\nabla f|(X_t)\\
&\le\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\left|2(a\nabla f+b\nabla P_tf)-\mathrm{e}^{-\frac{k_1+k_2}{2}t}\,//_{0,t}^{-1}\nabla f(X_t)\right|^2\\
&\quad+\mathrm{e}^{-(k_1+k_2)t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f|^2(X_t).
\end{aligned}$$

Thus, combining this inequality with (2.4), we obtain

$$\begin{aligned}
&\left|2(a\nabla f+b\nabla P_tf)-Q_t//_{0,t}^{-1}\nabla f(X_t)\right|^2\\
&\quad\le\mathrm{e}^{\frac{k_2-k_1}{2}t}\left|2(a\nabla f+b\nabla P_tf)-\mathrm{e}^{-\frac{k_2+k_1}{2}t}\,//_{0,t}^{-1}\nabla f(X_t)\right|^2\\
&\qquad+\mathrm{e}^{-(k_2+k_1)t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\mathrm{e}^{\frac{k_2-k_1}{2}t}\,|\nabla f|^2(X_t)\\
&\quad\le 4\,\mathrm{e}^{\frac{k_2-k_1}{2}t}\,|a\nabla f+b\nabla P_tf|^2-4\,\mathrm{e}^{-k_1t}\left\langle a\nabla f+b\nabla P_tf,//_{0,t}^{-1}\nabla f(X_t)\right\rangle+\mathrm{e}^{-2k_1t}\,|\nabla f|^2(X_t).
\end{aligned}$$

Expanding the terms above yields

$$\begin{aligned}
&\left|Q_t//_{0,t}^{-1}\nabla f(X_t)\right|^2-\mathrm{e}^{-2k_1t}\,|\nabla f|^2(X_t)\\
&\quad\le 4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|a\nabla f+b\nabla P_tf|^2\right.\\
&\qquad\left.+\left\langle a\nabla f+b\nabla P_tf,Q_t//_{0,t}^{-1}\nabla f(X_t)-\mathrm{e}^{-k_1t}\,//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right].
\end{aligned}\tag{2.5}$$

We observe that $|\nabla P_tf|^2\le\mathbb{E}(|Q_t//_{0,t}^{-1}\nabla f(X_t)|^2)$. Hence, by taking expectation on both sides of inequality (2.5), we arrive at

$$\begin{aligned}
&|\nabla P_tf|^2-\mathrm{e}^{-2k_1t}\,P_t|\nabla f|^2\\
&\le 4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|a\nabla f+b\nabla P_tf|^2+\left\langle a\nabla f+b\nabla P_tf,\nabla P_tf-\mathrm{e}^{-k_1t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right].
\end{aligned}\tag{2.6}$$

Thus, letting $a=1$, $b=0$, respectively $a=0$, $b=1$, we complete the proof of (ii) and (ii').

(b) (i) $\Rightarrow$ (iii), (iii'): By Itô's formula, we have

$$\begin{aligned}
\mathrm{d}(P_{t-s}f^{2/p})^p(X_s)&=\mathrm{d}M_s+(L+\partial_s)\left(P_{t-s}f^{2/p}(X_s)\right)^p\,\mathrm{d}s\\
&=\mathrm{d}M_s+p(p-1)\left(P_{t-s}f^{2/p}(X_s)\right)^{p-2}\,|\nabla P_{t-s}f^{2/p}|^2(X_s)\,\mathrm{d}s
\end{aligned}\tag{2.7}$$

where M_s is a local martingale. In addition,

$$\begin{aligned}\left|\nabla P_{t-s} f^{2/p}(X_s)\right|^2 &= \left|\mathbb{E}\left[//_{0,s}^{-1} Q_{s,t} //_{s,t}^{-1} \nabla f^{2/p}(X_t) | \mathscr{F}_s\right]\right|^2 \\ &= \frac{4}{p^2}\left|\mathbb{E}\left[f^{(2-p)/p}(X_t) //_{0,s}^{-1} Q_{s,t} //_{s,t}^{-1} \nabla f(X_t) | \mathscr{F}_s\right]\right|^2 \\ &\leq \frac{4}{p^2}(P_{t-s} f^{2(2-p)/p})(X_s)\, \mathbb{E}\left[|Q_{s,t} //_{s,t}^{-1} \nabla f(X_t)|^2 | \mathscr{F}_s\right], \end{aligned} \tag{2.8}$$

where for fixed $s \geq 0$, the two-parameter family $Q_{s,t}$ of random automorphisms of $T_{X_s}M$ solves the pathwise equation

$$\frac{\mathrm{d}Q_{s,t}}{\mathrm{d}t} = -Q_{s,t}\, \mathrm{Ric}^Z_{//_{s,t}}, \quad Q_{s,s} = \mathrm{id}_{T_{X_s}M}, \quad t \geq s.$$

Analogously to Eq. (2.2) we have $\mathrm{Ric}^Z_{//_{s,t}} = //_{s,t}^{-1} \circ \mathrm{Ric}^Z_{X_t} \circ //_{s,t}$.

As $2-p \in [0,1]$, by Jensen's inequality, we first observe that

$$P_{t-s} f^{2(2-p)/p} \leq (P_{t-s} f^{2/p})^{2-p}.$$

Combining this with (2.7) and (2.8), we obtain

$$\mathrm{d}(P_{t-s} f^{2/p})^p(X_s) \leq \mathrm{d}M_s + \frac{4(p-1)}{p}\, \mathbb{E}\left[|Q_{s,t} //_{s,t}^{-1} \nabla f(X_t)|^2 | \mathscr{F}_s\right] \mathrm{d}s.$$

Integrating both sides from 0 to t and taking expectation, we arrive at

$$\frac{p(P_t f^2 - (P_t f^{2/p})^p)}{4(p-1)} \leq \int_0^t \mathbb{E}\left[|Q_{s,t} //_{s,t}^{-1} \nabla f(X_t)|^2\right] \mathrm{d}s. \tag{2.9}$$

Now, using similar arguments as in **(a)**, we obtain

$$\begin{aligned} &\mathbb{E}\left[|Q_{s,t} //_{s,t}^{-1} \nabla f(X_t)|^2 | \mathscr{F}_s\right] \\ &\quad \leq \mathrm{e}^{-2k_1(t-s)} P_{t-s}|\nabla f|^2(X_s) + 4\left(\mathrm{e}^{\frac{k_2-k_1}{2}(t-s)} - 1\right)|\nabla f|^2(X_s) \\ &\qquad + 4\mathbb{E}\left[\left\langle \nabla f(X_s), \nabla P_{t-s} f(X_s) - \mathrm{e}^{-k_1(t-s)} //_{s,t}^{-1} \nabla f(X_t)\right\rangle \Big| \mathscr{F}_s\right] \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\mathbb{E}\left[|Q_{s,t} //_{s,t}^{-1} \nabla f(X_t)|^2 | \mathscr{F}_s\right] \\ &\quad \leq \mathrm{e}^{-2k_1(t-s)} P_{t-s}|\nabla f|^2(X_s) + 4\,\mathrm{e}^{\frac{k_2-k_1}{2}(t-s)} |\nabla P_{t-s} f|^2(X_s) \\ &\qquad - 4\,\mathrm{e}^{-k_1(t-s)}\, \mathbb{E}\left[\left\langle \nabla f(X_s), //_{s,t}^{-1} \nabla f(X_t)\right\rangle \big| \mathscr{F}_s\right]. \end{aligned} \tag{2.11}$$

Together with (2.9), the proof of (iii) and (iii') is completed.

(c) (i) $\Rightarrow$ (iv) and (iv'): By Itô's formula, we have

$$\begin{aligned} \mathrm{d}(P_{t-s} f^2)(X_s) \log(P_{t-s} f^2)(X_s) &= \mathrm{d}\tilde{M}_s + (L + \partial_s)(P_{t-s} f^2)(X_s) \log(P_{t-s} f^2)(X_s)\, \mathrm{d}s \\ &= \mathrm{d}\tilde{M}_s + \frac{1}{P_{t-s} f^2(X_s)} |\nabla P_{t-s} f^2|^2(X_s)\, \mathrm{d}s \end{aligned} \tag{2.12}$$

where $\tilde{M}_s$ is a local martingale. Furthermore, using the derivative formula, we have

$$\begin{aligned}|\nabla P_{t-s}f^2|^2(X_s) &= \left|\mathbb{E}\left[//_{0,s}^{-1}Q_{s,t}//_{s,t}^{-1}\nabla f^2(X_t)|\mathscr{F}_s\right]\right|^2\\ &\leq 4P_{t-s}f^2(X_s)\mathbb{E}\left[|Q_{s,t}//_{s,t}^{-1}\nabla f(X_t)|^2|\mathscr{F}_s\right].\end{aligned}$$

Combining this with (2.12), we obtain

$$\mathrm{d}(P_{t-s}f^2)(X_s)\log(P_{t-s}f^2)(X_s) \leq \mathrm{d}\tilde{M}_s + 4\mathbb{E}\left[|Q_{s,t}//_{s,t}^{-1}\nabla f(X_t)|^2|\mathscr{F}_s\right]\mathrm{d}s.$$

Using the estimates in (2.10) and (2.11) for $\mathbb{E}[|Q_{s,t}//_{s,t}^{-1}\nabla f(X_t)|^2|\mathscr{F}_s]$, we finish the proof by integrating from 0 to t and taking expectation on both sides. □

Remark 2.4 Actually, when $k_1 \neq k_2$, the following inequality can be derived by minimizing the upper bound in (2.6) over a,b under the restriction $a+b=1$:

$$\begin{aligned}&|\nabla P_t f|^2 - \mathrm{e}^{-2k_1t}P_t|\nabla f|^2\\ &\leq \Bigg\{4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f|^2 + \langle\nabla f,\nabla P_t f\rangle - \mathrm{e}^{-k_1t}\left\langle\nabla f,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right]\\ &\qquad - \frac{\left\langle\nabla P_t f-\nabla f, 2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\nabla f+\nabla P_t f - \mathrm{e}^{-k_1t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle^2}{\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla P_t f-\nabla f|^2}\Bigg\}\wedge 0\\ &= \Bigg\{4\left[\mathrm{e}^{\frac{k_2-k_1}{2}t}|\nabla P_t f|^2 - \mathrm{e}^{-k_1t}\left\langle\nabla P_t f,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right]\\ &\qquad - \frac{\left\langle\nabla P_t f-\nabla f, \left(2\,\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\nabla P_t f - \mathrm{e}^{-k_1t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle^2}{\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla P_t f-\nabla f|^2}\Bigg\}\wedge 0. \qquad (2.13)\end{aligned}$$

It is easy to see that this bound is sharper than the ones given in Theorem 2.1 (ii) and (ii').

