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Heat equation in vector bundles with time-dependent metric

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Dedicated to the memory of Professor Kiyosi Itô on the occasion of the 100th anniversary of his birth

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Abstract. We derive a stochastic representation formula for solutions of heat-type equations on vector bundles with time-dependent Riemannian metric over manifolds whose Riemannian metric is time-dependent as well. As a corollary we obtain a vanishing theorem for bounded ancient solutions under a curvature condition. Our results apply in particular to the case of differential forms.

1. Introduction.

In his talk at the ICM in Stockholm 1962 [**12**], Professor Kiyosi Itô's showed that the Levi-Civita parallel translation of tensors on a Riemannian manifold makes perfect sense along the trajectories of a Brownian motion. Several mathematicians, among them Paul Malliavin, James Eells and David Elworthy, understood quickly the significance of this construction and took up the new ideas [**4**], [**5**], [**16**], [**17**]. It turned out to be the starting point of a new mathematical field, stochastic differential geometry, born from a combination of Élie Cartan's method of moving frames and Kiyosi Itô's theory of diffusion processes. The first applications, fully in the tradition of Bochner's method, aimed at cohomology vanishing theorems under positivity conditions [**6**], [**7**], [**8**], [**13**], [**16**]. The idea is to use a stochastic representation of harmonic forms, or more generally of solutions of the heat equation on differential forms, in terms of a certain transport along Brownian motion which can be estimated in terms of curvature. It is the goal of this note to extend these ideas to the setting of Riemannian manifolds evolving under a geometric flow.

Let M be a d-dimensional differentiable manifold equipped with a family

$$(g(\tau))_{\tau\in[T_1,T_2]}$$

of Riemannian metrics depending smoothly on τ, and let E be a k-dimensional vector bundle over M, also equipped with a family $(g^E(\tau))_{\tau\in[T_1,T_2]}$ of Riemannian metrics depending smoothly on τ. Let $\nabla(\tau)$ be the Levi-Civita connection of $g(\tau)$, and let $\nabla^E(\tau)$

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be a covariant derivative on E which is compatible with $g^E(\tau)$ and which also depends smoothly on τ. The *Bochner Laplacian* (or *connection Laplacian*) Δ^τ with respect to $\nabla^E(\tau)$ and $g(\tau)$, acting on smooth sections ϑ of E, is defined as the trace (with respect to $g(\tau)$) of the second covariant derivative (with respect to $\nabla^E(\tau)$ and $\nabla(\tau)$) of ϑ.

One is often not only interested in the Bochner Laplacian, but more generally in operators allowing a Weitzenböck type decomposition with respect to the Bochner Laplacian of the form

$$\Delta^\tau - \mathcal{R}^\tau,$$

where $(\mathcal{R}^\tau)_{\tau\in[T_1,T_2]}$ is a family of symmetric (with respect to $g^E(\tau)$) endomorphisms of E, depending smoothly on τ.

The most important example is presumably $E = \Lambda^p M$, the bundle of differential p-forms on M, equipped with the family of Weitzenböck operators

$$\mathcal{R}^\tau = \Delta^\tau - \Box^\tau,$$

where $\Box^\tau$ denotes the Hodge-de Rham Laplacian on $\Lambda^p M$, see e.g. [**11**, Section 7.1]. In particular, if $p = 1$, $\mathcal{R}^\tau$ equals the Ricci tensor with respect to $g(\tau)$ considered as an endomorphism of $\Lambda^1 M = T^*M$, see e.g. [**11**, Corollary 7.1.4].

Extending the seminal ideas of Professor Itô [**12**] to the case of time-dependent geometry we prove in this paper a stochastic representation formula for solutions of the heat type equation

$$\frac{\partial\vartheta}{\partial\tau} = \frac{1}{2}(\Delta^\tau - \mathcal{R}^\tau)\vartheta \tag{1.1}$$

and apply it to prove a vanishing theorem for bounded ancient solutions under a positivity condition on

$$\mathcal{R}^\tau - \frac{\partial g^E}{\partial\tau}.$$

In the case of 1-forms this condition means that the metric of M evolves under uniformly strict super Ricci flow.

