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Martingales on Manifolds with Time-Dependent Connection

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Abstract We define martingales on manifolds with time-dependent connection, extending in this way the theory of stochastic processes on manifolds with time-changing geometry initiated by Arnaudon et al. (C R Acad Sci Paris Ser I 346:773–778, 2008). We show that some, but not all, properties of martingales on manifolds with a fixed connection extend to this more general setting.

Keywords Stochastic analysis on manifolds · Time-dependent geometry · Martingales

Mathematics Subject Classification (2010) 53C44 · 58J65 · 60G44 · 60G48

1 Introduction

Stochastic analysis on manifolds with a fixed connection or a fixed Riemannian metric has been studied for a long time, see e.g. the books by Hackenbroch and Thalmaier [9] and Hsu [10]. Motivated by Perelman's proof of the geometrization and hence the Poincaré conjecture using Ricci flow [14–16], Arnaudon, Coulibaly and

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Thalmaier [1] introduced Brownian motion on a manifold with a time-dependent Riemannian metric. Thanks to the subsequent papers by Coulibaly-Pasquier [5], Kuwada and Philipowski [11,12] and Paeng [13], Brownian motion in such a time-dependent framework is now well understood.

Stochastic analysis on manifolds, however, is not restricted to the study of Brownian motion. Another important topic is martingale theory, which in the case of a fixed connection is treated in depth in e.g. [6,7,9,10], but which has not yet been studied in the case of a time-dependent connection. The aim of the present paper was to fill this gap.

The results of this paper will be fundamental for various geometric applications in subsequent papers which include a study of the harmonic map heat flow on manifolds with time-dependent metric, stochastic representations of harmonic forms in a time-dependent setting, as well as new entropy formulas for positive solutions to the heat equation under Ricci flow.

2 Horizontal Lift, Stochastic Parallel Transport and Stochastic Development on Manifolds with Time-Dependent Connection

Let M be a d-dimensional differentiable manifold, $\pi : \mathcal{F}(M) \to M$ the frame bundle and $(\nabla(t))_{t\geq 0}$ a family of linear connections on M depending smoothly on t. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space. Throughout the whole paper, the notions of martingale, semimartingale, etc., are understood with respect to this filtration. Moreover, all processes are tacitly assumed to be continuous.

Definition 2.1 (cf. [9, Definition 7.135] for the case of a fixed connection) An $\mathcal{F}(M)$-valued semimartingale U is said to be $(\nabla(t))_{t\geq 0}$-*horizontal* if

$$\omega_t\,(\circ\ \mathrm{d}U_t) = 0, \tag{2.1}$$

where ω_t is the $\mathbb{R}^{d\times d}$-valued connection form with respect to $\nabla(t)$.

Proposition 2.2 (cf. [9, Satz 7.141] for the case of a fixed connection) *Let X be an M-valued semimartingale and U_0 an $\mathcal{F}_0$-measurable $\mathcal{F}(M)$-valued random variable with $\pi\, U_0 = X_0$. Then there exists a unique $(\nabla(t))_{t\geq 0}$-horizontal lift U of X starting at U_0, i.e. an $\mathcal{F}(M)$-valued semimartingale satisfying* (2.1) *and $\pi\, U = X$. Moreover, starting with an arbitrary lift $\tilde{U}$ of X satisfying $\tilde{U}_0 = U_0$ (which can be constructed using charts, see e.g.* [9, Proof of Satz 7.141]*) the horizontal lift U can be constructed in the following way: Let*

$$\gamma_t := \int_0^t \omega_s\left(\circ\ \mathrm{d}\tilde{U}_s\right), \tag{2.2}$$

and G the solution to the $\mathrm{GL}_d(\mathbb{R})$*-valued SDE*

$$\mathrm{d}G_t = -\sum_{\alpha,\beta=1}^{d} E_{\alpha\beta} G_t \circ \mathrm{d}\gamma_t^{\alpha\beta}, \qquad G_0 = I, \tag{2.3}$$

where $E_{\alpha\beta} \in \mathbb{R}^{d\times d}$ is the matrix whose (i, j)-entry is 1 if $i = \alpha$ and $j = \beta$, and 0 otherwise. Then

$$U_t = \tilde{U}_t G_t. \tag{2.4}$$

Proof We first show that the process U defined by (2.4) is indeed $(\nabla(t))_{t\geq 0}$-horizontal. Letting $\Phi\colon \mathcal{F}(M) \times \mathrm{GL}_d(\mathbb{R}) \to \mathcal{F}(M)$ defined by $\Phi(u, g) := ug$, we have

$$\left(\Phi^*\omega_t\right)_{(u,g)} = \left(R_g^*\omega_t\right)_u + \theta_g,$$

where $\theta_g := dL_g^{-1}$ (L_g and R_g denoting left resp. right multiplication with g). Since moreover by [9, Bemerkung 7.128 (ii)], $R_g^*\omega_t = \mathrm{Ad}(g^{-1})\omega_t$, we obtain

$$\begin{aligned}
\omega_t(\circ\ \mathrm{d}U_t) &= \omega_t\left(\circ\ \mathrm{d}\Phi(\tilde{U}_t, G_t)\right)\\
&= \left(\Phi^*\omega_t\right)\left(\circ\ d\left(\tilde{U}_t, G_t\right)\right)\\
&= \left(R_{G_t}^*\omega_t\right)\left(\circ\ \mathrm{d}\tilde{U}_t\right) + \theta_{G_t}\left(\circ\ \mathrm{d}G_t\right)\\
&= \mathrm{Ad}\left(G_t^{-1}\right)\omega_t\left(\circ\ \mathrm{d}\tilde{U}_t\right) + \mathrm{d}L_{G_t}^{-1}\left(\circ\ \mathrm{d}G_t\right)\\
&= \mathrm{Ad}\left(G_t^{-1}\right)\circ \mathrm{d}\gamma_t - \sum_{\alpha,\beta=1}^{d} G_t^{-1}E_{\alpha\beta}G_t \circ \mathrm{d}\gamma_t^{\alpha\beta} = 0.
\end{aligned}$$

To show uniqueness, assume that U' is another $(\nabla(t))_{t\geq 0}$-horizontal lift of X with $U'_0 = U_0$. Then, $U = U'G$ with a $\mathrm{GL}_d(\mathbb{R})$-valued semimartingale $G = (G_t)_{t\geq 0}$ starting at I. The above computation yields $dL_{G_t}^{-1}(\circ\ dG_t) = 0$ and hence $dG_t = 0$, so that $G_t = I$ for all $t \geq 0$. □

Definition 2.3 (cf. [9, Definition 7.144] for the case of a fixed connection) The $(\nabla(t))_{t\geq 0}$*-parallel transport* along an M-valued semimartingale X is the family of isomorphisms $//_{s,t} : T_{X_s}M \to T_{X_t}M$ $(0 \leq s \leq t)$ defined by

$$//_{s,t} := U_t\, U_s^{-1},$$

where U is an arbitrary $(\nabla(t))_{t\geq 0}$-horizontal lift of X. (As in the case of a fixed connection, the result does not depend on the choice of the horizontal lift.)

Definition 2.4 (cf. [9, Definition 7.136] for the case of a fixed connection) Let X be an M-valued semimartingale, U_0 an $\mathcal{F}_0$-measurable $\mathcal{F}(M)$-valued random variable with $\pi\, U_0 = X_0$, and U the unique $(\nabla(t))_{t\geq 0}$-horizontal lift of X starting at U_0. The $\mathbb{R}^d$-valued process

$$Z_t := \int_0^t \vartheta(\circ\, \mathrm{d}U_s) \tag{2.5}$$

is called the $(\nabla(t))_{t\geq 0}$-*antidevelopment* of X (or U) with initial frame U_0; here ϑ is the canonical $\mathbb{R}^d$-valued 1-form on $\mathcal{F}(M)$,

$$\vartheta_u(w) = u^{-1}(\mathrm{d}\pi\, w), \quad w \in T_u\mathcal{F}(M).$$

Remark 2.5 A $(\nabla(t))_{t\geq 0}$-horizontal semimartingale U can be recovered from its $(\nabla(t))_{t\geq 0}$-antidevelopment Z and its initial value U_0 as the solution to the SDE

$$\mathrm{d}U_t = \sum_{i=1}^d H_i^{\nabla(t)}(U_t) \circ \mathrm{d}Z_t^i, \tag{2.6}$$

where $(H_i^{\nabla(t)})_{i=1}^d$ are the standard $\nabla(t)$-horizontal vector fields on $\mathcal{F}(M)$, i.e.

$$H_i^{\nabla(t)}(u) = h_u^{\nabla(t)}(ue_i), \quad u \in \mathcal{F}(M),$$

where $h_u^{\nabla(t)}\colon\ T_{\pi(u)}M \to T_u\mathcal{F}(M)$ is the horizontal lift with respect to the connection $\nabla(t)$.