Proof Inequality (2.13) can be checked as follows. First recall estimate (2.6):

$$\begin{aligned}&|\nabla P_t f|^2 - \mathrm{e}^{-2k_1t}P_t|\nabla f|^2\\ &\leq 4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|a\nabla f+b\nabla P_t f|^2 + \left\langle a\nabla f+b\nabla P_t f,\nabla P_t f-\mathrm{e}^{-k_1t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right].\end{aligned}$$

Taking $b=1-a$ in the terms of the right-hand side, we get

$$\begin{aligned}&4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|a\nabla f+b\nabla P_t f|^2 + \left\langle a\nabla f+b\nabla P_t f,\nabla P_t f-\mathrm{e}^{-k_1t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right]\\ &= 4\Bigg[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f-\nabla P_t f|^2a^2\\ &\quad + \left\langle\nabla f-\nabla P_t f,(2\,\mathrm{e}^{\frac{k_2-k_1}{2}t}-1)\nabla P_t f-\mathrm{e}^{-k_1t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle a\\ &\quad + \mathrm{e}^{\frac{k_2-k_1}{2}t}|\nabla P_t f|^2 - \mathrm{e}^{-k_1t}\left\langle\nabla P_t f,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\Bigg]. \qquad (2.14)\end{aligned}$$

For the value

$$a=a_0=-\frac{\left\langle \nabla f-\nabla P_t f, (2\,\mathrm{e}^{\frac{k_2-k_1}{2}t}-1)\nabla P_t f-\mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle}{2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f-\nabla P_t f|^2}, \tag{2.15}$$

the expression in (2.14) reaches its minimum as a function of a:

$$\begin{aligned}&4\left[\mathrm{e}^{\frac{k_2-k_1}{2}t}\,|\nabla P_t f|^2-\mathrm{e}^{-k_1 t}\left\langle \nabla P_t f, \mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right]\\&\qquad-\frac{\left\langle \nabla f-\nabla P_t f, (2\,\mathrm{e}^{\frac{k_2-k_1}{2}t}-1)\nabla P_t f-\mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle^2}{\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f-\nabla P_t f|^2}.\end{aligned}$$

Similarly, substituting $a=1-b$ in the terms on the left-hand side of Eq. (2.14), we get

$$\begin{aligned}&4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|a\nabla f+b\nabla P_t f|^2+\left\langle a\nabla f+b\nabla P_t f, \nabla P_t f-\mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right]\\&=4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f-\nabla P_t f|^2 b^2\right.\\&\quad+\left\langle \nabla f-\nabla P_t f, 2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\nabla f+\nabla P_t f-\mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle b\\&\quad\left.+\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f|^2+\left\langle \nabla f, \nabla P_t f-\mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right].\end{aligned} \tag{2.16}$$

It is easy to see that for

$$b=1-a_0=-\frac{\left\langle \nabla f-\nabla P_t f, 2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\nabla f+\nabla P_t f-\mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle}{2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f-\nabla P_t f|^2},$$

expression (2.16) reaches its minimal value:

$$\begin{aligned}&4\left[\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla f|^2+\langle \nabla f, \nabla P_t f\rangle-\mathrm{e}^{-k_1 t}\left\langle \nabla f, \mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle\right]\\&\qquad-\frac{\left\langle \nabla P_t f-\nabla f, 2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)\nabla f+\nabla P_t f-\mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle^2}{\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|\nabla P_t f-\nabla f|^2}.\end{aligned}$$

As the minimum is unique, we conclude that the upper bounds (2.14) and (2.16) are indeed equivalent. □

To prove that the inequalities in (ii)-(iv), (ii')-(iv') imply condition (i), we use the following lemma.

Lemma 2.5 *For $x\in M$, let $X\in T_xM$ with $|X|=1$. Let $f\in C_0^\infty(M)$ such that $\nabla f(x)=X$ and* $\mathrm{Hess}_f(x)=0$*, and let $f_n=n+f$ for $n\geq 1$. Then,*

(i) *for any $p>0$,*

$$\mathrm{Ric}^Z(X,X)=\lim_{t\to 0}\frac{P_t|\nabla f|^p(x)-|\nabla P_t f|^p(x)}{pt};$$

(ii) *for any* $p > 1$,

$$\mathrm{Ric}^Z(X,X) = \lim_{n\to\infty}\lim_{t\to 0}\frac{1}{t}\left(P_t|\nabla f_n|^2 - \frac{p\left\{P_t f_n^2 - (P_t f_n^{2/p})^p\right\}}{4(p-1)t}\right)(x);$$

(iii) $\mathrm{Ric}^Z(X,X)$ *can be calculated as*

$$\mathrm{Ric}^Z(X,X) = \lim_{n\to\infty}\lim_{t\to 0}\frac{1}{4t^2}\left\{4tP_t|\nabla f_n|^2 + (P_t f_n^2)\log P_t f_n^2 - P_t f_n^2\log f_n^2\right\}(x);$$

(iv) $\mathrm{Ric}^Z(X,X)$ *is also given by the following two limits:*

$$\begin{aligned}\mathrm{Ric}^Z(X,X) =& \lim_{t\to 0}\frac{\left\{\left\langle\nabla f, \mathbb{E}[//_{0,t}^{-1}\nabla f(X_t)]\right\rangle - \langle\nabla f, \nabla P_t f\rangle\right\}(x)}{t}\\ =& \lim_{t\to 0}\frac{\left\{\left\langle\nabla P_t f, \mathbb{E}[//_{0,t}^{-1}\nabla f(X_t)]\right\rangle - |\nabla P_t f|^2\right\}(x)}{t}.\end{aligned}$$

Proof The formulae in (i)–(iii) can be found in [19, Theorem 2.2.4] (see also [2, 17]). Then, by Itô's formula and [19, Lemma 2.1.4.], for $f \in C_0^\infty(M)$ such that $\mathrm{Hess}_f(x) = 0$, we get

$$\begin{aligned}\mathbb{E}\left[//_{0,t}^{-1}\nabla f(X_t)\right] &= \nabla f(x) + \mathbb{E}\left[\int_0^{t\wedge\sigma_r} //_{0,s}^{-1}(\Box + \nabla_Z)(\nabla f)(X_s)\,\mathrm{d}s\right] + \mathrm{o}(t)\\ &= \nabla f(x) + //_{0,s}^{-1}(\Box + \nabla_Z)(\nabla f)(x)\,t + \mathrm{o}(t),\end{aligned}$$

where $\sigma_r = \inf\{t \geq 0 : X_t \notin B(x,r)\}$ and $\Box = -\nabla^*\nabla$ is the connection Laplacian (or rough Laplacian) acting on $\Gamma(TM)$. From this, the two expressions in (iv) are easily derived using Taylor expansions:

$$\begin{aligned}&\left\langle\nabla f, \mathbb{E}[//_{0,t}^{-1}\nabla f(X_t)]\right\rangle(x) - \langle\nabla f, \nabla P_t f\rangle(x)\\ &= [\langle\nabla f, (\Box + \nabla_Z)\nabla f\rangle(x) - \langle\nabla f, \nabla Lf\rangle(x)]t + \mathrm{o}(t)\\ &= \mathrm{Ric}^Z(\nabla f, \nabla f)(x)\,t + \mathrm{o}(t)\end{aligned}$$

and

$$\begin{aligned}&\left\langle\nabla P_t f, \mathbb{E}[//_{0,t}^{-1}\nabla f(X_t)]\right\rangle(x) - \langle\nabla P_t f, \nabla P_t f\rangle(x)\\ &= (\langle\nabla f, (\Box + \nabla_Z)\nabla f\rangle(x) - \langle\nabla f, \nabla Lf\rangle(x))t + \mathrm{o}(t)\\ &= \mathrm{Ric}^Z(\nabla f, \nabla f)(x)\,t + \mathrm{o}(t).\end{aligned}$$

Here, we use the fact that for $f \in C_0^\infty(M)$ such that $\mathrm{Hess}_f(x) = 0$, the following equation holds:

$$\mathrm{Ric}^Z(\nabla f, \nabla f)(x) = \langle(\Box + \nabla_Z)\nabla f, \nabla f\rangle(x) - \langle\nabla Lf, \nabla f\rangle(x).$$

$\square$

Using Lemma 2.5, we are now able to complete the proof of the main result.

Proof (Proof of Theorem 2.1.)

Part II "(ii) and (ii') ⇒ (i)":
Fix $x\in M$ and let $f\in C_0^\infty(M)$ such that $\mathrm{Hess}_f(x)=0$. Without explicit mention, the following computations are all taken implicitly at the point x. First, we rewrite the inequalities (ii) and (ii') as follows,

$$\begin{aligned}
&\frac{|\nabla P_t f|^2-P_t|\nabla f|^2}{2t}+\frac{1-\mathrm{e}^{-2k_1 t}}{2t}P_t|\nabla f|^2\\
&\leq \frac{2}{t}\left(\mathrm{e}^{\frac{k_2-k_1}{2}t}-1\right)|a\nabla f+b\nabla P_t f|^2\\
&\quad+2\frac{\langle a\nabla f+b\nabla P_t f,\nabla P_t f\rangle-\langle a\nabla f+b\nabla P_t f,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\rangle}{t}\\
&\quad+\frac{2}{t}\left(1-\mathrm{e}^{-k_1 t}\right)\mathbb{E}\langle a\nabla f+b\nabla P_t f,//_{0,t}^{-1}\nabla f(X_t)\rangle
\end{aligned}$$

where $a=1$, $b=0$ or $a=0$, $b=1$. Letting $t\to 0$, by Lemma 2.5, we obtain

$$-\mathrm{Ric}^Z(\nabla f,\nabla f)+k_1|\nabla f|^2\leq (k_2-k_1)|\nabla f|^2-2\,\mathrm{Ric}^Z(\nabla f,\nabla f)+2k_1|\nabla f|^2$$

which implies that

$$\mathrm{Ric}^Z(\nabla f,\nabla f)\leq k_2|\nabla f|^2.$$

"(iii), (iv), (iii'), (iv') ⇒ (i)": We only prove that "(iii) and (iii') imply (i)", as the inequalities (iv) and (iv') can be considered as limits of the inequalities (iii) and (iii') as $p\downarrow 1$.