2. Stochastic representation formula.

We denote by $\underline{\mathcal{R}}(\tau, y)$ the lowest eigenvalue of $\mathcal{R}^\tau_y - ((\partial g^E/\partial\tau)(\tau, y))^{\#g^E(\tau)}$. Here the superscript $^{\#g^E(\tau)}$ means that using the metric $g^E(\tau)$ we regard $\partial g^E/\partial\tau$ as an endomorphism of E.

Remark 2.1. Note that if $E = T^{p,q}M := (TM)^{\otimes p} \otimes (T^*M)^{\otimes q}$ is a tensor bundle over M and $g^E(\tau)$ the canonical metric induced from $g(\tau)$ on the base manifold M, we have for all $v_1, \dots, v_p \in T_yM$ and $\alpha_1, \dots, \alpha_q \in T^*_yM$,

$$\left(\frac{\partial g^E}{\partial\tau}(\tau,y)\right)^{\#g^E(\tau)}(v_1\otimes\cdots\otimes v_p\otimes\alpha_1\otimes\cdots\otimes\alpha_q)$$
$$=\sum_{i=1}^p v_1\otimes\cdots\otimes v_{i-1}\otimes\left(\frac{\partial g}{\partial\tau}(\tau,y)\right)^{\#g(\tau)}v_i\otimes v_{i+1}\otimes\cdots\otimes v_p\otimes\alpha_1\otimes\cdots\otimes\alpha_q$$
$$-\sum_{j=1}^q v_1\otimes\cdots\otimes v_p\otimes\alpha_1\otimes\cdots\otimes\alpha_{j-1}\otimes\left(\frac{\partial g}{\partial\tau}(\tau,y)\right)\big(\alpha_j^{\#},\,\cdot\,\big)\otimes\alpha_{j+1}\otimes\cdots\otimes\alpha_q. \tag{2.1}$$

The reason for the minus sign in formula (2.1) is as follows. Recall that for $\alpha,\beta\in T_y^*M$,

$$\frac{\partial g^{T^*M}(\tau,y)}{\partial\tau}(\alpha,\beta)=\frac{\partial g(\tau,y)(\alpha^{\#g(\tau)},\beta^{\#g(\tau)})}{\partial\tau}=-\left(\frac{\partial g(\tau,y)}{\partial\tau}\right)\big(\alpha^{\#g(\tau)},\beta^{\#g(\tau)}\big)$$

which is a consequence of

$$0=\frac{\partial}{\partial\tau}\alpha(\,\cdot\,)=\frac{\partial}{\partial\tau}g(\tau,y)\big(\alpha^{\#g(\tau)},\,\cdot\,\big)=\left(\frac{\partial g(\tau,y)}{\partial\tau}\right)\big(\alpha^{\#g(\tau)},\,\cdot\,\big)+g(\tau,y)\left(\frac{\partial\alpha^{\#g(\tau)}}{\partial\tau},\,\cdot\,\right).$$

In the special case of the (backward/forward) Ricci flow $\partial g/\partial\tau=\pm\operatorname{Ric}_{g(\tau)}$, we thus find

$$\left(\frac{\partial g^E}{\partial\tau}(\tau,y)\right)^{\#g^E(\tau)}(v_1\otimes\cdots\otimes v_p\otimes\alpha_1\otimes\cdots\otimes\alpha_q)$$
$$=\pm\sum_{i=1}^p v_1\otimes\cdots\otimes v_{i-1}\otimes\operatorname{Ric}(v_i,\,\cdot\,)^{\#}\otimes v_{i+1}\otimes\cdots\otimes v_p\otimes\alpha_1\otimes\cdots\otimes\alpha_q$$
$$\mp\sum_{j=1}^q v_1\otimes\cdots\otimes v_p\otimes\alpha_1\otimes\cdots\otimes\alpha_{j-1}\otimes\operatorname{Ric}\big(\alpha_j^{\#},\,\cdot\,\big)\otimes\alpha_{j+1}\otimes\cdots\otimes\alpha_q.$$