Proof To verify that U solves SDE (2.6), one has to show that

$$f(U_t) - f(U_0) = \sum_{i=1}^d \int_0^t H_i^{\nabla(s)} f(U_s) \circ \mathrm{d}Z_s^i$$

for all $f \in C^\infty(\mathcal{F}(M))$. As in [9, Proof of Satz 7.137], this can be done as follows: first, note that for $u \in \mathcal{F}(M)$ and $\xi \in T_u\mathcal{F}(M)$, we have

$$\begin{aligned}
\sum_{i=1}^d H_i^{\nabla(s)} f(u)\vartheta_u^i(\xi) &= \sum_{i=1}^d df(u) H_i^{\nabla(s)}(u)\vartheta_u^i(\xi) \\
&= \sum_{i=1}^d df(u) h_u^{\nabla(s)}(ue_i) \left(u^{-1}\pi_*\xi\right)^i \\
&= \sum_{i=1}^d df(u) h_u^{\nabla(s)}(\pi_*\xi).
\end{aligned}$$

Together with the horizontality of U and (2.5), this implies that

$$\begin{aligned} f(U_t) - f(U_0) &= \int_0^t df(\circ\, \mathrm{d}U_s) \\ &= \sum_{i=1}^d \int_0^t H_i^{\nabla(s)} f(U_s) \vartheta_u^i(\circ\, \mathrm{d}U_s) \\ &= \sum_{i=1}^d \int_0^t H_i^{\nabla(s)} f(U_s) \circ \mathrm{d}Z_s^i, \end{aligned}$$

as claimed. □

Corollary 2.6 *Let U be a $(\nabla(t))_{t\geq 0}$-horizontal semimartingale, and $X := \pi\, U$. Then we have the following Itô formulas:*

1. *For all smooth functions f on $\mathbb{R}_+ \times \mathcal{F}(M)$ we have*

$$\begin{aligned} d(f(t, U_t)) &= \frac{\partial f}{\partial t}(t, U_t)\, \mathrm{d}t + \sum_{i=1}^d H_i^{\nabla(t)} f(t, U_t) \circ \mathrm{d}Z_t^i \\ &= \frac{\partial f}{\partial t}(t, U_t)\, \mathrm{d}t + \sum_{i=1}^d H_i^{\nabla(t)} f(t, U_t)\, \mathrm{d}Z_t^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d H_i^{\nabla(t)} H_j^{\nabla(t)} f(t, U_t)\, d\langle Z^i, Z^j\rangle_t. \end{aligned}$$

2. *For all smooth functions f on $\mathbb{R}_+ \times M$ we have*

$$\begin{aligned} d(f(t, X_t)) &= \frac{\partial f}{\partial t}(t, X_t)\, \mathrm{d}t + \sum_{i=1}^d (U_t e_i) f(t, X_t) \circ \mathrm{d}Z_t^i \\ &= \frac{\partial f}{\partial t}(t, X_t)\, \mathrm{d}t + \sum_{i=1}^d (U_t e_i) f(t, X_t)\, \mathrm{d}Z_t^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \mathrm{Hess}^{\nabla(t)} f(U_t e_i, U_t e_j)\, d\langle Z^i, Z^j\rangle_t. \end{aligned}$$

Remark 2.7 In the situation of Proposition 2.2 let $\tilde{Z}_t := \int_0^t \vartheta(\circ\, \mathrm{d}\tilde{U}_s)$ where $\tilde{U}$ is an arbitrary lift of X with the same initial condition. Then,

$$\mathrm{d}Z_t = G_t^{-1} \circ \mathrm{d}\tilde{Z}_t. \tag{2.7}$$

Proof Since $\pi\, U_t = \pi\, \tilde{U}_t$, we have

$$\mathrm{d}Z_t = \vartheta(\circ\ \mathrm{d}U_t) = U_t^{-1} \circ \pi_* \mathrm{d}U_t = G_t^{-1} \tilde{U}_t^{-1} \circ \pi_* \mathrm{d}\tilde{U}_t = G_t^{-1} \vartheta(\circ\ \mathrm{d}\tilde{U}_t) = G_t^{-1} \circ \mathrm{d}\tilde{Z}_t.$$

□

Remark 2.8 In a formal way, the second part of Corollary 2.6 can be written as

$$\mathrm{d}X_t = U_t \circ \mathrm{d}Z_t, \quad \text{or equivalently,} \quad \mathrm{d}Z_t = U_t^{-1} \circ \mathrm{d}X_t. \tag{2.8}$$

In the same manner, the Itô differential $d^{\nabla(t)} X_t$ of X is defined as

$$d^{\nabla(t)} X_t = U_t\, \mathrm{d}Z_t, \quad \text{or equivalently,} \quad \mathrm{d}Z_t = U_t^{-1}\, d^{\nabla(t)} X_t. \tag{2.9}$$

We shall discuss the significance of these differentials in Sect. 5.

3 Alternative Definition of Horizontality in the Riemannian Case

In this section, we assume that for each $t \geq 0$, the connection $\nabla(t)$ is the Levi-Civita connection of a Riemannian metric $g(t)$ depending smoothly on t (we call this the Riemannian case). In this situation, it seems natural to require that each U_t takes values in the $g(t)$-orthonormal frames of M, i.e. $U_t \in \mathcal{O}_{g(t)}(M)$ for all $t \geq 0$. To ensure this, one has to add a correction term to (2.1).

Definition 3.1 An $\mathcal{F}(M)$-valued semimartingale U is said to be $(g(t))_{t\geq 0}$*-Riemann-horizontal* if $U_0 \in \mathcal{O}_{g(0)}(M)$ and

$$\omega_t(\circ\ \mathrm{d}U_t) = -\frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t} \left(t, U_t e_\alpha, U_t e_\beta\right) E_{\alpha\beta}\, \mathrm{d}t. \tag{3.1}$$

In Proposition 3.7 below, we will show that any $(g(t))_{t\geq 0}$-Riemann-horizontal semimartingale U satisfies indeed $U_t \in \mathcal{O}_{g(t)}(M)$ for all $t \geq 0$. Before doing so, we show that the results of the previous section carry over to $(g(t))_{t\geq 0}$-Riemann-horizontal processes with appropriate modifications:

Proposition 3.2 *Let X be an M-valued semimartingale and U_0 an $\mathcal{F}_0$-measurable $\mathcal{O}_{g(0)}(M)$-valued random variable with $\pi\, U_0 = X_0$. Then there exists a unique $(g(t))_{t\geq 0}$-Riemann-horizontal lift U of X starting at U_0, i.e. an $\mathcal{F}(M)$-valued semimartingale satisfying* (3.1) *and $\pi\, U = X$. Moreover, starting with an arbitrary lift $\tilde{U}$ of X satisfying $\tilde{U}_0 = U_0$, the $(g(t))_{t\geq 0}$-Riemann-horizontal lift U can be constructed in the following way: Let*

$$\gamma_t := \int_0^t \omega_s\left(\circ\ \mathrm{d}\tilde{U}_s\right), \tag{3.2}$$

and G the solution to the $\mathrm{GL}_d(\mathbb{R})$*-valued SDE*

$$\mathrm{d}G_t = -\sum_{\alpha,\beta=1}^{d} E_{\alpha\beta} G_t \circ \mathrm{d}\gamma_t^{\alpha\beta} - \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t} \left(t, \tilde{U}_t G_t e_\alpha, \tilde{U}_t G_t e_\beta\right) G_t E_{\alpha\beta}\, \mathrm{d}t, \quad G_0 = I, \tag{3.3}$$

Then

$$U_t = \tilde{U}_t G_t. \tag{3.4}$$

Proof We first show that the process U defined by (3.4) is indeed $(g(t))_{t\geq 0}$-Riemann-horizontal. As in the proof of Proposition 2.2 we obtain

$$\begin{aligned}
\omega_t(\circ \mathrm{d}U_t) &= \left(R^*_{G_t}\omega_t\right)\left(\circ \mathrm{d}\tilde{U}_t\right) + \theta_{G_t}(\circ d G_t) \\
&= \mathrm{Ad}\left(G_t^{-1}\right)\omega_t\left(\circ \mathrm{d}\tilde{U}_t\right) + dL^{-1}_{G_t}(\circ d G_t) \\
&= \mathrm{Ad}\left(G_t^{-1}\right) \circ \mathrm{d}\gamma_t - \sum_{\alpha,\beta=1}^{d} G_t^{-1} E_{\alpha\beta} G_t \circ \mathrm{d}\gamma_t^{\alpha\beta} \\
&\quad - \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}\left(t, \tilde{U}_t G_t e_\alpha, \tilde{U}_t G_t e_\beta\right) E_{\alpha\beta}\, \mathrm{d}t \\
&= -\frac{1}{2}\sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}\left(t, U_t e_\alpha, U_t e_\beta\right) E_{\alpha\beta}\, \mathrm{d}t.
\end{aligned}$$

Uniqueness of U can be proved in the same way as in Proposition 2.2 □.