For $x\in M$ and $f\in C_0^\infty(M)$ such that $\mathrm{Hess}_f(x)=0$, let $f_n:=f+n$ and rewrite (iii) as

$$\begin{aligned}
&\frac{1}{t^2}\left(\frac{p(P_t f_n^2-(P_t f_n^{2/p})^p)}{4(p-1)}-tP_t|\nabla f_n|^2\right)-\frac{1}{t^2}\int_0^t[1-\mathrm{e}^{-2k_1(t-s)}]\mathrm{d}s\times P_t|\nabla f_n|^2\\
&\leq \frac{4}{t^2}\int_0^t\left(\mathrm{e}^{\frac{k_2-k_1}{2}(t-r)}-1\right)P_r|\nabla f_n|^2\,\mathrm{d}r\\
&\quad+\frac{4}{t^2}\int_0^t\left(1-\mathrm{e}^{-k_1(t-r)}\right)\mathbb{E}\left\langle\nabla f_n(X_r),//_{r,t}^{-1}\nabla f_n(X_t)\right\rangle\,\mathrm{d}r\\
&\quad+\frac{4}{t^2}\int_0^t\mathbb{E}\left\langle\nabla f_n(X_r),\nabla P_{t-r}f_n(X_r)-//_{r,t}^{-1}\nabla f_n(X_t)\right\rangle\,\mathrm{d}r.
\end{aligned}\tag{2.17}$$

Now letting $t\to 0$, by Lemma 2.5 (ii), the terms on the right-hand side become

$$-\mathrm{Ric}^Z(\nabla f,\nabla f)+k_1|\nabla f|^2.$$

For the terms on the left-hand side of (2.17), we have the following expansions:

$$\begin{aligned}
&\frac{4}{t^2}\int_0^t\left(\mathrm{e}^{\frac{k_2-k_1}{2}(t-r)}-1\right)P_r|\nabla f_n|^2\,\mathrm{d}r\\
&\quad=\frac{4}{t^2}\int_0^t\left(\mathrm{e}^{\frac{k_2-k_1}{2}(t-r)}-1\right)(|\nabla f_n|^2+\mathrm{o}(1))\,\mathrm{d}r\\
&\quad=(k_2-k_1)|\nabla f|^2+\mathrm{o}(1);\\
&\frac{4}{t^2}\int_0^t\left(1-\mathrm{e}^{-k_1(t-r)}\right)\mathbb{E}\left\langle\nabla f_n(X_r),//_{r,t}^{-1}\nabla f_n(X_t)\right\rangle\,\mathrm{d}r\\
&\quad=\frac{4}{t^2}\int_0^t\left(1-\mathrm{e}^{-k_1(t-r)}\right)(|\nabla f_n|^2+\mathrm{o}(1))\,\mathrm{d}r\\
&\quad=2k_1|\nabla f|^2+\mathrm{o}(1);
\end{aligned}$$

$$
\begin{aligned}
&\frac{4}{t^2}\int_0^t \mathbb{E}\left\langle \nabla f_n(X_r), \nabla P_{t-r} f_n(X_r) - //_{r,t}^{-1}\nabla f_n(X_t)\right\rangle \mathrm{d}r \\
&\quad = \frac{4}{t^2}\int_0^t (\mathrm{Ric}^Z(\nabla f_n, \nabla f_n)(t-r) + \mathrm{o}(t) + \mathrm{o}(r))\,\mathrm{d}r \\
&\quad = 2\,\mathrm{Ric}^Z(\nabla f, \nabla f) + \mathrm{o}(1).
\end{aligned}
$$

Therefore, letting $t \to 0$ in (2.17), we arrive at

$$
-\mathrm{Ric}^Z(\nabla f, \nabla f) + k_1|\nabla f|^2 \le (-2\,\mathrm{Ric}^Z(\nabla f, \nabla f) + (k_2+k_1)|\nabla f|^2) \wedge 0,
$$

i.e.,

$$
k_1|\nabla f|^2 \le \mathrm{Ric}^Z(\nabla f, \nabla f) \le k_2|\nabla f|^2.
$$

The proof of "(iii') implies (i)" is similar. We skip the details here. □

Remark 2.6 In the proof of Theorem 2.1 "(ii) (ii') $\Rightarrow$ (i)", we take into account that for a and b satisfying $a+b=1$, trivially $\lim_{t\to 0}(a\nabla f + b\nabla P_t f) = \nabla f$ holds. However, when choosing $a = a_0$ as in (2.15) for the proof of inequality (2.6), obviously a_0 depends on t, and thus we get

$$
\begin{aligned}
&\lim_{t\to 0}(a_0\nabla f + (1-a_0)\nabla P_t f) \\
&= \lim_{t\to 0}(\nabla f + (1-a_0)(\nabla P_t f - \nabla f)) \\
&= \nabla f - \lim_{t\to 0}\frac{\left\langle \nabla f - \nabla P_t f, 2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t} - 1\right)\nabla f + \nabla P_t f - \mathrm{e}^{-k_1 t}\,\mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle}{2\left(\mathrm{e}^{\frac{k_2-k_1}{2}t} - 1\right)|\nabla f - \nabla P_t f|^2}(\nabla P_t f - \nabla f) \\
&= \nabla f + \lim_{t\to 0}\frac{\langle (\nabla Lf)t + \mathrm{o}(t), k_2\nabla f t + (\nabla Lf)t - ((\Box + \nabla_Z)\nabla f)t + \mathrm{o}(t)\rangle}{(k_2-k_1)|\nabla Lf|^2 t^3 + \mathrm{o}(t^3)}(\nabla Lf)t \\
&= \nabla f + \frac{\langle \nabla Lf, k_2\nabla f + \nabla Lf - (\Box + \nabla_Z)\nabla f\rangle}{(k_2-k_1)|\nabla Lf|^2}\nabla Lf \neq \nabla f.
\end{aligned}
$$

Actually, dividing both hands of inequality (2.13) by $2t$ and letting $t \to 0$, we obtain

$$
k_1|\nabla f|^2 \le \mathrm{Ric}(\nabla f, \nabla f) \le k_2|\nabla f|^2 - \frac{\langle \nabla Lf, k_2\nabla f + \nabla Lf - (\Box + \nabla_Z)\nabla f\rangle^2}{(k_2-k_1)|\nabla Lf|^2}\ (\le k_2|\nabla f|^2).
$$

3 Pointwise characterizations of curvature bounds

Consider a Riemannian manifold M possibly with non-empty boundary ∂M, and let X_t be a reflecting diffusion processes generated by $L = \Delta + Z$. We assume that X_t is non-explosive. It is well known that the reflecting process X_t solves the equation

$$
\mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z(X_t)\mathrm{d}t + N(X_t)\mathrm{d}l_t,
$$

where u_t is a horizontal lift of X_t to the orthonormal frame bundle, N the inward normal unit vector field on ∂M and l_t the local time of X_t supported on ∂M, see [19] for details. Again,

$$
//_{r,s} = u_s \circ u_r^{-1} \colon T_{X_r}M \to T_{X_s}M, \quad r \le s,
$$

denotes parallel transport along $t \mapsto X_t$. Finally, let II be the second fundamental form of the boundary:

$$\mathrm{II}(X,Y) = -\langle \nabla_X N, Y\rangle, \quad X,Y \in T_x\partial M,\ x \in \partial M.$$

In this section, we extend the results of Section 2 in order to characterize pointwise bounds on Ric^Z and II. To this end, for continuous functions K_1, K_2, σ_1 and σ_2 on M, let

$$\mathbb{K}_1(X_{[s,t]}) = \int_s^t K_1(X_r)\,\mathrm{d}r + \sigma_1(X_r)\,\mathrm{d}l_r, \quad \mathbb{K}_2(X_{[s,t]}) = \int_s^t K_2(X_r)\,\mathrm{d}r + \sigma_2(X_r)\,\mathrm{d}l_r$$

where $X_{[s,t]} = \{X_r\colon r \in [s,t]\}$. Furthermore, let

$$C_N^\infty(M) := \{f \in C_0^\infty(M)\colon Nf|_{\partial M} = 0\}.$$

Finally let

$$(P_t f)(x) = \mathbb{E}[f(X_t^x)], \quad f \in C_b(M),$$

be the semigroup with Neumann boundary condition generated by L.

The result of this section can be presented as follows.

Theorem 3.1 *We keep the assumptions and notations from above. Let $x \mapsto K_1(x)$ and $x \mapsto K_2(x)$ be two continuous functions on M such that $K_1 \le K_2$. In addition, let $x \mapsto \sigma_1(x)$ and $x \mapsto \sigma_2(x)$ be two continuous functions on ∂M such that $\sigma_1 \le \sigma_2$. Assume that*

$$\mathbb{E}\big[\mathrm{e}^{-(2+\varepsilon)\mathbb{K}_1(X_{[s,t]})} + \mathrm{e}^{(\frac{1}{2}+\varepsilon)(\mathbb{K}_2-\mathbb{K}_1)(X_{[s,t]})}\big] < \infty, \quad \textit{for some } \varepsilon > 0 \textit{ and all } t > s \ge 0. \tag{3.1}$$

The following statements are equivalent:

(i) *Curvature* Ric^Z *and second fundamental form* II *satisfy the bounds*

$$K_1(x) \le \mathrm{Ric}^Z(x) \le K_2(x), \quad x \in M, \quad \textit{and} \quad \sigma_1(x) \le \mathrm{II}(x) \le \sigma_2(x), \quad x \in \partial M.$$

(ii) *For $f \in C_N^\infty(M)$ and $t > 0$,*

$$\begin{aligned}&|\nabla P_t f|^2 - \mathbb{E}\left[\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}|\nabla f|^2(X_t)\right]\\ &\le 4\bigg[\Big(\mathbb{E}\,\mathrm{e}^{\frac{1}{2}(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]}))}-1\Big)|\nabla f|^2 + \langle \nabla f, \nabla P_t f\rangle\\ &\quad - \Big\langle \nabla f, \mathbb{E}\big[\mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})}//_{0,t}^{-1}\nabla f(X_t)\big]\Big\rangle\bigg] \wedge 0.\end{aligned}$$

(ii') *For $f \in C_N^\infty(M)$ and $t > 0$,*

$$\begin{aligned}&|\nabla P_t f|^2 - \mathbb{E}\,\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}|\nabla f|^2(X_t)\\ &\le 4\bigg[\mathbb{E}\,\mathrm{e}^{\frac{1}{2}(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]}))}|\nabla P_t f|^2 - \Big\langle \nabla P_t f, \mathbb{E}\big[\mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})}//_{0,t}^{-1}\nabla f(X_t)\big]\Big\rangle\bigg] \wedge 0.\end{aligned}$$

(iii) *For $f \in C_N^\infty(M)$, $p \in (1,2]$ and $t > 0$,*

$$\begin{aligned}&\frac{p(P_t f^2 - (P_t f^{2/p})^p)}{4(p-1)} - \mathbb{E}\left[\int_0^t \mathrm{e}^{-2\mathbb{K}_1(X_{[r,t]})}\,\mathrm{d}r \times |\nabla f|^2(X_t)\right]\\ &\le 4\int_0^t \bigg[\Big(\mathbb{E}\,\mathrm{e}^{\frac{1}{2}(\mathbb{K}_2(X_{[r,t]})-\mathbb{K}_1(X_{[r,t]}))}-1\Big)P_r|\nabla f|^2\\ &\quad + \mathbb{E}\Big\langle \nabla f(X_r), \nabla P_{t-r}f(X_r) - \mathrm{e}^{-\mathbb{K}_1(X_{[r,t]})}//_{r,t}^{-1}\nabla f(X_t)\Big\rangle\bigg]\,\mathrm{d}r \wedge 0.\end{aligned}$$

(iii') *For $f \in C_N^\infty(M)$, $p \in (1,2]$ and $t > 0$,*

$$\begin{aligned}&\frac{p(P_t f^2 - (P_t f^{2/p})^p)}{4(p-1)} - \mathbb{E}\left[\int_0^t \mathrm{e}^{-2\mathbb{K}_1(X_{[r,t]})}\,\mathrm{d}r \times |\nabla f|^2(X_t)\right]\\ &\le 4\int_0^t \Bigg[\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}(\mathbb{K}_2(X_{[r,t]})-\mathbb{K}_1(X_{[r,t]}))}\right] P_r|\nabla P_{t-r} f|^2\\ &\quad - \mathbb{E}\left[\mathrm{e}^{-\mathbb{K}_1(X_{[r,t]})}\left\langle \nabla f(X_r), //_{r,t}^{-1}\nabla f(X_t)\right\rangle\right]\Bigg]\,\mathrm{d}r \wedge 0.\end{aligned}$$