We now fix $x\in M$ and let $X=(X_t)_{0\le t\le T_2-T_1}$ be a $(g(T_2-t))_{0\le t\le T_2-T_1}$-Brownian motion on M starting at x [**1**], [**3**], [**9**], [**10**], [**14**], [**15**], [**18**]. Throughout the paper we assume that it cannot explode (see [**14**], [**18**] for sufficient criteria). We denote by $//_{0,t}^E\colon E_x\to E_{X_t}$ the parallel transport in E along X. This is a random isometry which will be defined in the proof of Lemma 5.1.

THEOREM 2.2 (Stochastic representation formula). *Assume that $\underline{\mathcal{R}}$ is bounded from below, and let $\Phi_t\colon E_x\to E_x$ be the solution to the (random) ODE*

$$\frac{d\Phi_t}{dt}=-\frac12\,\Phi_t(//_{0,t}^E)^{-1}\left(\mathcal{R}_{X_t}^{T_2-t}-\left(\frac{\partial g^E}{\partial\tau}(T_2-t,X_t)\right)^{\#g^E(T-t)}\right)//_{0,t}^E \tag{2.2}$$

with initial value $\Phi_0=\mathrm{Id}_{E_x}$. (This is a linear ODE, hence its solution cannot explode.)

Then any bounded solution $\vartheta\colon [T_1, T_2] \to \Gamma(E)$ *to the heat type equation* (1.1) *has the stochastic representation*

$$\vartheta(T_2, x) = \mathbb{E}\big[\Phi_{T_2-T_1}(//^E_{0,T_2-T_1})^{-1}\,\vartheta(T_1, X_{T_2-T_1})\big]. \tag{2.3}$$

Moreover, the following estimate holds:

$$|\vartheta(T_2,x)|_{g^E(T_2)} \le \mathbb{E}\bigg[\exp\bigg(-\frac{1}{2}\int_0^{T_2-T_1} \underline{\mathcal{R}}(T_2-s, X_s)\,ds\bigg)|\vartheta(T_1, X_{T_2-T_1})|_{g^E(T_1)}\bigg]. \tag{2.4}$$

The key ingredient to the proof of Theorem 2.2 is the following proposition whose proof is given in Section 5.

PROPOSITION 2.3. *Let* $\vartheta\colon [T_1, T_2] \to \Gamma(E)$ *be any smooth time-dependent section of* E (*not necessarily bounded, and not necessarily a solution of Equation* (1.1)). *Then the* E_x*-valued stochastic process*

$$N_t := \Phi_t(//^E_{0,t})^{-1}\,\vartheta(T_2-t, X_t) \tag{2.5}$$

satisfies

$$\begin{aligned} dN_t &= -\Phi_t(//^E_{0,t})^{-1}\bigg(\frac{\partial\vartheta}{\partial\tau} - \frac{1}{2}\big(\Delta^{T_2-t} - \mathcal{R}^{T_2-t}_{X_t}\big)\vartheta\bigg)(T_2-t, X_t)\,dt \\ &\quad + \Phi_t\sum_{i=1}^d (//^E_{0,t})^{-1}\nabla^{E,T_2-t}_{U_te_i}\vartheta(T_2-t,X_t)\,dB^i_t, \end{aligned}$$

where $(U_t)_{0\le t\le T_2-T_1}$ *is the* $(g(T_2-t))_{t\ge 0}$*-horizontal lift of* X *with respect to an arbitrary initial frame* $U_0 \in \mathscr{O}^{g(T_2)}_x(M)$ (*see Equation* (5.2) *below*) *and* $(B_t)_{0\le t\le T_2-T_1}$ *is the corresponding anti-development* (*which is a standard* $\mathbb{R}^d$*-valued Brownian motion*).