Definition 3.3 The $(g(t))_{t\geq 0}$*-Riemann-parallel transport* along an M-valued semimartingale X is the family of isomorphisms $//_{s,t} \colon T_{X_s}M \to T_{X_t}M\ (0 \leq s \leq t)$ defined by

$$//_{s,t} := U_t U_s^{-1},$$

where U is an arbitrary $(g(t))_{t\geq 0}$-Riemann-horizontal lift of X.

Definition 3.4 Let X be an M-valued semimartingale, U_0 an $\mathcal{F}_0$-measurable $\mathcal{O}_{g(0)}(M)$-valued random variable with $\pi\, U_0 = X_0$, and U the unique $(g(t))_{t\geq 0}$-Riemann-horizontal lift of X starting at U_0. The $\mathbb{R}^d$-valued process

$$Z_t := \int_0^t \vartheta(\circ\ \mathrm{d}U_s)$$

is called the $(g(t))_{t\geq 0}$-Riemann-antidevelopment of X (or U) with initial frame U_0.

Remark 3.5 A $(g(t))_{t\geq 0}$-Riemann-horizontal process U can be recovered from its $(g(t))_{t\geq 0}$-antidevelopment X and its initial value U_0 as the solution to the SDE

$$\mathrm{d}U_t = \sum_{i=1}^{d} H_i^{\nabla(t)}(U_t) \circ \mathrm{d}Z_t^i - \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t} \left(t, U_t e_\alpha, U_t e_\beta\right) V_{\alpha\beta}(U_t)\, \mathrm{d}t,$$

where $(V_{\alpha,\beta})_{\alpha,\beta=1}^{d}$ are the canonical vertical vector fields defined as

$$V^{\alpha\beta} f(u) = \frac{d}{\mathrm{d}s}\Big|_{s=0} f\left(u(I + sE_{\alpha\beta})\right)$$

(I denoting the identity matrix).

Proof Noting that $\omega_t(V_{\alpha\beta}) = E_{\alpha\beta}$ (by the definition of ω_t), this can be proved in the same way as Remark 2.5. □

Corollary 3.6 *Let U be a $(g(t))_{t\geq 0}$-Riemann-horizontal semimartingale, and $X := \pi U$. Then we have the following Itô formulas:*

1. *For all smooth functions f on $\mathbb{R}_+ \times \mathcal{F}(M)$ we have*

$$\begin{aligned} d\left(f(t, U_t)\right) &= \frac{\partial f}{\partial t}(t, U_t)\, \mathrm{d}t + \sum_{i=1}^{d} H_i^{\nabla(t)} f(t, U_t) \circ \mathrm{d}Z_t^i \\ &\quad - \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}\left(t, U_t e_\alpha, U_t e_\beta\right) V_{\alpha\beta} f(t, U_t)\, \mathrm{d}t \\ &= \frac{\partial f}{\partial t}(t, U_t)\, \mathrm{d}t + \sum_{i=1}^{d} H_i^{\nabla(t)} f(t, U_t)\, \mathrm{d}Z_t^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{d} H_i^{\nabla(t)} H_j^{\nabla(t)} f(t, U_t)\, d\langle Z^i, Z^j\rangle_t \\ &\quad - \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}\left(t, U_t e_\alpha, U_t e_\beta\right) V_{\alpha\beta} f(t, U_t)\, \mathrm{d}t. \end{aligned} \tag{3.5}$$

2. *For all smooth functions f on $\mathbb{R}_+ \times M$ we have*

$$d(f(t, X_t)) = \frac{\partial f}{\partial t}(t, X_t)\, \mathrm{d}t + \sum_{i=1}^{d} (U_t e_i) f(t, X_t) \circ \mathrm{d}Z_t^i$$

$$= \frac{\partial f}{\partial t}(t, X_t)\,\mathrm{d}t + \sum_{i=1}^{d}(U_t e_i) f(t, X_t)\,\mathrm{d}Z_t^i$$
$$+ \frac{1}{2}\sum_{i,j=1}^{d} \mathrm{Hess}^{\nabla(t)} f(U_t e_i, U_t e_j)\, d\langle Z^i, Z^j\rangle_t.$$

Proposition 3.7 *Let U be a $(g(t))_{t\geq 0}$-Riemann-horizontal semimartingale. If $U_0 \in \mathcal{O}_{g(0)}(M)$, then $U_t \in \mathcal{O}_{g(t)}(M)$ for all $t \geq 0$.*

Proof We have to show that $\langle U_t e_i, U_t e_j\rangle_{g(t)}$ is constant for all $i, j \in \{1, \ldots, d\}$. To do so, we fix $i, j \in \{1, \ldots, d\}$ and apply Itô's formula (3.5) to the function $f(t, u) := \langle u e_i, u e_j\rangle_{g(t)}$. Obviously,

$$\frac{\partial f}{\partial t}(t, u) = \frac{\partial g}{\partial t}(u e_i, u e_j).$$

Since f is constant along horizontal curves in $\mathcal{F}(M)$, we have

$$H_i^{\nabla(t)} f = H_i^{\nabla(t)} H_j^{\nabla(t)} f = 0.$$

Finally, for $u \in \mathcal{O}_{g(t)}(M)$,

$$\begin{aligned}
V^{\alpha\beta} f(t, u) &= \frac{d}{\mathrm{d}s}\Big|_{s=0} f\left(t, u(I + sE_{\alpha\beta})\right)\\
&= \frac{d}{\mathrm{d}s}\Big|_{s=0} \langle u\left(I + sE_{\alpha\beta}\right) e_i, u\left(I + sE_{\alpha\beta}\right) e_j\rangle_{g(t)}\\
&= \frac{d}{\mathrm{d}s}\Big|_{s=0} \langle \left(I + sE_{\alpha\beta}\right) e_i, \left(I + sE_{\alpha\beta}\right) e_j\rangle_{\mathbb{R}^d}\\
&= \langle E_{\alpha\beta} e_i, e_j\rangle_{\mathbb{R}^d} + \langle e_i, E_{\alpha\beta} e_j\rangle_{\mathbb{R}^d}\\
&= \begin{cases} 2 & \text{if } \alpha = \beta = i = j,\\ 1 & \text{if } i \neq j \text{ and } (\alpha = i, \beta = j \text{ or } \alpha = j, \beta = i),\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

so that

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(t, u e_\alpha, u e_\beta) V_{\alpha\beta} f(t, u) = \frac{\partial g}{\partial t}(u e_i, u e_j) = -\frac{\partial f}{\partial t}(t, u).$$

□

Remark 3.8 In the situation of Proposition 3.2 let $\tilde{Z}_t := \int_0^t \vartheta(\circ\ \mathrm{d}\tilde{U}_s)$. Then

$$\mathrm{d}Z_t = G_t^{-1} \circ \mathrm{d}\tilde{Z}_t. \tag{3.6}$$

Proof This can be proved in the same way as Remark 2.7. □

Remark 3.9 Let X be an M-valued semimartingale and U_0 an $\mathcal{F}_0$-measurable $\mathcal{O}_{g(0)}(M)$-valued random variable with $\pi\, U_0 = X_0$. Then, X has on the one hand a unique $(\nabla(t))_{t\geq 0}$-horizontal lift U starting at U_0, $(\nabla(t))_{t\geq 0}$-parallel transports $//_{s,t}$ $(0 \leq s \leq t)$ and a $(\nabla(t))_{t\geq 0}$-antidevelopment Z, and on the other hand a unique $(g(t))_{t\geq 0}$-Riemann-horizontal lift U^{Riem} starting at U_0, $(g(t))_{t\geq 0}$-Riemann-parallel transports $//^{\mathrm{Riem}}_{s,t}$ $(0 \leq s \leq t)$ and a $(g(t))_{t\geq 0}$-Riemann-antidevelopment Z^{Riem}. Proposition 3.2 implies that

$$d\left(U_t^{-1} U_t^{\mathrm{Riem}}\right) = -\frac{1}{2} U_t^{-1} \left(\frac{\partial g}{\partial t}\right)^{\#} U_t^{\mathrm{Riem}} \mathrm{d}t$$

and

$$d\left(//_{0,t}^{-1} //_{0,t}^{\mathrm{Riem}}\right) = -\frac{1}{2} //_{0,t}^{-1} \left(\frac{\partial g}{\partial t}\right)^{\#} //_{0,t}^{\mathrm{Riem}} \mathrm{d}t.$$

Moreover, in this case, the process γ defined in (2.2) resp. (3.2) and therefore also the process G defined in (2.3) resp. (3.3) is of finite variation, so that the Stratonovich differential appearing in (2.7) resp. (3.6) may be replaced by an Itô differential.