(iv) *For $f \in C_N^\infty(M)$ and $t > 0$,*

$$\begin{aligned}&\frac{1}{4}\left(P_t(f^2\log f^2) - P_t f^2 \log P_t f^2\right) - \mathbb{E}\left[\int_0^t \mathrm{e}^{-2\mathbb{K}_1(X_{[r,t]})}\,\mathrm{d}r \times |\nabla f|^2(X_t)\right]\\ &\le 4\int_0^t \Bigg[\left(\mathbb{E}\,\mathrm{e}^{\frac{1}{2}(\mathbb{K}_2(X_{[r,t]})-\mathbb{K}_1(X_{[r,t]}))} - 1\right) P_r|\nabla f|^2\\ &\quad + \mathbb{E}\left\langle \nabla f(X_r), \nabla P_{t-r} f(X_r) - \mathrm{e}^{-\mathbb{K}_1(X_{[r,t]})}\, //_{r,t}^{-1}\nabla f(X_t)\right\rangle\Bigg]\,\mathrm{d}r \wedge 0.\end{aligned}$$

(iv') *For $f \in C_N^\infty(M)$ and $t > 0$,*

$$\begin{aligned}&\frac{1}{4}\left(P_t(f^2\log f^2) - P_t f^2 \log P_t f^2\right) - \mathbb{E}\left[\int_0^t \mathrm{e}^{-2\mathbb{K}_1(X_{[r,t]})}\,\mathrm{d}r \times |\nabla f|^2(X_t)\right]\\ &\le 4\int_0^t \Bigg[\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}(\mathbb{K}_2(X_{[r,t]})-\mathbb{K}_1(X_{[r,t]}))}\right] P_r|\nabla P_{t-r} f|^2\\ &\quad - \mathbb{E}\left[\mathrm{e}^{-\mathbb{K}_1(X_{[r,t]})}\left\langle \nabla f(X_r), //_{r,t}^{-1}\nabla f(X_t)\right\rangle\right]\Bigg]\,\mathrm{d}r \wedge 0.\end{aligned}$$

To prove the theorem, we need the following lemmas.

Lemma 3.2 *([19, Lemma 3.1.2]) Let X_t^x be the reflecting diffusion process generated by L such that $X_0 = x$ and l_t^x the corresponding local time on the boundary.*

(i) *For any $x \in M$ and $r_0 > 0$, there exists a constant $c > 0$ such that*

$$\mathbb{P}\{\sigma_r \le t\} \le \mathrm{e}^{-cr^2/t}, \quad \textit{for all } r \in [0, r_0] \textit{ and } t > 0,$$

where $\sigma_r = \inf\{s \ge 0\colon\ \rho(x, X_s^x) \ge r\}$.

(ii) *Let $x \in \partial M$ and r as above. Then:*
  (a) *$\mathbb{E}^x[\mathrm{e}^{\lambda l_{t\wedge\sigma_r}}] < \infty$ for any $\lambda > 0$ and there exists $c > 0$ such that $\mathbb{E}^x[l_{t\wedge\sigma_r}^2] \le c(t+t^2)$;*
  (b) *$\mathbb{E}^x[l_{t\wedge\sigma_r}] = \frac{2\sqrt{t}}{\sqrt{\pi}} + \mathrm{o}(t^{1/2})$ holds for small $t > 0$.*

By means of Lemma 3.2, we can derive pointwise formulae for Ric^Z and II.

Lemma 3.3 *Let $x \in \mathring{M} =: M \setminus \partial M$ and $X \in T_x M$ with $|X| = 1$. Let $f \in C_0^\infty(M)$ such that $Nf|_{\partial M} = 0$, $\mathrm{Hess}_f(x) = 0$ and $\nabla f(x) = X$ and let $f_n = f + n$ for $n \ge 1$. Then all assertions of Lemma 2.5 hold.*

Proof Let $r > 0$ be such that $B(x,r) \subset \mathring{M}$ and $|\nabla f| \ge \frac{1}{2}$ on $B(x,r)$. Due to Lemma 3.2, the proof of Lemma 2.5 applies to the present situation, using $t \wedge \sigma_r$ to replace t, so that the boundary condition is avoided. We refer the reader to the proof of [19, Theorem 3.2.3] for more explanation. $\square$

Lemma 3.4 *Let $x \in \partial M$ and $X \in T_xM$ with $|X| = 1$.*

(1) *For any $f \in C_0^\infty(M)$ such that $\nabla f(x) = X$, and for any $p > 0$, we have*

$$\begin{aligned}
\mathrm{II}(X,X) &= \lim_{t\downarrow 0} \frac{\sqrt{\pi}}{2p\sqrt{t}} \left\{P_t|\nabla f|^p - |\nabla f|^p\right\}(x) \\
&= \lim_{t\downarrow 0} \frac{\sqrt{\pi}}{2p\sqrt{t}} \left\{P_t|\nabla f|^p - |\nabla P_t f|^p\right\}(x) \\
&= \lim_{t\to 0} \frac{\sqrt{\pi}\left\{\left\langle \nabla f, \mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle - \langle \nabla f, \nabla P_t f\rangle\right\}(x)}{2\sqrt{t}} \qquad (3.2)\\
&= \lim_{t\to 0} \frac{\sqrt{\pi}\left\{\left\langle \nabla P_t f, \mathbb{E}//_{0,t}^{-1}\nabla f(X_t)\right\rangle - |\nabla P_t f|^2\right\}(x)}{2\sqrt{t}}. \qquad (3.3)
\end{aligned}$$

(2) *If moreover $f > 0$, then for any $p \in [1,2]$,*

$$\begin{aligned}
\mathrm{II}(X,X) &= -\lim_{t\downarrow 0} \frac{3}{8}\sqrt{\frac{\pi}{t}} \left\{|\nabla f|^2 + \frac{p[(P_t f^{2/p})^p - P_t f^2]}{4(p-1)t}\right\}(x) \\
&= -\lim_{t\downarrow 0} \frac{3}{8}\sqrt{\frac{\pi}{t}} \left\{|\nabla P_t f|^2 + \frac{p[(P_t f^{2/p})^p - P_t f^2]}{4(p-1)t}\right\}(x),
\end{aligned}$$

where when $p = 1$, we interpret the quotient $\dfrac{(P_t f^{2/p})^p - P_t f^2}{p-1}$ as the limit

$$\lim_{p\downarrow 1} \frac{(P_t f^{2/p})^p - P_t f^2}{p-1} = (P_t f^2)\log P_t f^2 - P_t(f^2 \log f^2).$$

Proof We only need to prove formulas (3.2) and (3.3). For the remaining statements we refer to [19, Theorem 3.2.4]. Let $r > 0$ such that $|\nabla f| \geq 1/2$ on $B(x,r)$, and let

$$\sigma_r := \inf\{s \geq 0 : X_s \notin B(x,r)\}.$$

Then, by Itô's formula and Lemma 3.2 (ii) (b), we get

$$\begin{aligned}
&\mathbb{E}\left[//_{0,t}^{-1}\nabla f(X_t)\right] \\
&\quad = \nabla f(x) + \mathbb{E}\left[\int_0^{t\wedge\sigma_r} //_{0,s}^{-1}(\Box + \nabla_Z)(\nabla f)(X_s)\,\mathrm{d}s + //_{0,s}^{-1}\nabla_N(\nabla f)(X_s)\,\mathrm{d}l_s\right] + \mathrm{o}(t) \\
&\quad = \nabla f(x) + \nabla_N(\nabla f)(x)\frac{2\sqrt{t}}{\sqrt{\pi}} + \mathrm{o}(\sqrt{t}).
\end{aligned}$$

It follows that the formulae in (3.2) and (3.3) are obtained by taking into account the expansions:

$$\left\langle \mathbb{E}\left[//_{0,t}^{-1}\nabla f(X_t)\right], \nabla f\right\rangle = |\nabla f|^2 + \mathrm{II}(\nabla f, \nabla f)\frac{2\sqrt{t}}{\sqrt{\pi}} + \mathrm{o}(\sqrt{t}),$$

resp.

$$\left\langle \mathbb{E}\left[//_{0,t}^{-1}\nabla f(X_t)\right], \nabla P_t f\right\rangle = |\nabla f|^2 + \mathrm{II}(\nabla f, \nabla f)\frac{2\sqrt{t}}{\sqrt{\pi}} + \mathrm{o}(\sqrt{t}).$$

$\square$

Proof (Proof of Theorem 3.1.) Let $\mathrm{Ric}^Z(x) \geq K_1(x)$ and $\mathrm{II}(x) \geq \sigma_1(x)$. Furthermore, assume that

$$\mathbb{E}\left[\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}\right] < \infty, \quad \text{for } t > 0.$$

By [19, Theorem 3.2.1], there exists a unique two-parameter family of random endomorphisms $Q_{s,t} \in \mathrm{End}(T_{X_s}M)$ solving, for $s \geq 0$ fixed, the following equation in $t \geq s$,

$$\mathrm{d}Q_{s,t} = -Q_{s,t}\left(\mathrm{Ric}^Z_{//_{s,t}}\,\mathrm{d}t + \mathrm{II}_{//_{s,t}}\,\mathrm{d}l_t\right)(\mathrm{id} - \mathbb{1}_{\{X_t\in\partial M\}}P_{//_{s,t}}), \quad Q_{s,s} = \mathrm{id},$$

where by definition, for $u \in \partial\,\mathrm{O}(M) := \{u \in \mathrm{O}(M) : \mathbf{p}u \in \partial M\}$,

$$P(uy, uz) = \langle uy, N\rangle \langle uz, N\rangle, \quad y, z \in \mathbb{R}^d.$$

Recall that

$$\mathrm{Ric}^Z_{//_{s,t}} = //_{s,t}^{-1} \circ \mathrm{Ric}^Z_{X_t} \circ //_{s,t}, \quad \mathrm{II}_{//_{s,t}} = //_{s,t}^{-1} \circ \mathrm{II}_{X_t} \circ //_{s,t}, \quad P_{//_{s,t}} = //_{s,t}^{-1} \circ P_{X_t} \circ //_{s,t},$$

where as usual bilinear forms on TM, resp. on $T\partial M$, are understood fiberwise as linear endomorphisms via the metric. Moreover, by [19, Theorem 3.2.1], we have

$$\nabla P_{t-s}f(X_s) = //_{0,s}\mathbb{E}[//_{0,s}^{-1}Q_{s,t}//_{s,t}^{-1}\nabla f(X_t)|\mathscr{F}_s]. \tag{3.4}$$

By using derivative formula (3.4), the proofs are similar to that of Theorem 2.1. We only prove the equivalence "(i) $\Leftrightarrow$ (ii) or (iii)" to explain the idea.