REMARK 2.4. In the special case where E is a tensor bundle over M and $g^E(\tau)$ is the usual extension of $g(\tau)$, Proposition 2.3 is due to Chen et al. [**2**, Equation (3.7)]. Our Proposition 2.3 is considerably more general in the following two respects:

- It is not restricted to tensor bundles, but holds on arbitrary vector bundles E over M.
- The metrics on E need not be related in any way to the metrics on M.

For the proof of Theorem 2.2 we also need the following lemma:

LEMMA 2.5. *For all* $t \in [0, T_2 - T_1]$ *we have*

$$|\Phi_t|_{g^E(T_2),g^E(T_2)} \le \exp\bigg(-\frac{1}{2}\int_0^t \underline{\mathcal{R}}(T_2-s, X_s)ds\bigg). \tag{2.6}$$

PROOF. Fix $v \in E_x$ and let $f(t) := |\Phi_t^* v|^2_{g^E(T_2)}$. Then Equation (2.2) and the definition of $\underline{\mathcal{R}}$ imply that $f'(t) \leq -\underline{\mathcal{R}}(T_2 - t, X_t) f(t)$ and hence

$$f(t) \leq \exp\Big(-\int_0^t \underline{\mathcal{R}}(T_2 - s, X_s)ds\Big) f(0),$$

i.e.

$$|\Phi_t^* v|_{g^E(T_2)} \leq \exp\Big(-\frac{1}{2}\int_0^t \underline{\mathcal{R}}(T_2 - s, X_s)ds\Big) |v|_{g^E(T_2)}.$$

Since $|\Phi_t^*|_{g^E(T_2),g^E(T_2)} = |\Phi_t|_{g^E(T_2),g^E(T_2)}$ this gives the claim. $\square$

PROOF OF THEOREM 2.2. Since ϑ satisfies Equation (1.1), Proposition 2.3 implies that the process N defined in (2.5) is a local martingale. Since moreover ϑ and Φ are bounded (the latter by Lemma 2.5 and the assumption that $\underline{\mathcal{R}}$ is bounded from below), N is seen to be a true martingale. Taking expectations of N at $t = 0$ and $t = T_2 - T_1$ yields (2.3).

The estimate (2.4) follows from (2.3), (2.6) along with the fact that $//^E_{0,T_2-T_1}$ is an isometry from $(E_x, g(T_2))$ to $(E_{X_{T_2-T_1}}, g(T_1))$. $\square$

3. Solution flow domination.

Let f be a bounded solution to the scalar heat-type equation

$$\frac{\partial f}{\partial \tau} = \frac{1}{2}\big(\Delta^{g(\tau)} - \underline{\mathcal{R}}(\tau, \cdot)\big) f \tag{3.1}$$

with $f(T_1, \cdot) = |\vartheta(T_1, \cdot)|_{g^E(T_1)}$. Then Equation (2.2) reduces to the scalar ODE

$$\frac{d\Phi_t}{dt} = -\frac{1}{2}\,\Phi_t\,\underline{\mathcal{R}}(T_2 - t, X_t)$$

with initial value $\Phi_0 = 1$. It follows that

$$\Phi_t = \exp\Big(-\frac{1}{2}\int_0^t \underline{\mathcal{R}}(T_2 - s, X_s)ds\Big),$$

so that Theorem 2.2 implies

$$f(T_2, x) = \mathbb{E}\Big[\exp\Big(-\frac{1}{2}\int_0^{T_2-T_1} \underline{\mathcal{R}}(T_2 - s, X_s)ds\Big) |\vartheta(T_1, X_{T_2-T_1})|_{g^E(T_1)}\Big]. \tag{3.2}$$

As a consequence we can reformulate estimate (2.4) as follows:

THEOREM 3.1 (Solution flow domination). *We have*

$$|\vartheta(T_2, x)|_{g^E(T_2)} \leq f(T_2, x).$$

4. Vanishing theorem for bounded ancient solutions.

THEOREM 4.1 (Vanishing theorem). *Let $(g(\tau))_{-\infty<\tau\le T_2}$ be an ancient family of Riemannian metrics such that $\underline{\mathcal{R}}$ is bounded from below on $(-\infty,T_2]\times M$ and*

$$\liminf_{T_1\to-\infty}\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^{T_2-T_1}\underline{\mathcal{R}}(T_2-s,X_s)ds\right)\right]=0 \tag{4.1}$$

for all $x\in M$ (x enters condition (4.1) *as the starting point of the Brownian motion X). Then every bounded ancient solution $\vartheta\colon(-\infty,T_2]\to\Gamma(E)$ to the heat type Equation* (1.1) *vanishes.*

PROOF. It clearly suffices to prove that $\vartheta(T_2,x)=0$ for all $x\in M$. By (2.4), for all $T_1\in(-\infty,T_2]$,

$$|\vartheta(T_2,x)|_{g^E(T_2)}\le\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^{T_2-T_1}\underline{\mathcal{R}}(T_2-s,X_s)ds\right)\right]\sup_{y\in M}|\vartheta(T_1,y)|_{g^E(T_1)}.$$

Letting $T_1\to-\infty$, the claim follows from (4.1) and the boundedness of ϑ. □

REMARK 4.2. Trivially condition (4.1) holds if $\underline{\mathcal{R}}(\tau,y)\ge C>0$ for all $(\tau,y)\in(-\infty,T_2]\times M$. In the special case of 1-forms, this means that the metric of M evolves under uniformly strict super Ricci flow, i.e.

$$\frac{\partial g}{\partial\tau}+\mathrm{Ric}_{g(\tau)}\ge C$$

for some $C>0$. Here we used that in this case (see Remark 2.1 above) for $\alpha\in T^*M$,

$$\mathcal{R}^\tau(\alpha)=\mathrm{Ric}_{g(\tau)}\left(\alpha^\#,\,\cdot\right)\quad\text{and}\quad\frac{\partial g^E}{\partial\tau}(\alpha)=-\left(\frac{\partial g}{\partial\tau}\right)\left(\alpha^\#,\,\cdot\right).$$

REMARK 4.3. Since the endomorphisms $\mathcal{R}^\tau$ may depend on ϑ, our results can also be applied to nonlinear equations. As observed by Chen et al. [**2**] such nonlinear equations arise naturally in the context of geometric flows such as the Ricci flow or the mean curvature flow.

5. Proof of Proposition 2.3.

To keep notation simple, we assume in this section without loss of generality that $T_1=0$. Moreover we write T instead of T_2. We need the following lemma.

LEMMA 5.1 (cf. [**2**, Proposition 2.1] for the case that E is a tensor bundle and ϑ independent of time). *The E_x-valued process*

$$\tilde{N}_t:=(//^E_{0,t})^{-1}\vartheta(T-t,X_t)$$

(*where* $//^E_{0,t}\colon E_x \to E_{X_t}$ *is defined in the proof*) *satisfies*

$$\begin{aligned} d\tilde{N}_t &= -(//^E_{0,t})^{-1}\biggl(\frac{\partial\vartheta}{\partial\tau} - \frac{1}{2}\Delta^{T-t}\vartheta + \frac{1}{2}\biggl(\frac{\partial g^E}{\partial\tau}\biggr)^{\#g^E(T-t)}\vartheta\biggr)(T-t, X_t)\,dt \\ &\quad + \sum_{i=1}^d (//^E_{0,t})^{-1}\nabla^{E,T-t}_{U_t e_i}\vartheta(T-t, X_t)\,dB^i_t. \end{aligned}$$