Remark 3.10 For the Stratonovich differential $\mathrm{d}X_t$, resp. Itô differential $d^{\nabla(t)} X_t$ of X, as introduced in Remark 2.8, one observes that

$$\begin{aligned} \mathrm{d}X_t &= U_t \circ \mathrm{d}Z_t = U_t^{\mathrm{Riem}} \circ \mathrm{d}Z_t^{\mathrm{Riem}}, \quad \text{and} \\ d^{\nabla(t)} X_t &= U_t\, \mathrm{d}Z_t = U_t^{\mathrm{Riem}}\, \mathrm{d}Z_t^{\mathrm{Riem}}. \end{aligned}$$

This follows directly from Remark 3.8 or from a comparison of Corollaries 2.6 and 3.6.

4 Quadratic Variation and Integration of 1-Forms

Proposition 4.1 *Let X be an M-valued semimartingale, U a $(\nabla(t))_{t\geq 0}$-horizontal or $(g(t))_{t\geq 0}$-Riemann-horizontal lift of X, and $Z_t := \int_0^t \vartheta(\circ\, \mathrm{d}U_s)$ the corresponding $(\nabla(t))_{t\geq 0}$-antidevelopment resp. $(g(t))_{t\geq 0}$-Riemann-antidevelopment. Then for every adapted $T^*M \otimes T^*M$-valued process B above X (i.e. $B_t \in T^*_{X_t}M \otimes T^*_{X_t}M$ for all $t \geq 0$) we have*

$$\int_0^t B_s(\mathrm{d}X_s, \mathrm{d}X_s) = \sum_{i,j=1}^{d} \int_0^t B_s\left(U_s e_i, U_s e_j\right) d\langle Z^i, Z^j\rangle_s.$$

Proof By [9, Lemma 7.56 (iv)] there exist $\ell \in \mathbb{N}$, real-valued adapted processes $(B^{\mu\nu})_{\mu,\nu=1}^{\ell}$ and functions $h_1, \ldots, h_\ell \in C^\infty(M)$ such that $B_t = \sum_{\mu,\nu=1}^{\ell} B_t^{\mu\nu} (dh_\mu \otimes dh_\nu)(X_t)$ for all $t \geq 0$. It follows that

$$\int_0^t B_s\,(\mathrm{d}X_s,\mathrm{d}X_s) = \sum_{\mu,\nu=1}^{\ell} \int_0^t \left(B_s^{\mu\nu} dh_\mu \otimes dh_\nu\right)(\mathrm{d}X_s,\mathrm{d}X_s)$$
$$= \sum_{\mu,\nu=1}^{\ell} \int_0^t B_s^{\mu\nu}\, d\langle h_\mu(X), h_\nu(X)\rangle_s.$$

Since by Itô's formula (Corollary 2.6 resp. Corollary 3.6)

$$d\langle h_\mu(X), h_\nu(X)\rangle_s = \sum_{i,j=1}^{d} (U_s e_i) h_\mu(X_s)(U_s e_j) h_\nu(X_s)\, d\langle Z^i, Z^j\rangle_s$$
$$= \sum_{i,j=1}^{d} \left(dh_\mu \otimes dh_\nu\right)\left(U_s e_i, U_s e_j\right)\, d\langle Z^i, Z^j\rangle_s,$$

the claim follows. □

By choosing $B_s = \mathrm{Hess}^{\nabla(s)} f(X_s)$ or (in the Riemannian case) $B_s = g(s, X_s)$, we obtain the following two corollaries:

Corollary 4.2 *For all smooth functions f on $\mathbb{R}_+ \times M$ we have*

$$d\,(f(t,X_t)) = \frac{\partial f}{\partial t}(t,X_t)\,\mathrm{d}t + \sum_{i=1}^{d} (U_t e_i) f(t,X_t)\,\mathrm{d}Z_t^i + \frac{1}{2}\,\mathrm{Hess}^{\nabla(t)} f(\mathrm{d}X_t, \mathrm{d}X_t). \tag{4.1}$$

Corollary 4.3 (Riemannian quadratic variation) *In the Riemannian case,*

$$\int_0^t g(s)(\mathrm{d}X_s, \mathrm{d}X_s) = \sum_{i=1}^{d} \langle Z^i, Z^i\rangle_t.$$

Remark 4.4 For a smooth function f on M (independent of time), using that

$$\sum_{i=1}^{d} (U_t e_i) f(X_t)\,\mathrm{d}Z_t^i = \sum_{i=1}^{d} (df)_{X_t}(U_t e_i)\,\mathrm{d}Z_t^i = (df)_{X_t}(U_t\,\mathrm{d}Z_t)$$
$$= (df)_{X_t}(d^{\nabla(t)} X_t),$$

formula (4.1) reads as

$$d(f(X_t)) = (df)_{X_t}(d^{\nabla(t)} X_t) + \frac{1}{2}\,\mathrm{Hess}^{\nabla(t)} f(\mathrm{d}X_t, \mathrm{d}X_t), \tag{4.2}$$

or more generally, replacing df by a general 1-form $\alpha \in \Gamma(T^*M)$ we obtain

$$\alpha(\circ\ \mathrm{d}X_t) = \alpha(d^{\nabla(t)}X_t) + \frac{1}{2}(\nabla(t)\alpha)(\mathrm{d}X_t, \mathrm{d}X_t). \tag{4.3}$$

Formula (4.3) gives the relation between Itô and Stratonovich differential. In local coordinates, we have formulas analogous to the time-independent case (see for instance [3, p. 423]):

$$\alpha(\circ\ \mathrm{d}X_t) = \sum_i \alpha_i(X_t)\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j}\frac{\partial\alpha_i}{\partial x^j}(X_t)\,d\langle X_t^i, X_t^j\rangle, \tag{4.4}$$

$$\alpha(d^{\nabla(t)}X_t) = \sum_i \alpha_i(X_t)\left(\mathrm{d}X_t^i + \frac{1}{2}\sum_{j,k}\Gamma^i_{jk}(t, X_t)\,d\langle X_t^i, X_t^j\rangle\right) \tag{4.5}$$

where $\Gamma^i_{jk}(t,\ \cdot\ )$ are the Christoffel symbols with respect of $\nabla(t)$.

Proposition 4.5 *Let X be an M-valued semimartingale, U a $(\nabla(t))_{t\geq 0}$-horizontal or $(g(t))_{t\geq 0}$-Riemann-horizontal lift of X, and $Z_t := \int_0^t \vartheta(\circ\ \mathrm{d}U_s)$ the corresponding $(\nabla(t))_{t\geq 0}$-antidevelopment resp. $(g(t))_{t\geq 0}$-Riemann-antidevelopment. Then for every adapted T^*M-valued process Ψ above X (i.e. $\Psi_t \in T^*_{X_t}M$ for all $t \geq 0$) we have*

$$\int_0^t \Psi_s(\circ\ \mathrm{d}X_s) = \sum_{i=1}^d \int_0^t \Psi_s(U_s e_i)\circ \mathrm{d}Z_s^i.$$

Proof By [9, Lemma 7.56 (v)], there exist $\ell \in \mathbb{N}$, real-valued adapted processes $\Psi^1, \ldots, \Psi^\ell$ and functions $h_1, \ldots, h_\ell \in C^\infty(M)$ such that $\Psi_t = \sum_{\nu=1}^\ell \Psi_t^\nu dh_\nu(X_t)$ for all $t \geq 0$. It follows that

$$\begin{aligned}\int_0^t \Psi_s(\circ\ \mathrm{d}X_s) &= \sum_{\nu=1}^\ell \int_0^t \left(\Psi_s^\nu dh_\nu\right)(\circ\ \mathrm{d}X_s)\\ &= \sum_{\nu=1}^\ell \int_0^t \Psi_s^\nu \circ dh_\nu(X_s).\end{aligned}$$

Since by Itô's formula (Corollary 2.6 resp. Corollary 3.6)

$$dh_\nu(X_s) = \sum_{i=1}^d dh_\nu(U_s e_i)\circ \mathrm{d}Z_s^i,$$

the claim follows. □

Remark 4.6 In the situation of Proposition 4.5, repeating the calculation with Itô differentials and taking into account (4.2), resp. (4.2), we obtain the analogous formula for the Itô integral:

$$\int_0^t \Psi_s(d^{\nabla(s)} X_s) = \sum_{i=1}^d \int_0^t \Psi_s(U_s e_i)\, \mathrm{d}Z_s^i.$$

See Émery [6, Chapter VII] for the general framework.