"(i) $\Rightarrow$ (ii)": First, from the derivative formula and the lower bound on the curvature, we get

$$|\nabla P_t f|^2 \leq \mathbb{E}\left[\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}|\nabla f|^2(X_t)\right]. \tag{3.5}$$

Next, it is easy to see that

$$\begin{aligned}
&2\nabla f - Q_t//_{0,t}^{-1}\nabla f(X_t)\\
&\quad = 2\nabla f - \mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}//_{0,t}^{-1}\nabla f(X_t)\\
&\qquad + \left(\mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}\mathrm{id} - Q_t\right)//_{0,t}^{-1}\nabla f(X_t)
\end{aligned} \tag{3.6}$$

where $Q_t := Q_{0,t}$, which implies that

$$\begin{aligned}
&\left|2\nabla f - Q_t//_{0,t}^{-1}\nabla f(X_t)\right|\\
&\quad \leq \left|2\nabla f - \mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}//_{0,t}^{-1}\nabla f(X_t)\right|\\
&\qquad + \left|(\mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}\mathrm{id} - Q_t)//_{0,t}^{-1}\nabla f(X_t)\right|.
\end{aligned}$$

We start by estimating the last term on the right-hand side,

$$\begin{aligned}
&\left|\left(\mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}\mathrm{id} - Q_t\right)//_{0,t}^{-1}\nabla f(X_t)\right|\\
&\quad \leq \mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}\left\|\mathrm{id} - \mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}Q_t\right\||\nabla f(X_t)|.
\end{aligned}$$

Observe that we may rewrite

$$
\begin{aligned}
&\mathrm{id}-\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}Q_t\\
&=-\int_0^t \frac{\mathrm{d}\left[\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,s]})+\mathbb{K}_1(X_{[0,s]})\right)}Q_s\right]}{\mathrm{d}s}\,\mathrm{d}s\\
&=\int_0^t \mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,s]})+\mathbb{K}_1(X_{[0,s]})\right)}Q_s\Bigg[\left(\mathrm{Ric}^Z_{//_{0,s}}-\frac{K_1(X_s)+K_2(X_s)}{2}\mathrm{id}\right)\left(\mathrm{id}-\mathbb{1}_{\{X_s\in\partial M\}}P_{//_{0,s}}\right)\mathrm{d}s\\
&\qquad+\left(\mathrm{II}_{//_{0,s}}-\frac{\sigma_1(X_s)+\sigma_2(X_s)}{2}\mathrm{id}\right)\left(\mathrm{id}-\mathbb{1}_{\{X_s\in\partial M\}}P_{//_{0,s}}\right)\mathrm{d}l_s\Bigg].
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&\left\|\mathrm{id}-\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}Q_t\right\|\\
&\quad\le\int_0^t \mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,s]})+\mathbb{K}_1(X_{[0,s]})\right)}\|Q_s\|\left(\left|\mathrm{Ric}^Z_{//_{0,s}}-\frac{K_1(X_s)+K_2(X_s)}{2}\mathrm{id}\right|\mathrm{d}s\right.\\
&\qquad\left.+\left|\mathrm{II}_{//_{0,s}}-\frac{\sigma_1(X_s)+\sigma_2(X_s)}{2}\mathrm{id}\right|\mathrm{d}l_s\right)\\
&\quad\le\int_0^t \mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,s]})-\mathbb{K}_1(X_{[0,s]})\right)}\left(\frac{K_2(X_s)-K_1(X_s)}{2}\,\mathrm{d}s+\frac{\sigma_2(X_s)-\sigma_1(X_s)}{2}\,\mathrm{d}l_s\right)\\
&\quad=\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}-1
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left|2\nabla f-Q_t//_{0,t}^{-1}\nabla f(X_t)\right|^2\\
&\le\Bigg[\left|2\nabla f-\mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}//_{0,t}^{-1}\nabla f(X_t)\right|\\
&\qquad+\mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}\left(\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}-1\right)|\nabla f|(X_t)\Bigg]^2\\
&\le\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}\left|2\nabla f-\mathrm{e}^{-\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}//_{0,t}^{-1}\nabla f(X_t)\right|^2\\
&\qquad+\mathrm{e}^{-\left(\mathbb{K}_2(X_{[0,t]})+\mathbb{K}_1(X_{[0,t]})\right)}\left(\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}-1\right)\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}|\nabla f|^2(X_t)\\
&=4\,\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}|\nabla f|^2-4\,\mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})}\left\langle\nabla f,//_{0,t}^{-1}\nabla f(X_t)\right\rangle\\
&\qquad+\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}|\nabla f|^2(X_t).
\end{aligned}
$$

By expanding the terms above, we get

$$
\begin{aligned}
&|Q_t//_{0,t}^{-1}\nabla f(X_t)|^2-\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}|\nabla f|^2(X_t)\\
&\quad\le 4\left(\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}-1\right)|\nabla f|^2+4\left\langle\nabla f,Q_t//_{0,t}^{-1}\nabla f(X_t)\right\rangle\\
&\qquad-4\,\mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})}\left\langle\nabla f,//_{0,t}^{-1}\nabla f(X_t)\right\rangle.
\end{aligned}
$$

We observe that $|\nabla P_t f|^2 \le \mathbb{E}[|Q_t //_{0,t}^{-1} \nabla f(X_t)|^2]$ and take expectation on both sides of the inequality above, to obtain

$$
\begin{aligned}
&|\nabla P_t f|^2 - \mathbb{E}\left[\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}\,|\nabla f|^2(X_t)\right]\\
&\le 4\left(\mathbb{E}\,\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}-1\right)|\nabla f|^2+4\mathbb{E}\left\langle \nabla f, \nabla P_t f-\mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})}\,//_{0,t}^{-1}\nabla f(X_t)\right\rangle.
\end{aligned}
$$

Combining this with (3.5) completes the proof of "(i) $\Rightarrow$ (ii)".

"(i) $\Rightarrow$ (iii)": It is well known that if $f \in C_N^\infty(M)$, then $NP_t f = 0$ for $t > 0$. Combined with Itô's formula, we obtain

$$
\begin{aligned}
\mathrm{d}(P_{t-s}f^{2/p})^p(X_s) &= \mathrm{d}M_s + (L+\partial_s)(P_{t-s}f^{2/p}(X_s))^p\,\mathrm{d}s\\
&= \mathrm{d}M_s + p(p-1)(P_{t-s}f^{2/p}(X_s))^{p-2}|\nabla P_{t-s}f^{2/p}|^2(X_s)\,\mathrm{d}s\\
&\quad + p(P_{t-s}f^{2/p})^{p-1}NP_{t-s}f^{2/p}(X_s)\,\mathrm{d}l_s\\
&= \mathrm{d}M_s + p(p-1)(P_{t-s}f^{2/p}(X_s))^{p-2}|\nabla P_{t-s}f^{2/p}|^2(X_s)\,\mathrm{d}s
\end{aligned}
$$

where M_s is a local martingale. The rest of the argument is then similar to the proof of Theorem 2.1; we skip it here.

"(ii) $\Rightarrow$ (i)": Conversely, for $x \in \mathring{M}$ and $f \in C_N^\infty(M)$ such that $\mathrm{Hess}_f(x) = 0$, we have

$$
\begin{aligned}
&\frac{|\nabla P_t f|^2 - P_t|\nabla f|^2}{t} + \mathbb{E}\left[\frac{1-\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}}{t}|\nabla f|^2(X_t)\right]\\
&\le 4\Bigg(\frac{\mathbb{E}[\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}-1]}{t}|\nabla f|^2 + \frac{\left\langle \nabla f, \nabla P_t f - //_{0,t}^{-1}\nabla f(X_t)\right\rangle}{t}\\
&\qquad + \left\langle \nabla f, \mathbb{E}\left[\frac{1}{t}\left(1-\mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})}\right)//_{0,t}^{-1}\nabla f(X_t)\right]\right\rangle\Bigg)\wedge 0.
\end{aligned}
\tag{3.7}
$$

By Lemma 2.5(i) and condition (3.1), there exists $r > 0$ such that $B(x,r) \subseteq \mathring{M}$ and

$$
\begin{aligned}
&\lim_{t\to 0}\mathbb{E}\left[\mathbb{1}_{\{\sigma_r \le t\}}\frac{|1-\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}|}{t}|\nabla f|^2(X_t)\right]\\
&\le \lim_{t\to 0}\left(\frac{1}{t}\mathbb{P}\{\sigma_r \le t\}\,\|\nabla f\|_\infty^2 + \frac{1}{t}\mathbb{P}\{\sigma_r \le t\}^{\frac{\varepsilon}{2+\varepsilon}}\,\mathbb{E}\left[\mathrm{e}^{-(2+\varepsilon)\mathbb{K}_1(X_{[0,t]})}\right]^{\frac{2}{2+\varepsilon}}\|\nabla f\|_\infty^2\right) = 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\lim_{t\to 0}\mathbb{E}\left[\frac{1-\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}}{t}|\nabla f|^2(X_t)\right] &= \lim_{t\to 0}\mathbb{E}\left[\mathbb{1}_{\{t<\sigma_r\}}\frac{1-\mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}}{t}|\nabla f|^2(X_t)\right]\\
&= 2K_1(x)|\nabla f|^2(x).
\end{aligned}
$$

Similarly, we have

$$
\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)}-1\right] = \frac{K_2(x)-K_1(x)}{2},
$$

and

$$
\lim_{t\to 0}\left\langle \nabla f, \mathbb{E}\left[\frac{(1-\mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})})}{t}//_{0,t}^{-1}\nabla f(X_t)\right]\right\rangle = K_1(x)|\nabla f|^2(x).
$$

Thus, letting $t \to 0$ on both sides of (3.7) and using Lemma 3.3, we obtain

$$\begin{aligned}&-2\,\mathrm{Ric}^Z(\nabla f, \nabla f) + 2K_1(x)|\nabla f|^2 \\ &\quad \le \left[2(K_2(x) - K_1(x))|\nabla f|^2 - 4\,\mathrm{Ric}^Z(\nabla f, \nabla f) + 4K_1(x)|\nabla f|^2\right] \wedge 0,\end{aligned}$$

i.e.,

$$K_1(x)|\nabla f|^2 \le \mathrm{Ric}^Z(\nabla f, \nabla f) \le K_2(x)|\nabla f|^2.$$

We choose $x \in \partial M$ and $f \in C_N^\infty(M)$. We can rewrite the inequality in item (ii) as

$$\begin{aligned}&\frac{\sqrt{\pi}\,(|\nabla P_t f|^2 - P_t|\nabla f|^2)}{2\sqrt{t}} + \mathbb{E}\left[\frac{\sqrt{\pi}\left(1 - \mathrm{e}^{-2\mathbb{K}_1(X_{[0,t]})}\right)}{2\sqrt{t}}\,|\nabla f|^2(X_t)\right] \\ &\le 4\left[\frac{\sqrt{\pi}\,\mathbb{E}\left[\mathrm{e}^{\frac{1}{2}\left(\mathbb{K}_2(X_{[0,t]})-\mathbb{K}_1(X_{[0,t]})\right)} - 1\right]}{2\sqrt{t}}|\nabla f|^2 + \frac{\sqrt{\pi}\left\langle \nabla f, \nabla P_t f - //_{0,t}^{-1}\nabla f(X_t)\right\rangle}{2\sqrt{t}}\right. \\ &\qquad \left. + \left\langle \nabla f, \mathbb{E}\left[\frac{\sqrt{\pi}\,(1 - \mathrm{e}^{-\mathbb{K}_1(X_{[0,t]})})}{2\sqrt{t}}//_{0,t}^{-1}\nabla f(X_t)\right]\right\rangle\right] \wedge 0.\end{aligned}$$

Now letting $t \to 0$, by Lemma 3.4 and Lemma 3.2, we obtain

$$\begin{aligned}&-2\,\mathrm{II}(\nabla f, \nabla f) + 2\sigma_1(x)|\nabla f|^2 \\ &\quad \le [-4\,\mathrm{II}(\nabla f, \nabla f) + 2(\sigma_2(x) - \sigma_1(x))|\nabla f|^2 + 4\sigma_1(x)|\nabla f|^2] \wedge 0,\end{aligned}$$

i.e.,

$$\sigma_1(x)|\nabla f|^2(x) \le \mathrm{II}(\nabla f, \nabla f)(x) \le \sigma_2(x)|\nabla f|^2(x).$$

Similarly, using Lemmas 3.3 and 3.4, one can prove "(iii) $\Rightarrow$ (i)"; we skip the details here. □

4 Extension to evolving manifolds

In this section, we deal with the case that the underlying manifold carries a geometric flow of complete Riemannian metrics. More precisely, for some $T_c \in (0, \infty]$, we consider the situation of a d-dimensional differentiable manifold M equipped with a C^1 family of complete Riemannian metrics $(g_t)_{t\in[0,T_c)}$. Let ∇^t be the Levi-Civita connection and Δ_t the Laplace-Beltrami operator associated with the metric g_t. In addition, let $(Z_t)_{t\in[0,T_c)}$ be a C^1-family of vector fields on M. For the sake of brevity, we write

$$\mathscr{R}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y\right\rangle_t - \frac{1}{2}\partial_t g_t(X,Y), \quad X, Y \in T_x M,\ x \in M,$$

where Ric_t is the Ricci curvature tensor with respect to the metric g_t and $\langle\cdot,\cdot\rangle_t := g_t(\cdot,\cdot)$.