Proof. Let $\pi\colon \mathscr{F}(E) \to M$ the frame bundle of E (so that for $x \in M$ the fiber $\mathscr{F}(E)_x$ is the set of linear isomorphisms from $\mathbb{R}^k$ to E_x). Moreover let $\mathscr{F}(M) \times_M \mathscr{F}(E)$ be the product of the fiber bundles $\mathscr{F}(M)$ and $\mathscr{F}(E)$, i.e.,

$$\mathscr{F}(M) \times_M \mathscr{F}(E) = \{(u,\psi) \in \mathscr{F}(M) \times \mathscr{F}(E) \mid \pi u = \pi\psi\}.$$

Note that for all $(u,\psi) \in \mathscr{F}(M) \times_M \mathscr{F}(E)$ we have the following canonical identification:

$$T_{(u,\psi)}(\mathscr{F}(M) \times_M \mathscr{F}(E)) \simeq \bigl\{(X_1, X_2) \in T_u\mathscr{F}(M) \times T_\psi\mathscr{F}(E) \bigm| \pi_* X_1 = \pi_* X_2\bigr\}. \tag{5.1}$$

For $\tau \in [T_1, T_2]$ and $i \in \{1, \ldots, d\}$ we define the i-th standard horizontal vector field

$$\mathcal{H}^\tau_i = \mathcal{H}^{\nabla(\tau), \nabla^E(\tau)}_i$$

on $\mathscr{F}(M) \times_M \mathscr{F}(E)$ with respect to $\nabla(\tau)$ and $\nabla^E(\tau)$ as follows: In the sense of the identification (5.1), for $(u,\psi) \in \mathscr{F}(M) \times_M \mathscr{F}(E)$ the first component of $\mathcal{H}^\tau_i(u,\psi)$ is the $\nabla(\tau)$-horizontal lift of ue_i to $T_u\mathscr{F}(M)$, and the second component is the $\nabla^E(\tau)$-horizontal lift of ue_i to $T_\psi\mathscr{F}(E)$.

Let now Ψ_0 be an arbitrary element of $\mathscr{O}^{g^E(T)}_x(E)$, and let $(U_t, \Psi_t)_{0\le t\le T}$ be the solution to the Stratonovich SDE

$$\begin{aligned} d(U_t, \Psi_t) &= \sum_{i=1}^d \mathcal{H}^{T-t}_i(U_t, \Psi_t) * dB^i_t \\ &\quad + \frac{1}{2}\biggl(\frac{\partial g}{\partial\tau}(T-t)\biggr)^{\#g(T-t)} \circ U_t\,dt \\ &\quad + \frac{1}{2}\biggl(\frac{\partial g^E}{\partial\tau}(T-t)\biggr)^{\#g(T-t)} \circ \Psi_t\,dt \end{aligned} \tag{5.2}$$

with initial value (U_0, Ψ_0). Note that by construction $\pi U_t = \pi\Psi_t = X_t$ for all $t \in [0,T]$. In addition, we have $U_t \in \mathscr{O}^{g(T-t)}_x(M)$ and $\Psi_t \in \mathscr{O}^{g^E(T-t)}_x(E)$ for all $t \in [0,T]$. We then define

$$//^E_{0,t} := \Psi_t \circ \Psi_0^{-1}.$$

Similarly to [**11**, Section 2.2] we define the scalarization of ϑ as the map

$$\tilde{\vartheta}\colon [0,T] \times \mathscr{F}(E) \to \mathbb{R}^k$$

given by

$$\tilde{\vartheta}(\tau,\psi) := \psi^{-1}\vartheta(\tau,\pi\psi).$$

Clearly, $\tilde{N}_t = \Psi_0\tilde{\vartheta}(T-t,\Psi_t)$ so that

$$\begin{aligned} d\tilde{N}_t &= \Psi_0 d\tilde{\vartheta}(T-t,\Psi_t) \\ &= -\Psi_0 \frac{\partial\tilde{\vartheta}}{\partial\tau}(T-t,\Psi_t)\,dt + \frac{1}{2}\Psi_0\sum_{i=1}^d (\mathcal{H}_i^{T-t})^2\tilde{\vartheta}(T-t,U_t,\Psi_t)\,dt \\ &\quad + \frac{1}{2}\Psi_0\left(\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(T-t)} \circ \Psi_t\right)\tilde{\vartheta}(T-t,\Psi_t)\,dt \\ &\quad + \Psi_0\sum_{i=1}^d \mathcal{H}_i^{T-t}\tilde{\vartheta}(T-t,U_t,\Psi_t)\,dB_t^i. \end{aligned}$$