By choosing $\Psi_s = \alpha(X_s)$ for $\alpha \in \Gamma(T^*M)$, we have the following corollary:

Corollary 4.7 (Itô and Stratonovich integration of 1-forms along a semimartingale) *Let X be an M-valued semimartingale, U a $\nabla(t)$-horizontal lift and Z the $\nabla(t)$-antidevelopment of X, resp. U^{Riem} a $g(t)$-Riemann-horizontal lift and Z^{Riem} the corresponding $g(t)$-antidevelopment of X. Then for each $\alpha \in \Gamma(T^*M)$ the following formulas hold:*

$$\begin{aligned}
\int_0^t \alpha(\circ\ \mathrm{d}X_s) &= \sum_{i=1}^d \int_0^t \alpha(X_s)(U_s e_i) \circ \mathrm{d}Z_s^i \\
&= \sum_{i=1}^d \int_0^t \alpha\left(X_s\right)\left(U_s^{\text{Riem}} e_i\right) \circ d\left(Z^{\text{Riem}}\right)_s^i \\
\int_0^t \alpha(d^{\nabla(s)} X_s) &= \sum_{i=1}^d \int_0^t \alpha(X_s)(U_s e_i)\, \mathrm{d}Z_s^i \\
&= \sum_{i=1}^d \int_0^t \alpha\left(X_s\right)\left(U_s^{\text{Riem}} e_i\right) d\left(Z^{\text{Riem}}\right)_s^i.
\end{aligned}$$

5 Martingales on Manifolds with Time-Dependent Connection

Proposition 5.1 (cf. [9, Satz 7.147 (i)] for the case of a fixed connection) *Let X be an M-valued semimartingale. Then the following conditions are equivalent:*

1. *The $(\nabla(t))_{t\geq 0}$-antidevelopment of X is an $\mathbb{R}^d$-valued local martingale.*
2. *For any smooth $f : M \to \mathbb{R}$ the process*

$$f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \mathrm{Hess}^{\nabla(s)} f(\mathrm{d}X_s, \mathrm{d}X_s), \quad t \geq 0,$$

is a real-valued local martingale.

3. *For any $\alpha \in \Gamma(T^*M)$ the process*

$$\int_0^t \alpha(d^{\nabla(s)} X_s), \quad t \geq 0,$$

is a real-valued local martingale.

Moreover, in the Riemannian case, these conditions are equivalent to the condition that the $(g(t))_{t\geq 0}$-Riemann-antidevelopment of X is an $\mathbb{R}^d$-valued local martingale.

Definition 5.2 X is called a $(\nabla(t))_{t\geq 0}$*-martingale* if the equivalent conditions of Proposition 5.1 are satisfied.

Proof of Proposition 5.1 Let Z be the $(\nabla(t))_{t\geq 0}$-antidevelopment or $(g(t))_{t\geq 0}$-Riemann-antidevelopment of X, and $f \in C^\infty(M)$. Then by Corollary 4.2

$$f(X_t) - f(X_0) - \frac{1}{2}\int_0^t \mathrm{Hess}^{\nabla(s)} f(\mathrm{d}X_s, \mathrm{d}X_s) = \sum_{i=1}^d \int_0^t (U_s e_i) f(X_s) \mathrm{d}Z_s^i.$$

This is a local martingale for all $f \in C^\infty(M)$ if and only if Z is an $\mathbb{R}^d$-valued local martingale. The equivalence with the third item is clear from Corollary 4.7. □

Proposition 5.3 (Local expression) *A semimartingale X is a $(\nabla(t))_{t\geq 0}$-martingale if and only if in local coordinates*

$$\mathrm{d}X_t^i = -\frac{1}{2}\sum_{jk} \Gamma^i_{jk}(t, X_t)\, d\langle X^j, X^k\rangle_t$$

up to the differential of a local martingale.

Proof This can be proved in the same way as in the case of a fixed connection (see e.g. [7, Proposition 3.7]), or derived directly from the representation (4.5) in local coordinates. □

Example 5.4 Let $M = \mathbb{R}$ equipped with the standard metric g_0, and let u be a strictly positive smooth function on $\mathbb{R}_+ \times \mathbb{R}$. Define the metric $g(t, \cdot)$ by $g(t, x) = u(t, x) g_0(x)$, and let $\nabla(t)$ be its Levi-Civita connection. Let b and σ be smooth functions on $\mathbb{R}_+ \times \mathbb{R}$, and X the solution to the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t) dW_t,$$

where W is a standard one-dimensional Brownian motion. Then, X is a $(\nabla(t))_{t\geq 0}$-martingale if and only if

$$b = -\frac{u'\sigma^2}{4u}$$

on $\{(t, X_t) \mid t \geq 0\}$ (the prime denotes differentiation with respect to x).

Proof Taking into account that the unique Christoffel symbol of $\nabla(t)$ equals $u'/(2u)$, the claim follows immediately from Proposition 5.3. □

6 Convergence of Martingales

6.1 Local Convergence

Proposition 6.1 (cf. [9, Lemma 7.187] or [10, Theorem 2.5.6] for the case of a fixed connection) *Let $U \subseteq M$ be an open subset with the following property: There exists a smooth function $\varphi = (\varphi^1, \dots, \varphi^d)\colon M \to \mathbb{R}^d$ such that*

- *$\varphi|_U$ is bounded,*
- *$\varphi|_U$ is a diffeomorphism onto its image, and*
- *$\operatorname{Hess}^{\nabla(t)} \varphi^i(x) \geq 0$ for all $i \in \{1, \dots, d\}$, all $x \in U$ and all $t \geq 0$.*

Then each $(\nabla(t))_{t\geq 0}$-martingale X converges almost surely on the set

$$\Omega_0 := \{X \textit{ lies eventually in } U\}.$$

Remark 6.2 In the case of a fixed connection, each point $x \in M$ has a neighbourhood U with that property (see e.g. [9, Lemma 7.187] or [10, Theorem 2.5.6]).

Proof of Proposition 6.1 By Definition 5.2 for each $i \in \{1, \dots, d\}$, there exists a real-valued local martingale M^i such that

$$\varphi^i(X_t) = \varphi^i(X_0) + M^i_t + A^i_t,$$

where $A^i_t := \frac{1}{2}\int_0^t \operatorname{Hess}^{\nabla(s)}\varphi^i(\mathrm{d}X_s, \mathrm{d}X_s)$.

Since $\operatorname{Hess}^{\nabla(s)} \varphi^i \geq 0$ on U, the process A is eventually non-decreasing and in particular bounded from below on Ω_0. Since $\varphi^i|_U$ is bounded, it follows that the local martingale M^i is bounded from above and hence convergent on Ω_0 (because it is a time-changed Brownian motion). This implies that the process A^i is bounded and hence convergent on Ω_0 (since it is eventually non-decreasing). Consequently, the process $\varphi^i(X)$ converges on Ω_0, and, since $\varphi|_U$ is a diffeomorphism onto its image, so does the process X.

6.2 Darling–Zheng

An important result of martingale theory in the case of a fixed connection is the convergence theorem of Darling and Zheng (see e.g. [9, Satz 7.190]): let X be an M-valued martingale with respect to a fixed connection ∇, and g_0 an arbitrary Riemannian metric on M. Then

$$\left\{X_\infty \text{ exists in } M\right\} \subset \left\{\int_0^\infty g_0(\mathrm{d}X_s, \mathrm{d}X_s) < \infty\right\} \subset \left\{X_\infty \text{ exists in } \hat{M}\right\}, \quad (6.1)$$

where $\hat{M}$ is the Alexandrov compactification of M. In the case of a time-dependent connection, at least the second inclusion does not hold. To see this, consider the following example:

Example 6.3 In the situation of Example 5.4 take $u(t,x) = \exp(a(t)x)$, $\sigma(t,x) = \sigma(t)$ and $b(t,x) = -\frac{1}{4}a(t)\sigma^2(t)$ with smooth functions $a, \sigma : \mathbb{R}_+ \to \mathbb{R}$. Then, X is a $(\nabla(t))_{t\geq 0}$-martingale, and

$$X_t = X_0 - \frac{1}{4}\int_0^t a(s)\sigma(s)^2 \mathrm{d}s + \int_0^t \sigma(s) dW_s,$$

so that $\int_0^t g_0(\mathrm{d}X_s, \mathrm{d}X_s) = \int_0^t \sigma(s)^2 \mathrm{d}s$. If σ is chosen in such a way that $\int_0^\infty \sigma(s)^2 \mathrm{d}s < \infty$, then $\int_0^\infty g_0(\mathrm{d}X_s, \mathrm{d}X_s) < \infty$, but the function a (being arbitrary) can be chosen in such a way that X does not converge in $\hat{\mathbb{R}}$.