In what follows, for real-valued functions ϕ, ψ on $[0, T_c) \times M$, we write $\psi \le \mathscr{R}^Z \le \phi$, if

$$\psi_t|X|_t^2 \le \mathscr{R}_t^Z(X,X) \le \phi_t|X|_t^2$$

holds for all $X \in TM$ and $t \in [0, T_c)$, where by definition $|X|_t := \sqrt{g_t(X,X)}$. Let X_t be the diffusion process generated by $L_t := \Delta_t + Z_t$ (called L_t-diffusion) which is assumed to be non-explosive up to time T_c.

We first introduce some notations and recall the construction of X_t. Let $F(M)$ be the frame bundle over M and $\mathrm{O}_t(M)$ the orthonormal frame bundle over M with respect to the metric g_t. We denote by $\pi\colon F(M)\to M$ the projection from $F(M)$ onto M. For $u\in F(M)$, let

$$T_{\pi u}M\to T_uF(M),\quad X\mapsto H^t_X(u),$$

be the ∇^t-horizontal lift. In particular, we consider the standard-horizontal vector fields H^t_i on $F(M)$ given by

$$H^t_i(u)=H^t_{ue_i}(u),\quad i=1,2,\dots,d$$

where $\{e_i\}_{i=1}^d$ denotes the canonical orthonormal basis of $\mathbb{R}^d$. Let $\{V_{\alpha,\beta}\}_{\alpha,\beta=1}^d$ be the standard-vertical vector fields on $F(M)$,

$$V_{\alpha,\beta}(u):=T\ell_u(\exp(E_{\alpha,\beta})),\quad u\in F(M),$$

where $E_{\alpha,\beta}$ is a basis of the real $d\times d$ matrices, and

$$\ell_u\colon\ \mathrm{GL}(d;\mathbb{R})\to F(M),\quad g\mapsto u\cdot g,$$

is defined via left multiplication of the general linear group $\mathrm{GL}(d;\mathbb{R})$ on $F(M)$.

Let $B_t=(B^1_t,\dots,B^d_t)$ be a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega,\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$. To construct the L_t-diffusion X_t, we first construct the corresponding horizontal diffusion process u_t by solving the following Stratonovich SDE on $F(M)$:

$$\begin{cases}\mathrm{d}u_t=\sqrt{2}\sum_{i=1}^d H^t_i(u_t)\circ \mathrm{d}B^i_t+H^t_{Z_t}(u_t)\mathrm{d}t-\dfrac{1}{2}\sum_{\alpha,\beta=1}^d\mathscr{G}_{\alpha,\beta}(t,u_t)V_{\alpha\beta}(u_t)\mathrm{d}t,\\[2ex] u_s\in \mathrm{O}_s(M),\ \pi(u_s)=x,\ s\in[0,T_c),\end{cases}\tag{4.1}$$

where $\mathscr{G}_{\alpha,\beta}(t,u_t):=\partial_t g_t(u_te_\alpha,u_te_\beta)$. As explained in [1], the last term is crucial to ensure $u_t\in \mathrm{O}_t(M)$. Since $\{H^t_{Z_t}\}_{t\in[0,T_c)}$ is $C^{1,\infty}$-smooth, Eq. (4.1) has a unique solution up to its lifetime $\zeta:=\lim\limits_{n\to\infty}\zeta_n$ where

$$\zeta_n:=\inf\{t\in[s,T_c):\rho_t(\pi(u_s),\pi(u_t))\geq n\},\quad n\geq 1,\quad \inf\varnothing:=T_c,\tag{4.2}$$

and where ρ_t stands for the Riemannian distance induced by the metric g_t. Then $X^{(s,x)}_t=\pi(u_t)$ solves the equation

$$\mathrm{d}X^{(s,x)}_t=\sqrt{2}u_t\circ\mathrm{d}B_t+Z_t(X^{(s,x)}_t)\,\mathrm{d}t,\quad X^{(s,x)}_s=x:=\pi(u_s),$$

up to the lifetime ζ. By Itô's formula, for any $f\in C^2_0(M)$,

$$f(X^{(s,x)}_t)-f(x)-\int_s^t L_rf(X^{(s,x)}_r)\mathrm{d}r=\sqrt{2}\int_s^t\left\langle //_{s,r}^{-1}\nabla^r f(X^{(s,x)}_r),u^x_s\mathrm{d}B_r\right\rangle_s,\quad t\in[s,T_c),$$

is a martingale up to ζ. In other words, $X^{(s,x)}_t$ is a diffusion process with generator L_t. In case $s=0$, if there is no risk of confusion, we write again X^x_t instead of $X^{(0,x)}_t$.

Throughout this section, we assume that the diffusion process X_t generated by L_t is non-explosive up to time T_c (see [15] for sufficient conditions ensuring non-explosion). Then this process gives rise to an inhomogeneous Markov semigroup $\{P_{s,t}\}_{0\leq s\leq t<T_c}$ on $\mathscr{B}_b(M)$ by

$$P_{s,t}f(x):=\mathbb{E}\big[f(X^{(s,x)}_t)\big]=\mathbb{E}^{(s,x)}\left[f(X_t)\right],\quad x\in M,\ f\in\mathscr{B}_b(M),$$

which is called the diffusion semigroup generated by L_t.

We are now in position to present the main result of this section.

Theorem 4.1 *Let $(t,x)\mapsto K_1(t,x)$ and $(t,x)\mapsto K_2(t,x)$ be two continuous functions on $[0,T_c)\times M$ such that $K_1\le K_2$. Suppose that*

$$\begin{cases}\mathbb{E}\left[\mathrm{e}^{-(2+\varepsilon)\int_s^t K_1(r,X_r)\,\mathrm{d}r}+\mathrm{e}^{(\frac{1}{2}+\varepsilon)\int_s^t (K_2(r,X_r)-K_1(r,X_r))\,\mathrm{d}r}\right]<\infty,\\ \text{for some } \varepsilon>0 \text{ and all } t>s\ge 0.\end{cases}\tag{4.3}$$

The following statements are equivalent to each other:

(i) *the curvature $\mathscr{R}_t^Z$ satisfies*

$$K_1(t,x)\le \mathscr{R}_t^Z(x)\le K_2(t,x),\quad (t,x)\in[0,T_c)\times M;$$

(ii) *for $f\in C_0^\infty(M)$ and $0\le s\le t<T_c$,*

$$\begin{aligned}&|\nabla^s P_{s,t}f|_s^2-\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-2\int_s^t K_1(r,X_r)\,\mathrm{d}r}\,|\nabla^t f|_t^2(X_t)\right]\\&\le 4\bigg[\left(\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\,\mathrm{d}r}-1\right)|\nabla^s f|_s^2+\langle\nabla^s f,\nabla^s P_{s,t}f\rangle_s\\&\quad-\left\langle\nabla^s f,\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_s^t K_1(r,X_r)\,\mathrm{d}r}\,//_{s,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_s\bigg]\wedge 0;\end{aligned}$$

(ii') *for $f\in C_0^\infty(M)$ and $0\le s\le t<T_c$,*

$$\begin{aligned}&|\nabla^s P_{s,t}f|_s^2-\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-2\int_s^t K_1(r,X_r)\,\mathrm{d}r}\,|\nabla^t f|_t^2(X_t)\right]\\&\le 4\bigg[\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\,\mathrm{d}r}\,|\nabla^s P_{s,t}f|_s^2\\&\quad-\left\langle\nabla^s P_{s,t}f,\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_s^t K_1(r,X_r)\,\mathrm{d}r}\,//_{s,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_s\bigg]\wedge 0;\end{aligned}$$

(iii) *for $f\in C_0^\infty(M)$, $p\in(1,2]$ and $0\le s\le t<T_c$,*

$$\begin{aligned}&\frac{p(P_{s,t}f^2-(P_{s,t}f^{2/p})^p)}{4(p-1)}-\mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K_1(\tau,X_\tau)\,\mathrm{d}\tau}\,\mathrm{d}r\times|\nabla^t f|_t^2(X_t)\right]\\&\le 4\int_s^t\bigg[\left(\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))\mathrm{d}\tau}-1\right)P_{s,r}|\nabla^r f|_r^2\\&\quad+\mathbb{E}^{(s,x)}\left\langle\nabla^r f(X_r),\nabla^r P_{r,t}f(X_r)-\mathrm{e}^{-\int_r^t K_1(\tau,X_\tau)\mathrm{d}\tau}\,//_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\bigg]\,\mathrm{d}r\wedge 0;\end{aligned}$$

(iii') *for $f\in C_0^\infty(M)$, $p\in(1,2]$ and $0\le s\le t<T_c$,*

$$\begin{aligned}&\frac{p(P_{s,t}f^2-(P_{s,t}f^{2/p})^p)}{4(p-1)}-\mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K_1(\tau,X_\tau)\,\mathrm{d}\tau}\,\mathrm{d}r\times|\nabla^t f|_t^2(X_t)\right]\\&\le 4\int_s^t\bigg[\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))\mathrm{d}\tau}\,P_{s,r}|\nabla^r P_{r,t}f|_r^2\\&\quad-\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_r^t K_1(\tau,X_\tau)\mathrm{d}\tau}\left\langle\nabla^r P_{r,t}f(X_r),//_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\right]\bigg]\,\mathrm{d}r\wedge 0;\end{aligned}$$