The claim now follows from Lemma 5.2, Lemma 6.1 and Corollary 6.3 below. □

LEMMA 5.2. *For all $\tau \in [0,t]$ and all $\psi \in \mathscr{F}(E)$,*

$$\left(\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(\tau)} \circ \psi\right)\tilde{\vartheta}(\tau,\psi) = -\psi^{-1}\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(\tau)}\vartheta(\tau,\pi\psi).$$

PROOF. We have

$$\begin{aligned} \left(\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(\tau)} \circ \psi\right)\tilde{\vartheta}(\tau,\psi) &= \frac{d}{ds}\bigg|_{s=0}\tilde{\vartheta}\left(\tau,\psi + s\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(\tau)} \circ \psi\right) \\ &= \frac{d}{ds}\bigg|_{s=0}\left(\psi + s\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(\tau)} \circ \psi\right)^{-1}\vartheta(\tau,\pi\psi) \\ &= \psi^{-1}\frac{d}{ds}\bigg|_{s=0}\left(\mathrm{Id}_{E_{\pi\psi}} + s\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(\tau)}\right)^{-1}\vartheta(\tau,\pi\psi) \\ &= -\psi^{-1}\left(\frac{\partial g^E}{\partial\tau}\right)^{\#g^E(\tau)}\vartheta(\tau,\pi\psi), \end{aligned}$$

as claimed. □

Proof of Proposition 2.3. We have $N_t = \Phi_t \tilde{N}_t$. Using Lemma 5.1 this implies

$$\begin{aligned}
dN_t &= (d\Phi_t)(//^E_{0,t})^{-1}\vartheta(T-t, X_t) + \Phi_t d\tilde{N}_t \\
&= -\frac{1}{2}\,\Phi_t(//^E_{0,t})^{-1}\left(\mathcal{R}^{T-t}_{X_t} - \left(\frac{\partial g^E}{\partial \tau}\right)^{\#g^E(T-t)}\right)\vartheta(T-t, X_t)\,dt \\
&\quad - \Phi_t(//^E_{0,t})^{-1}\left(\frac{\partial \vartheta}{\partial \tau} - \frac{1}{2}\Delta^{T-t}\vartheta + \frac{1}{2}\left(\frac{\partial g^E}{\partial \tau}\right)^{\#g^E(T-t)}\vartheta\right)(T-t, X_t)\,dt \\
&\quad + \Phi_t \sum_{i=1}^d (//^E_{0,t})^{-1}\nabla^{E,T-t}_{U_t e_i}\vartheta(T-t, X_t)\,dB^i_t \\
&= -\Phi_t(//^E_{0,t})^{-1}\left(\frac{\partial \vartheta}{\partial \tau} - \frac{1}{2}\big(\Delta^{T-t} - \mathcal{R}^{T-t}_{X_t}\big)\vartheta\right)(T-t, X_t)\,dt \\
&\quad + \Phi_t \sum_{i=1}^d (//^E_{0,t})^{-1}\nabla^{E,T-t}_{U_t e_i}\vartheta(T-t, X_t)\,dB^i_t,
\end{aligned}$$

as claimed. □

6. Appendix.

In this appendix we fix $\tau \in [0, T]$ and, to simplify notation, suppress it in the sequel.