In the Riemannian case, one might hope that the second inclusion of (6.1) holds if we replace the arbitrary fixed metric g_0 with the given metrics $(g(s))_{s\geq 0}$, i.e. that

$$\left\{ \int_0^\infty g(s)(\mathrm{d}X_s, \mathrm{d}X_s) < \infty \right\} \subset \left\{ X_\infty \text{ exists in } \hat{M} \right\}.$$

This, however, turns out to be wrong as well:

Example 6.4 In the situation of Example 5.4 take $u(t,x) = u(t)$, $\sigma(t,x) \equiv 1$ and $b(t,x) \equiv 0$. Then X is a $(\nabla(t))_{t\geq 0}$-martingale, and

$$X_t = X_0 + W_t,$$

so that $\int_0^t g(s)(\mathrm{d}X_s, \mathrm{d}X_s) = \int_0^t u(s)\mathrm{d}s$. If u is chosen in such a way that $\int_0^\infty u(s)\mathrm{d}s < \infty$, then $\int_0^\infty g(s)(\mathrm{d}X_s, \mathrm{d}X_s) < \infty$, but obviously X does not converge in $\hat{\mathbb{R}}$.

7 Uniqueness of Martingales with Given Terminal Value

Proposition 7.1 (cf. [9, Lemma 7.204] for the case of a fixed connection) *Let M_0 be a submanifold of M which is totally geodesic in M with respect to $\nabla(t)$ for all t. Then for each $x_0 \in M_0$ and each $T \geq 0$ there exist an open neighbourhood V of x_0 in M and a non-negative function $f \in C^\infty(V)$ satisfying*

$$f(x) = 0 \Longleftrightarrow x \in M_0$$

and

$$\mathrm{Hess}^{\nabla(s)} f(x) \geq 0 \tag{7.1}$$

for all $s \in [0, T]$ and all $x \in V$.

Proof Let $d_0 := \dim M_0$. Choose coordinates $x_1, \dots, x_d$ for M on a neighbourhood O of x_0 in such a way that

$$O \cap M_0 = O \cap \left\{x_{d_0+1} = \dots = x_d = 0\right\}. \tag{7.2}$$

We will show that for sufficiently small $c > 0$ the function

$$f(x) := \frac{1}{2}\left(c^2 + |\tilde{x}|^2\right)|\hat{x}|^2,$$

where $\tilde{x} := (x_1, \dots, x_{d_0})$ and $\hat{x} := (x_{d_0+1}, \dots, x_d)$ does the job on a possibly smaller neighbourhood V of x_0. All we have to show is that (7.1) holds provided one chooses c and V small enough.

Let $\Gamma_{ij}^k(s, x)$ be the Christoffel symbols with respect to $\nabla(s)$. Since M_0 is totally geodesic and because of (7.2) one has

$$\Gamma_{ij}^k(s, x) = 0, \quad i, j \le d_0, \quad k \ge d_0 + 1$$

for all $s \ge 0$ and all $x \in O \cap M_0$. By the compactness of $[0, T]$, this implies the existence of a constant $C < \infty$ such that

$$|\Gamma_{ij}^k(s, x)| \le C|\hat{x}|, \quad i, j \le d_0, \quad k \ge d_0 + 1. \tag{7.3}$$

Since

$$\mathrm{Hess}^{\nabla(s)} f(x) = \sum_{i,j=1}^{d} H_{ij}(s, x)\,\mathrm{d}x_i \otimes \mathrm{d}x_j,$$

where

$$H_{ij}(s, x) := \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{k=1}^{d} \Gamma_{ij}^k(s, x)\frac{\partial f}{\partial x_k}(x),$$

it suffices to show that the matrix $H(s, x)$ is positive definite for all $s \in [0, T]$ and all $x \in V \setminus M_0$, provided that c and V are chosen small enough. Using the decomposition of $\{1, \dots, d\}$ into $I = \{1, \dots, d_0\}$ and $J = \{d_0 + 1, \dots, d\}$, this is true if and only if the same statement holds for the block matrix $H^*(s, x)$ defined by

$$H^*(s, x) := \begin{pmatrix} \frac{1}{|\hat{x}|^2}(H_{ij}(s, x))_{(i,j)\in I\times I} & \frac{1}{c\,|\hat{x}|}(H_{ij}(s, x))_{(i,j)\in I\times J} \\ \frac{1}{c\,|\hat{x}|}(H_{ij}(s, x))_{(i,j)\in J\times I} & \frac{1}{c^2}(H_{ij}(s, x))_{(i,j)\in J\times J} \end{pmatrix}$$

(use the general fact that for any $c > 0$, a symmetric block matrix $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ is positive definite if and only if the matrix $\begin{pmatrix} A & cB \\ cB & c^2C \end{pmatrix}$ has this property).

Since

$$\frac{\partial f}{\partial x_k}(x) = \begin{cases} x_k|\hat{x}|^2 & 1 \le k \le d_0, \\ x_k(c^2 + |\tilde{x}|^2) & d_0 + 1 \le k \le d, \end{cases}$$

and

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \begin{cases} \delta_{ij}|\hat{x}|^2 & 1 \le i, j \le d_0, \\ 2x_i x_j & 1 \le i \le d_0 \text{ and } d_0 + 1 \le j \le d, \\ \delta_{ij}(c^2 + |\tilde{x}|^2) & d_0 + 1 \le i, j \le d, \end{cases}$$

we obtain on $[0, T] \times (O \setminus M_0)$, using (7.3),

$$\begin{aligned} H^*(s, x)_{ij} &= \delta_{ij} - \sum_{k=1}^{d_0} \Gamma_{ij}^k(s, x)x_k - \frac{(c^2 + |\tilde{x}|^2)}{|\hat{x}|^2} \sum_{k=d_0+1}^{d} \Gamma_{ij}^k(s, x)x_k \\ &= \delta_{ij} + \mathrm{O}\left(|\tilde{x}| + c^2\right) \qquad \text{if } i, j \le d_0, \\ H^*(s, x)_{ij} &= \frac{2x_i x_j}{c|\hat{x}|} - \frac{|\hat{x}|}{c} \sum_{k=1}^{d_0} \Gamma_{ij}^k(s, x)x_k - \left(\frac{c}{|\hat{x}|} + \frac{|\tilde{x}|^2}{c|\hat{x}|}\right) \sum_{k=d_0+1}^{d} \Gamma_{ij}^k(s, x)x_k \\ &= \mathrm{O}\left(|\tilde{x}| + \frac{|\tilde{x}||\hat{x}|}{c} + c|\hat{x}|\right) \qquad \text{if } i \le d_0,\ j \ge d_0 + 1, \\ H^*(s, x)_{ij} &= \delta_{ij}\left(1 + \frac{|\tilde{x}|^2}{c^2}\right) - \frac{|\hat{x}|^2}{c^2} \sum_{k=1}^{d_0} \Gamma_{ij}^k(s, x)x_k - \left(1 + \frac{|\tilde{x}|^2}{|\hat{x}|^2}\right) \sum_{k=d_0+1}^{d} \Gamma_{ij}^k(s, x)x_k \\ &= \delta_{ij} + \mathrm{O}\left(\frac{|\tilde{x}|^2}{c^2} + \frac{|\tilde{x}||\hat{x}|^2}{c^2} + |\hat{x}|^2\right) \qquad \text{if } i, j \ge d_0 + 1. \end{aligned}$$

This implies that

$$\lim_{c \to 0} \lim_{x \to x_0} H^*(s, x) = I$$

uniformly in $s \in [0, T]$, so that on $[0, T] \times (V \setminus M_0)$ the matrix $H^*(s, x)$ is positive definite provided that c and V are small enough (the choice of V has to depend on the choice of c). □

Corollary 7.2 *Let M_0 be submanifold of M which is totally geodesic in M with respect to $\nabla(t)$ for all t. Then given $T > 0$ each point $x_0 \in M_0$ has an open neighbourhood V in M with the following property: If X is a V-valued $(\nabla(t))_{t \ge 0}$-martingale such that a.s. $X_T \in M_0$, then a.s. $X_t \in M_0$ for all $t \in [0, T]$.*

Proof Choose V and f as in Proposition 7.1. Then $f(X)$ is a non-negative submartingale with $f(X_T) = 0$ a.s., hence $f(X) \equiv 0$ a.s. on $[0, T]$. □

Corollary 7.3 (Uniqueness of $(\nabla(t))_{t\geq 0}$-martingales with given terminal value) *Given $T > 0$ each point $x \in M$ has an open neighbourhood V with the following property: If X and Y are two V-valued $(\nabla(t))_{t\geq 0}$-martingales such that a.s. $X_T = Y_T$, then a.s. $X_t = Y_t$ for all $t \in [0, T]$.*

Proof Apply Corollary 7.2 to the diagonal embedding of M into $M \times M$ equipped with the product connections $\nabla(t) \otimes \nabla(t)$. □