(iv) *for* $f \in C_0^\infty(M)$ *and* $0 \le s \le t < T_c$,

$$\begin{aligned}
&\frac{1}{4}\left(P_{s,t}(f^2\log f^2) - P_{s,t}f^2\log P_{s,t}f^2\right) - \mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K_1(\tau,X_\tau)\,\mathrm{d}\tau}\,\mathrm{d}r \times |\nabla^t f|_t^2(X_t)\right]\\
&\le 4\int_s^t\left[\left(\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))\mathrm{d}\tau} - 1\right)P_{s,r}|\nabla^r f|_r^2\right.\\
&\qquad\left. + \mathbb{E}^{(s,x)}\left\langle \nabla^r f(X_r), \nabla^r P_{r,t}f(X_r) - \mathrm{e}^{-\int_r^t K_1(\tau,X_\tau)\mathrm{d}\tau}//_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\right]\mathrm{d}r \wedge 0;
\end{aligned}$$

(iv') *for* $f \in C_0^\infty(M)$ *and* $0 \le s \le t < T_c$,

$$\begin{aligned}
&\frac{1}{4}(P_{s,t}(f^2\log f^2) - P_{s,t}f^2\log P_{s,t}f^2) - \mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K_1(\tau,X_\tau)\,\mathrm{d}\tau}\,\mathrm{d}r \times |\nabla^t f|_t^2(X_t)\right]\\
&\le 4\int_s^t\left[\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_r^t (K_2(\tau,X_\tau)-K_1(\tau,X_\tau))\mathrm{d}\tau}P_{s,r}|\nabla^r P_{r,t}f|_r^2\right.\\
&\qquad\left. - \mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_r^t K_1(\tau,X_\tau)\mathrm{d}\tau}\left\langle \nabla^r P_{r,t}f(X_r), //_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\right]\right]\mathrm{d}r \wedge 0.
\end{aligned}$$

Remark 4.2 By [6], the integral condition (4.3) can be satisfied if $K_2(t,\cdot)/\rho_t^2 \to 0$ as $\rho_t^2 \to \infty$ and one of the following conditions is satisfied:

(A1) there exists a non-negative continuous function C on $[0,T_c)$ such that for all $t \in [0,T_c)$,

$$\mathscr{R}_t^Z \ge -C(t);$$

(A2) there exist two non-negative continuous functions C_1, C_2 on $[0,T_c)$ such that for all $t \in [0,T_c)$,

$$\mathrm{Ric}_t \ge -C_1(t)(1+\rho_t^2) \quad \text{and} \quad \partial_t\rho_t + \left\langle Z_t, \nabla^t\rho_t\right\rangle_t \le C_2(t)(1+\rho_t).$$

To prove the theorem, we need the following lemmas: the derivative formula and characterization formulae for $\mathscr{R}_t^Z$. For $s \le t$, let

$$\mathscr{R}_{//_{s,t}}^Z := //_{s,t}^{-1}\circ\mathscr{R}_t^Z(X_t)\circ //_{s,t}.$$

Lemma 4.3 *([6, Theorem 3.1]) Let* $\mathscr{R}_t^Z(x) \ge K(t,x)$ *for all* $t \in [0,T_c)$ *and suppose that*

$$\mathbb{E}\left[\mathrm{e}^{-\int_s^t K(r,X_r)\mathrm{d}r}\right] < \infty$$

for $0 \le s \le t < T_c$. *Then, for* $0 \le s \le t < T_c$,

$$\nabla^s P_{s,t}f(x) = \mathbb{E}^{(s,x)}\left[Q_{s,t}//_{s,t}^{-1}\nabla^t f(X_t)\right],$$

where for fixed $s \ge 0$, *the random family* $Q_{s,t} \in \mathrm{Aut}(T_{X_s}M)$ *is constructed for* $t \ge s$ *as solution to the equation:*

$$\frac{\mathrm{d}Q_{s,t}}{\mathrm{d}t} = -Q_{s,t}\mathscr{R}_{//_{s,t}}^Z, \quad Q_{s,s} = \mathrm{id}. \tag{4.4}$$

Lemma 4.4 *For* $s \in [0,T_c)$ *and* $x \in M$, *let* $X \in T_xM$ *with* $|X|_s = 1$. *Furthermore, let* $f \in C_0^\infty(M)$ *be such that* $\nabla^s f(x) = X$ *and* $\mathrm{Hess}_f^s(x) = 0$, *and set* $f_n = n + f$ *for* $n \ge 1$. *Then,*

(i) *for any* $p>0$,

$$\mathscr{R}_s^Z(X,X)=\lim_{t\downarrow s}\frac{P_{s,t}|\nabla^t f|_t^p(x)-|\nabla^s P_{s,t}f|_s^p(x)}{p(t-s)};$$

(ii) *for any* $p>1$,

$$\begin{aligned}\mathscr{R}_s^Z(X,X)&=\lim_{n\to\infty}\lim_{t\downarrow s}\frac{1}{t-s}\left(\frac{p\big(P_{s,t}f_n^2-(P_{s,t}f_n^{2/p})^p\big)}{4(p-1)(t-s)}-|\nabla^s P_{s,t}f_n|_s^2\right)(x)\\&=\lim_{n\to\infty}\lim_{t\downarrow s}\frac{1}{t-s}\left(P_{s,t}|\nabla^t f|_t^2-\frac{p\big(P_{s,t}f_n^2-(P_{s,t}f_n^{2/p})^p\big)}{4(p-1)(t-s)}\right)(x);\end{aligned}\tag{4.5}$$

(iii) $\mathscr{R}_s^Z(X,X)$ *is equal to each of the following limits:*

$$\begin{aligned}&\mathscr{R}_s^Z(X,X)\\&=\lim_{n\to\infty}\lim_{t\downarrow s}\frac{1}{(t-s)^2}\left\{(P_{s,t}f_n)\big[P_{s,t}(f_n\log f_n)-(P_{s,t}f_n)\log P_{s,t}f_n\big]-(t-s)|\nabla^s P_{s,t}f|_s^2\right\}(x)\\&=\lim_{n\to\infty}\lim_{t\downarrow s}\frac{1}{4(t-s)^2}\left\{4(t-s)P_{s,t}|\nabla^t f|_t^2+(P_{s,t}f_n^2)\log P_{s,t}f_n^2-P_{s,t}f_n^2\log f_n^2\right\}(x);\end{aligned}$$

(iv) $\mathscr{R}_s^Z(X,X)$ *can also be calculated via the following limits:*

$$\begin{aligned}\mathscr{R}_s^Z(X,X)&=\lim_{t\downarrow s}\frac{\left\{\left\langle\nabla^s f,\mathbb{E}^{(s,x)}//_{s,t}^{-1}\nabla^t f(X_t)\right\rangle_s-\langle\nabla^s f,\nabla^s P_{s,t}f\rangle_s\right\}(x)}{t-s}\\&=\lim_{t\downarrow s}\frac{\left\{\left\langle\nabla^s P_{s,t}f,\mathbb{E}^{(s,x)}//_{s,t}^{-1}\nabla^t f(X_t)\right\rangle_s-|\nabla^s P_{s,t}f|_s^2\right\}(x)}{t-s}.\end{aligned}$$

Proof Without loss of generality, we prove (iv) only for $s=0$. For the remaining formulae, the reader is referred to [6]. We have

$$\begin{aligned}&\lim_{t\downarrow 0}\frac{\langle\nabla^0 f,\mathbb{E}//_{0,t}^{-1}\nabla^t f(X_t)\rangle_0-\left\langle\nabla^0 f,\mathbb{E}Q_t//_{0,t}^{-1}\nabla^t f(X_t)\right\rangle_0}{t}\\&=\lim_{t\downarrow 0}\left\langle\nabla^0 f,\mathbb{E}\left[\frac{(\mathrm{id}-Q_t)}{t}//_{0,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_0\\&=\lim_{t\downarrow 0}\left\langle\nabla^0 f,\mathbb{E}\left[\frac{1}{t}\int_0^t Q_s\mathscr{R}^Z_{//_{0,s}}\,\mathrm{d}s\,//_{0,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_0\\&=\mathscr{R}_0^Z(\nabla^0 f,\nabla^0 f).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\lim_{t\downarrow 0}\frac{\left\langle\nabla^0 P_t f,\mathbb{E}//_{0,t}^{-1}\nabla^t f(X_t)\right\rangle_0-\left\langle\nabla^0 P_t f,\mathbb{E}Q_t//_{0,t}^{-1}\nabla^t f(X_t)\right\rangle_0}{t}\\&=\lim_{t\downarrow 0}\left\langle\nabla^0 P_t f,\mathbb{E}\left[\frac{(\mathrm{id}-Q_t)}{t}//_{0,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_0\\&=\lim_{t\downarrow 0}\left\langle\nabla^0 P_t f,\mathbb{E}\left[\frac{1}{t}\int_0^t Q_s\mathscr{R}^Z_{//_{0,s}}\,\mathrm{d}s\,//_{0,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_0\\&=\mathscr{R}_0^Z(\nabla^0 f,\nabla^0 f).\end{aligned}$$

□

Proof (Proof of Theorem 4.1.) We give the proof of the equivalence (i) and (ii), resp. (ii').

"(i) implies (ii) and (ii')": By (4.4), we know that

$$\begin{aligned}
&\left\|\mathrm{id}-\mathrm{e}^{\frac{1}{2}\int_s^t (K_1(r,X_r)+K_2(r,X_r))\,\mathrm{d}r}Q_{s,t}\right\| \\
&=\left\|\int_s^t \mathrm{e}^{\frac{1}{2}\int_s^r (K_1(u,X_u)+K_2(u,X_u))\mathrm{d}u}\,Q_{s,r}\left(\mathscr{R}^Z_{//_{s,r}}-\frac{K_1(r,X_r)+K_2(r,X_r)}{2}\mathrm{id}\right)\mathrm{d}r\right\| \\
&\le \int_s^t \mathrm{e}^{\frac{1}{2}\int_s^r (K_1(u,X_u)+K_2(u,X_u))\mathrm{d}u}\,\|Q_{s,r}\|\,\frac{K_2(r,X_r)-K_1(r,X_r)}{2}\,\mathrm{d}r \\
&\le \int_s^t \mathrm{e}^{\frac{1}{2}\int_s^r (K_2(u,X_u)-K_1(u,X_u))\mathrm{d}u}\,\frac{K_2(r,X_r)-K_1(r,X_r)}{2}\,\mathrm{d}r \\
&=\mathrm{e}^{\frac{1}{2}\int_s^t (K_2(u,X_u)-K_1(u,X_u))\mathrm{d}u}-1.
\end{aligned}$$