Lemma 6.1 (cf. [**11**, Proposition 2.2.1]). *Let ϑ be a section of E, $x \in M$, $\psi \in \mathscr{F}(E)_x$, $\xi \in T_xM$ and $\bar{\xi}$ the ∇^E-horizontal lift of ξ to $T_\psi \mathscr{F}(E)$. Then*

$$\bar{\xi}\tilde{\vartheta} = \widetilde{\nabla^E_\xi \vartheta}(\psi).$$

Proof. Let $(\psi_t)_{t\in\mathbb{R}}$ be a horizontal curve (with respect to ∇^E) in $\mathscr{F}(E)$ with $\psi_0 = \psi$ and $\dot{\psi}_0 = \bar{\xi}$, and let $x_t := \pi\psi_t$ (so that $x_0 = x$ and $\dot{x}_0 = \xi$). Then

$$//_{0,t} := \psi_t\psi_0^{-1} : E_x \to E_{x_t}$$

is the parallel transport (with respect to ∇^E) along the curve $(x_t)_{t\in\mathbb{R}}$. Consequently,

$$\begin{aligned}
\bar{\xi}\tilde{\vartheta} &= \left.\frac{d}{dt}\right|_{t=0} \tilde{\vartheta}(\psi_t) \\
&= \left.\frac{d}{dt}\right|_{t=0} \psi_t^{-1}\vartheta(\pi\psi_t) \\
&= \psi_0^{-1}\left[\left.\frac{d}{dt}\right|_{t=0} //^{-1}_{0,t}\vartheta(x_t)\right] \\
&= \psi^{-1}\nabla^E_\xi \vartheta \\
&= \widetilde{\nabla^E_\xi \vartheta}(\psi),
\end{aligned}$$

as claimed. □

LEMMA 6.2 (cf. [**11**, Equation (2.2.3)]). *For* $(u,\psi)\in\mathscr{F}(M)\times_M\mathscr{F}(E)$ *and* $i,j\in\{1,\dots,n\}$ *we have*

$$(\mathcal{H}_i\mathcal{H}_j\tilde{\vartheta})(u,\psi)=\psi^{-1}\operatorname{Hess}\vartheta(ue_i,ue_j).$$

PROOF. As in the proof of Lemma 6.1 we obtain

$$(\mathcal{H}_j\tilde{\vartheta})(u,\psi)=\psi^{-1}\nabla^E_{ue_j}\vartheta$$

for all $(u,\psi)\in\mathscr{F}(M)\times_M\mathscr{F}(E)$. We now fix $(u,\psi)\in\mathscr{F}(M)\times_M\mathscr{F}(E)$ and let $(u_t,\psi_t)_{t\in\mathbb{R}}$ be a horizontal curve in $\mathscr{F}(M)\times_M\mathscr{F}(E)$ such that $(u_0,\psi_0)=(u,\psi)$ and $(\dot{u}_0,\dot{\psi}_0)=\mathcal{H}_i(u,\psi)$. Let $x_t:=\pi u_t=\pi\psi_t$ (so that $\dot{x}_0=ue_i$). Then

$$//_{0,t}=\psi_t\psi_0^{-1}:E_x\to E_{x_t}$$

is the parallel transport (with respect to ∇^E) along the curve $(x_t)_{t\in\mathbb{R}}$. Consequently,

$$\begin{aligned}(\mathcal{H}_i\mathcal{H}_j\tilde{\vartheta})(u,\psi)&=\frac{d}{dt}\bigg|_{t=0}\psi_t^{-1}\nabla^E_{u_te_j}\vartheta\\&=\psi_0^{-1}\frac{d}{dt}\bigg|_{t=0}//_{0,t}^{-1}\nabla^E_{u_te_j}\vartheta\\&=\psi^{-1}\operatorname{Hess}\vartheta(ue_i,ue_j),\end{aligned}$$

as claimed. □

COROLLARY 6.3 (Horizontal Laplacian, cf. [**11**, Proposition 3.1.2]). *For all* $(u,\psi)\in\mathscr{O}(M)\times\mathscr{F}(E)$*, we have*

$$\sum_{i=1}^d(\mathcal{H}_i^2\tilde{\vartheta})(u,\psi)=\psi^{-1}\Delta\vartheta(\pi\psi).$$

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