8 Behaviour of Semimartingales Under Maps

Proposition 8.1 (cf. [9, Satz 7.156] for the case of fixed connections) *Let N be another manifold, also equipped with a smooth family of connections $(\tilde{\nabla}(t))_{t\geq 0}$, and let $f \in C^\infty(\mathbb{R}_+ \times M, N)$. Let X be a semimartingale on M, U a $(\nabla(t))_{t\geq 0}$-horizontal or $(g(t))_{t\geq 0}$-Riemann-horizontal lift of X, and Z the corresponding $(\nabla(t))_{t\geq 0}$-antidevelopment or $(g(t))_{t\geq 0}$-Riemann-antidevelopment. Moreover, let $\tilde{U}$ be a $(\tilde{\nabla}(t))_{t\geq 0}$-horizontal or $(\tilde{g}(t))_{t\geq 0}$-Riemann-horizontal lift of the image process $\tilde{X}_t := f(t, X_t)$, and $\tilde{Z}$ the corresponding $(\tilde{\nabla}(t))_{t\geq 0}$-antidevelopment or $(\tilde{g}(t))_{t\geq 0}$-Riemann-antidevelopment. Then the following formula holds:*

$$\begin{aligned} \mathrm{d}\tilde{Z}_t &= \tilde{U}_t^{-1} \frac{\partial f}{\partial t}(t, X_t)\,\mathrm{d}t + \tilde{U}_t^{-1} df\, U_t\, \mathrm{d}Z_t \\ &\quad + \frac{1}{2}\tilde{U}_t^{-1}\,\mathrm{Hess}^{\nabla(t),\tilde{\nabla}(t)} f(t, X_t)(\mathrm{d}X_t, \mathrm{d}X_t). \end{aligned} \tag{8.1}$$

Proof Let $n := \dim N$ and $\varphi \in C^\infty(N)$. Then by Corollary 4.2 and the pullback formula for the quadratic variation (see e.g. [9, Satz 7.61]),

$$\begin{aligned} \mathrm{d}\varphi(\tilde{X}_t) &= \sum_{k=1}^{n} \left(\tilde{U}_t e_k\right) \varphi(\tilde{X}_t)\,\mathrm{d}\tilde{Z}_t^k + \frac{1}{2}\,\mathrm{Hess}^{\tilde{\nabla}(t)}\varphi\left(\mathrm{d}\tilde{X}_t, \mathrm{d}\tilde{X}_t\right) \\ &= \sum_{k=1}^{n} \left(\tilde{U}_t e_k\right) \varphi(\tilde{X}_t)\,\mathrm{d}\tilde{Z}_t^k + \frac{1}{2}\left(f^*\,\mathrm{Hess}^{\tilde{\nabla}(t)}\varphi\right)(dX_t, \mathrm{d}X_t)\,. \end{aligned} \tag{8.2}$$

On the other hand, using the Hessian composition formula

$$\mathrm{Hess}^{\nabla(t)}(\varphi \circ f) = \mathrm{d}\varphi \circ \mathrm{Hess}^{\nabla(t),\tilde{\nabla}(t)} f + f^*\,\mathrm{Hess}^{\tilde{\nabla}(t)}\,\varphi$$

(see e.g. [9, Satz 7.155]), we obtain

$$\begin{aligned} \mathrm{d}\varphi(\tilde{X}_t) &= d(\varphi \circ f)(t, X_t) \\ &= \frac{\partial(\varphi \circ f)}{\partial t}(t, X_t)\,\mathrm{d}t + \sum_{i=1}^{d}(U_t e_i)(\varphi \circ f)(t, X_t)\,\mathrm{d}Z_t^i \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i,j=1}^{d}\mathrm{Hess}^{\nabla(t)}(\varphi\circ f)(\mathrm{d}X_t,\mathrm{d}X_t)\\
&=\frac{\partial(\varphi\circ f)}{\partial t}(t,X_t)\,\mathrm{d}t+\sum_{i=1}^{d}(U_te_i)(\varphi\circ f)(t,X_t)\,\mathrm{d}Z_t^i\\
&+\frac{1}{2}\left(\mathrm{d}\varphi\circ\mathrm{Hess}^{\nabla(t),\tilde\nabla(t)}f\right)(\mathrm{d}X_t,\mathrm{d}X_t)\\
&+\frac{1}{2}\left(f^*\,\mathrm{Hess}^{\tilde\nabla(t)}\varphi\right)(\mathrm{d}X_t,\mathrm{d}X_t). \qquad (8.3)
\end{aligned}
$$

Combining (8.2) and (8.3) we obtain

$$
\begin{aligned}
\sum_{k=1}^{n}\left(\tilde U_te_k\right)\varphi(\tilde X_t)\,\mathrm{d}\tilde Z_t^k&=\frac{\partial(\varphi\circ f)}{\partial t}(t,X_t)\,\mathrm{d}t+\sum_{i=1}^{d}(U_te_i)(\varphi\circ f)(t,X_t)\,\mathrm{d}Z_t^i\\
&+\frac{1}{2}\left(d\varphi\circ\mathrm{Hess}^{\nabla(t),\tilde\nabla(t)}f\right)(\mathrm{d}X_t,\mathrm{d}X_t).
\end{aligned}
$$

Since this holds for all $\varphi\in C^\infty(N)$, it follows that

$$
\begin{aligned}
\sum_{k=1}^{n}\left(\tilde U_te_k\right)\,\mathrm{d}\tilde Z_t^k&=\frac{\partial f}{\partial t}(t,X_t)\,\mathrm{d}t+\sum_{i=1}^{d}df(t,U_te_i)\,\mathrm{d}Z_t^i\\
&+\frac{1}{2}\,\mathrm{Hess}^{\nabla(t),\tilde\nabla(t)}f(\mathrm{d}X_t,\mathrm{d}X_t)
\end{aligned}
$$

and hence

$$
\mathrm{d}\tilde Z_t=\tilde U_t^{-1}\frac{\partial f}{\partial t}(t,X_t)\,\mathrm{d}t+\tilde U_t^{-1}df\,U_t\,\mathrm{d}Z_t+\frac{1}{2}\tilde U_t^{-1}\,\mathrm{Hess}^{\nabla(t),\tilde\nabla(t)}f(t,X_t)(\mathrm{d}X_t,\mathrm{d}X_t).
$$

□

Corollary 8.2 *If the connections $\nabla(t)$ are the Levi-Civita connections of Riemannian metrics $g(t)$ and if X is a $(g(t))_{t\geq0}$-Brownian motion (whose $(g(t))_{t\geq0}$-Riemann-antidevelopment W is a Euclidean Brownian motion), then*

$$
\mathrm{d}\tilde Z_t=\tilde U_t^{-1}\left(\frac{\partial f}{\partial t}+\frac{1}{2}\Delta^{g(t),\tilde\nabla(t)}f\right)(t,X_t)\mathrm{d}t+\tilde U_t^{-1}df\,U_t\,dW_t, \qquad (8.4)
$$

where $\Delta^{g(t),\tilde\nabla(t)}u$ is the tension field of u with respect to $g(t)$ and $\tilde\nabla(t)$.

Corollary 8.3 *The function f maps $(g(t))_{t\geq0}$-Brownian motions to $(\nabla(t))_{t\geq0}$-martingales if and only if*

$$
\frac{\partial f}{\partial t}+\frac{1}{2}\Delta^{g(t),\tilde\nabla(t)}f=0
$$

for all $t \geq 0$.

Remark 8.4 In the situation of Proposition 8.1, one may consider the "intrinsic" anti-developments of X, respectively $\tilde{X}$, defined by

$$\mathcal{A}_t := U_0 Z_t, \quad \text{respectively} \quad \tilde{\mathcal{A}}_t := \tilde{U}_0 \tilde{Z}_t,$$

which take values in $T^M_{X_0}$, respectively $T_{\tilde{X}_0}N$. Note that,

$$\mathrm{d}\mathcal{A}_t = //_{0,t}^{-1} \circ \mathrm{d}X_t, \quad \text{respectively} \quad \mathrm{d}\tilde{\mathcal{A}}_t = \tilde{//}_{0,t}^{-1} \circ \mathrm{d}\tilde{X}_t,$$

where $//_{0,t} \equiv U_t U_0^{-1}$ and $\tilde{//}_{0,t} \equiv \tilde{U}_t \tilde{U}_0^{-1}$ denote the parallel transports along X, respectively $\tilde{X}$. Then, formula (8.1) reads more intrinsically as

$$\begin{aligned} d\tilde{\mathcal{A}}_t &= \tilde{//}_{0,t}^{-1} \frac{\partial f}{\partial t}(t, X_t)\,\mathrm{d}t + \tilde{//}_{0,t}^{-1}\, df\, //_{0,t} \mathrm{d}\mathcal{A}_t \\ &\quad + \frac{1}{2} \tilde{//}_{0,t}^{-1}\, \mathrm{Hess}^{\nabla(t),\tilde{\nabla}(t)} f(t, X_t)(\mathrm{d}X_t, \mathrm{d}X_t). \end{aligned} \tag{8.5}$$

The same remark applies to formula (8.4) which then reads as

$$\mathrm{d}\tilde{\mathcal{A}}_t = \tilde{//}_{0,t}^{-1} \left(\frac{\partial f}{\partial t} + \frac{1}{2} \Delta^{g(t),\tilde{\nabla}(t)} f \right)(t, X_t)\mathrm{d}t + \tilde{//}_{0,t}^{-1}\, df\, //_{0,t} \mathrm{d}\mathcal{A}_t\,.$$

Recall that in this formula, $\mathcal{A}_t = U_0 W_t$ is a Euclidean Brownian motion in $T_{X_0}M$.