By a similar discussion as in the proof of Theorem 2.1, we have

$$\begin{aligned}
&\left|2a\nabla^s f+2b\nabla^s P_{s,t}f-Q_{s,t}\,//_{s,t}^{-1}\nabla^t f(X_t)\right|_s^2 \\
&\le \mathrm{e}^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\mathrm{d}r}\left|2a\nabla^s f+2b\nabla^s P_{s,t}f-\mathrm{e}^{-\frac{1}{2}\int_s^t (K_1(r,X_r)+K_2(r,X_r))\mathrm{d}r}\,//_{s,t}^{-1}\nabla^t f(X_t)\right|_s^2 \\
&\quad+\mathrm{e}^{-\int_s^t (K_1(r,X_r)+K_2(r,X_r))\mathrm{d}r}\left(\mathrm{e}^{\int_s^t (K_2(r,X_r)-K_1(r,X_r))\mathrm{d}r}-\mathrm{e}^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\mathrm{d}r}\right)|\nabla^t f(X_t)|_t^2 \\
&=\mathrm{e}^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\mathrm{d}r}\,|2a\nabla^s f+2b\nabla^s P_{s,t}f|_s^2 \\
&\quad-2\,\mathrm{e}^{-\int_s^t K_1(r,X_r)\mathrm{d}r}\left\langle 2a\nabla^s f+2b\nabla^s P_{s,t}f,//_{s,t}^{-1}\nabla^t f(X_t)\right\rangle_s+\mathrm{e}^{-2\int_s^t K_1(r,X_r)\mathrm{d}r}\,|\nabla^t f(X_t)|_t^2
\end{aligned}$$

where a,b are constants such that $a+b=1$. From this, we obtain

$$\begin{aligned}
&\mathbb{E}^{(s,x)}\left|Q_{s,t}//_{s,t}^{-1}\nabla^t f(X_t)\right|_s^2-\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-2\int_s^t K_1(r,X_r)\mathrm{d}r}\,|\nabla^t f(X_t)|_t^2\right] \\
&\le\left(\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\mathrm{d}r}-1\right)|2a\nabla^s f+2b\nabla^s P_{s,t}f|_s^2 \\
&\quad-2\left\langle 2a\nabla^s f+2b\nabla^s P_{s,t}f,\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_s^t K_1(r,X_r)\mathrm{d}r}\,//_{s,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_s \\
&\quad+2\left\langle 2a\nabla^s f+2b\nabla^s P_{s,t}f,\nabla^s P_{s,t}f\right\rangle_s.
\end{aligned}\tag{4.6}$$

Moreover, by the derivative formula (Lemma 4.3), we have

$$|\nabla^s P_{s,t}f|_s^2(x)\le\mathbb{E}^{(s,x)}|Q_{s,t}//_{s,t}^{-1}\nabla^t f(X_t)|_s^2$$

which combines with (4.6) implies

$$\begin{aligned}
&|\nabla^s P_{s,t}f|_s^2-\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-2\int_s^t K_1(r,X_r)\mathrm{d}r}\,|\nabla^t f(X_t)|_t^2\right] \\
&\le\left(\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac{1}{2}\int_s^t (K_2(r,X_r)-K_1(r,X_r))\mathrm{d}r}-1\right)|2a\nabla^s f+2b\nabla^s P_{s,t}f|_s^2 \\
&\quad-2\left\langle 2a\nabla^s f+2b\nabla^s P_{s,t}f,\mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_s^t K_1(r,X_r)\mathrm{d}r}\,//_{s,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_s \\
&\quad+2\left\langle 2a\nabla^s f+2b\nabla^s P_{s,t}f,\nabla^s P_{s,t}f\right\rangle_s.
\end{aligned}$$

Hence taking $a=1$, $b=0$ and $a=0$, $b=1$ in the above inequalities, we complete the proof of "(i) $\Rightarrow$ (ii)(ii')".

"(i) ⇒ (iii)": By Itô's formula, for $f \in C_0^\infty(M)$,

$$\begin{aligned} \mathrm{d}(P_{s,t}f^{2/p})^p(X_s) &= \mathrm{d}M_s + (L_s + \partial_s)(P_{s,t}f^{2/p}(X_s))^p \,\mathrm{d}s \\ &= \mathrm{d}M_s + p(p-1)(P_{s,t}f^{2/p}(X_s))^{p-2}|\nabla^s P_{s,t}f^{2/p}|_s^2(X_s)\,\mathrm{d}s \\ &= \mathrm{d}M_s + p(p-1)(P_{s,t}f^{2/p}(X_s))^{p-2}|\nabla^s P_{s,t}f^{2/p}|_s^2(X_s)\,\mathrm{d}s \end{aligned}$$

where M_s is a local martingale. The rest of the proof then is similar to the one of Theorem 2.1; we skip the details here.

"(ii) and (ii') ⇒ (i)":

$$\begin{aligned} &\frac{|\nabla^s P_{s,t}f|_s^2 - P_{s,t}|\nabla^t f|_t^2}{t-s} + \mathbb{E}^{(s,x)}\left[\frac{1-\mathrm{e}^{-2\int_s^t K_1(r,X_r)\mathrm{d}r}}{t-s}|\nabla^t f|_t^2(X_t)\right] \\ &\le 4\Bigg[\frac{\mathbb{E}^{(s,x)}\,\mathrm{e}^{\frac12\int_s^t (K_2(r,X_r)-K_1(r,X_r))\mathrm{d}r}-1}{t-s}|\nabla^s f|_s^2 + \frac{\left\langle \nabla^s f, \nabla^s P_{s,t}f - \mathbb{E}//_{s,t}^{-1}\nabla^t f(X_t)\right\rangle_s}{t-s} \\ &\quad - \left\langle \nabla^s f, \mathbb{E}^{(s,x)}\left[\frac{\mathrm{e}^{-\int_s^t K_1(r,X_r)\mathrm{d}r}-1}{t-s}//_{s,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_s\Bigg] \wedge 0; \end{aligned}$$

Letting $t \downarrow s$ and using Lemma 4.4 (i) (iv), we have

$$\begin{aligned} &-2\mathscr{R}_s^Z(\nabla^s f, \nabla^s f) + 2K_1(s,x)|\nabla^s f|_s^2 \\ &\quad \le 4\left[\frac12 (K_2(s,x)-K_1(s,x))|\nabla^s f|_s^2 - \mathscr{R}_s^Z(\nabla^s f,\nabla^s f) + K_1(s,x)|\nabla^s f|_s^2\right] \wedge 0, \end{aligned}$$

that is

$$K_1(s,x)|\nabla^s f|_s^2(x) \le \mathscr{R}_s^Z(\nabla^s f, \nabla^s f)(x) \le K_2(s,x)|\nabla^s f|_s^2(x).$$

Similarly, (ii') implies (i) as well. We skip the details here. □

Based on our characterizations for pinched curvature on evolving manifolds, we can characterize solutions to some geometric flows.

Corollary 4.5 *Let $(t,x) \mapsto K(t,x)$ be some function on $[0,T_c)\times M$. The following statements are equivalent to each other:*

(i) *the family $(M,g_t)_{t\in[0,T_c)}$ evolves by*

$$\frac12 \partial_t g_t = \mathrm{Ric}_t - \nabla^t Z_t - K(t,\cdot)g_t, \quad t \in [0,T_c);$$

(ii) *for $f \in C_0^\infty(M)$ and $0 \le s \le t < T_c$,*

$$\begin{aligned} &|\nabla^s P_{s,t}f|_s^2 - \mathbb{E}^{(s,x)}\left[\mathrm{e}^{-2\int_s^t K(r,X_r)\mathrm{d}r}|\nabla^t f|_t^2(X_t)\right] \\ &\le 4\left[\langle \nabla^s f, \nabla^s P_{s,t} f\rangle_s - \left\langle \nabla^s f, \mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_s^t K(r,X_r)\mathrm{d}r}//_{s,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_s\right] \wedge 0; \end{aligned}$$

(ii') *for $f \in C_0^\infty(M)$ and $0 \le s \le t < T_c$,*

$$\begin{aligned} &|\nabla^s P_{s,t}f|_s^2 - \mathbb{E}^{(s,x)}\left[\mathrm{e}^{-2\int_s^t K(r,X_r)\mathrm{d}r}|\nabla^t f|_t^2(X_t)\right] \\ &\le 4\left[|\nabla^s P_{s,t}f|_s^2 - \left\langle \nabla^s P_{s,t} f, \mathbb{E}^{(s,x)}\left[\mathrm{e}^{-\int_s^t K(r,X_r)\mathrm{d}r}//_{s,t}^{-1}\nabla^t f(X_t)\right]\right\rangle_s\right] \wedge 0; \end{aligned}$$

(iii) *for* $f \in C_0^\infty(M)$, $p \in (1,2]$ *and* $0 \le s \le t < T_c$,

$$\frac{p(P_{s,t}f^2 - (P_{s,t}f^{2/p})^p)}{4(p-1)} - \mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K(\tau,X_\tau)\mathrm{d}\tau}\mathrm{d}r \times |\nabla^t f|_t^2(X_t)\right]$$
$$\le 4\int_s^t \left[\mathbb{E}^{(s,x)}\left\langle \nabla^r f(X_r), \nabla^r P_{r,t}f(X_r) - \mathrm{e}^{-\int_r^t K(\tau,X_\tau)\mathrm{d}\tau} //_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\right]\mathrm{d}r \wedge 0;$$

(iii') *for* $f \in C_0^\infty(M)$, $p \in (1,2]$ *and* $0 \le s \le t < T_c$,

$$\frac{p(P_{s,t}f^2 - (P_{s,t}f^{2/p})^p)}{4(p-1)} - \mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K(\tau,X_\tau)\mathrm{d}\tau}\mathrm{d}r \times |\nabla^t f|_t^2(X_t)\right]$$
$$\le 4\int_s^t \left[P_{s,r}|\nabla^r P_{r,t}f|_r^2 - \mathbb{E}^{(s,x)}\left\langle \nabla^r P_{r,t}f(X_r), \mathrm{e}^{-\int_r^t K(\tau,X_\tau)\mathrm{d}\tau} //_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\right]\mathrm{d}r \wedge 0;$$

(iv) *for* $f \in C_0^\infty(M)$ *and* $0 \le s \le t < T_c$,

$$\frac{1}{4}\left(P_{s,t}(f^2\log f^2) - P_{s,t}f^2\log P_{s,t}f^2\right) - \mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K(\tau,X_\tau)\mathrm{d}\tau}\mathrm{d}r \times |\nabla^t f|_t^2(X_t)\right]$$
$$\le 4\int_s^t \left[\mathbb{E}^{(s,x)}\left\langle \nabla^r f(X_r), \nabla^r P_{r,t}f(X_r) - \mathrm{e}^{-\int_r^t K(\tau,X_\tau)\mathrm{d}\tau} //_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\right]\mathrm{d}r \wedge 0;$$

(iv') *for* $f \in C_0^\infty(M)$ *and* $0 \le s \le t < T_c$,

$$\frac{1}{4}(P_{s,t}(f^2\log f^2) - P_{s,t}f^2\log P_{s,t}f^2) - \mathbb{E}^{(s,x)}\left[\int_s^t \mathrm{e}^{-2\int_r^t K(\tau,X_\tau)\mathrm{d}\tau}\mathrm{d}r \times |\nabla^t f|_t^2(X_t)\right]$$
$$\le 4\int_s^t \left[P_{s,r}|\nabla^r P_{r,t}f|_r^2 - \mathbb{E}^{(s,x)}\left\langle \nabla^r P_{r,t}f(X_r), \mathrm{e}^{-\int_r^t K(\tau,X_\tau)\mathrm{d}\tau} //_{r,t}^{-1}\nabla^t f(X_t)\right\rangle_r\right]\mathrm{d}r \wedge 0.$$

Remark 4.6 In Corollary 4.5, if $Z_t \equiv 0$ and $K \equiv 0$, the results characterize solutions to the Ricci flow, see [11] for functional inequalities on path space characterizing Ricci flow.

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