Remark 8.5 In terms of Itô differentials (see Remark 3.10 above)

$$d^{\nabla(t)} X_t = //_{0,t}\, \mathrm{d}\mathcal{A}_t, \quad \text{respectively} \quad d^{\tilde{\nabla}(t)} \tilde{X}_t = \tilde{//}_{0,t}^{-1}\, \mathrm{d}\mathcal{A}_t,$$

formula (8.5) simplifies to

$$\begin{aligned} d^{\tilde{\nabla}(t)} \tilde{X}_t &= \frac{\partial f}{\partial t}(t, X_t)\,\mathrm{d}t + df \left(d^{\nabla(t)} X_t \right) \\ &\quad + \frac{1}{2} \mathrm{Hess}^{\nabla(t),\tilde{\nabla}(t)} f(t, X_t)(\mathrm{d}X_t, \mathrm{d}X_t). \end{aligned} \tag{8.6}$$

9 Derivative Processes, Martingales on the Tangent Bundle and Applications to the Non-linear Heat Equation

In this section, we assume for simplicity that the connections $\nabla(t)$ are torsion-free. Let $\nabla'(t)$ the complete and $\nabla^h(t)$ the horizontal lift of $\nabla(t)$ to the tangent bundle TM. In the same way as in [4], one can obtain the following results.

Theorem 9.1 (cf. [4, Theorem 3.1] for the case of a fixed connection) *Let I be an open interval containing 0 and $(X_t(s))_{t\geq 0, s\in I}$ a C^1-family of continuous M-valued $(\nabla(t))_{t\geq 0}$-martingales. Then the TM-valued derivative process $(X'_t)_{t\geq 0}$ defined by*

$$X'_t := \frac{\partial}{\partial s}\Big|_{s=0} X_t(s)$$

is a $(\nabla'(t))_{t\geq 0}$*-martingale.*

Theorem 9.2 (cf. [4, Corollary 4.4] for the case of a fixed connection) *A* TM*-valued semimartingale* J *is a* $(\nabla^h(t))_{t\geq 0}$*-martingale if and only if*

1. *its projection* X *to* M *is a* $(\nabla(t))_{t\geq 0}$*-martingale, and*
2. $d(//_{0,t}^{-1} J_t) \overset{\mathrm{m}}{=} 0$.

Theorem 9.3 (cf. [4, Theorem 4.12] for the case of a fixed connection) *A* TM*-valued semimartingale* J *is a* $(\nabla'(t))_{t\geq 0}$*-martingale if and only if*

1. *its projection* X *to* M *is a* $(\nabla(t))_{t\geq 0}$*-martingale, and*
2. $d(\Theta_{0,t}^{-1} J_t) \overset{\mathrm{m}}{=} 0$*, where* $\Theta_{0,t}\colon T_{X_0}M \to T_{X_t}M$ *denotes the damped parallel transport along* X*, defined by the covariant equation*

$$d\left(//_{0,t}^{-1}\Theta_{0,t}\right) = -\frac{1}{2}//_{0,t}^{-1} R^{\nabla(t)}\left(\Theta_{0,t}, \mathrm{d}X_t\right)\mathrm{d}X_t, \qquad \Theta_{0,0} = \mathrm{Id}_{T_{X_0}M}\,.$$

Remark 9.4 In the Riemannian case, the condition $d(//_{0,t}^{-1} J_t) \overset{\mathrm{m}}{=} 0$ in Theorem 9.2 can also be expressed using the Riemann-parallel transport $//_{0,t}^{\mathrm{Riem}}$; using Remark 3.9 one obtains that it is equivalent to

$$d\left(\left(//_{0,t}^{\mathrm{Riem}}\right)^{-1} J_t\right) \overset{\mathrm{m}}{=} \frac{1}{2}\left(//_{0,t}^{\mathrm{Riem}}\right)^{-1}\left(\frac{\partial g}{\partial t}\right)^{\#} J_t\,\mathrm{d}t.$$

Similarly, the equation defining the damped parallel transport is equivalent to

$$\begin{aligned} d\left(\left(//_{0,t}^{\mathrm{Riem}}\right)^{-1}\Theta_{0,t}\right) = \frac{1}{2}\left(//_{0,t}^{\mathrm{Riem}}\right)^{-1}\left(\frac{\partial g}{\partial t}\right)^{\#}\Theta_{0,t}\,\mathrm{d}t \\ -\frac{1}{2}\left(//_{0,t}^{\mathrm{Riem}}\right)^{-1} R^{g(t)}(\Theta_{0,t}, \mathrm{d}X_t)\mathrm{d}X_t. \end{aligned} \tag{9.1}$$

If X is a $(g(t))_{t\geq 0}$-Brownian motion, (9.1) simplifies to

$$d\left(\left(//_{0,t}^{\mathrm{Riem}}\right)^{-1}\Theta_{0,t}\right) = \frac{1}{2}\left(//_{0,t}^{\mathrm{Riem}}\right)^{-1}\left(\left(\frac{\partial g}{\partial t}\right)^{\#} - (\mathrm{Ric}^{g(t)})^{\#}\right)\Theta_{0,t}\mathrm{d}t, \tag{9.2}$$

which coincides with the expression given in [1, Definition 2.1] and [5, Definition 3.1].

Combining Theorems 9.1 and 9.3, one obtains

Corollary 9.5 *Let* I *be an open interval containing* 0*,* $(X_t(s))_{s\in I}$ *a* C^1*-family of continuous* M*-valued martingales,* $X_t := X_t(0)$*, and* $(X'_t)_{t\geq 0}$ *the* TM*-valued derivative process defined by* $X'_t := \frac{\partial}{\partial s}|_{s=0} X_t(s)$*. Then the process* $(\Theta_{0,t}^{-1} X'_t)_{t\geq 0}$ *is a* $T_{X_0}M$*-valued local martingale.*

Let now N be another differentiable manifold and $T_1 < T_2$. Let $(g(t))_{T_1 \le t \le T_2}$ be a smooth family of Riemannian metrics on M, $\nabla(t)$ the Levi-Civita connection of $g(t)$ and $(\tilde{\nabla}(t))_{T_1 \le t \le T_2}$ a smooth family of connections on N. Let $u\colon [T_1, T_2] \times M \to N$ be a solution of the non-linear heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta^{g(t), \tilde{\nabla}(t)} u. \tag{9.3}$$

We fix $x \in M$, let $(X_t)_{0 \le t \le T_2 - T_1}$ be an M-valued $(g(T_2 - t))_{0 \le t \le T_2 - T_1}$-Brownian motion starting at x, and define

$$\tilde{X}_t := u(T_2 - t, X_t), \quad 0 \le t \le T_2 - T_1.$$

Proposition 9.6 (cf. [4, (5.22)] for the case of fixed Riemannian metrics) *Let*

$$u\colon [T_1, T_2] \times M \to N$$

be a solution of Eq. (9.3). Let $\Theta_{0,t}\colon T_x M \to T_{X_t} M$ be the damped parallel transport along X, and $\tilde{\Theta}_{0,t}\colon T_{\tilde{X}_0} N \to T_{\tilde{X}_t} N$ the damped parallel transport along $\tilde{X}$, where X and $\tilde{X}$ are defined as above. Then for each $v \in T_x M$ the $T_{u(T,x)} N$-valued process

$$\tilde{\Theta}_{0,t}^{-1}\, \mathrm{d}u(T_2 - t, X_t)\, \Theta_{0,t}\, v, \quad 0 \le t \le T_2 - T_1,$$

is a local martingale.

Proof Let $\gamma\colon \mathbb{R} \to M$ be a smooth curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. By [2, Theorem 3.1] there exists a smooth family $(X_t(s))_{0 \le t \le T_2 - T_1,\, s \in \mathbb{R}}$ of M-valued $(g(T_2 - t))_{0 \le t \le T_2 - T_1}$-Brownian motions satisfying $X_0(s) = \gamma(s)$ for all $s \in \mathbb{R}$, $X_t(0) = X_t$ for all $t \in [0, T_2 - T_1]$ and

$$\frac{\partial}{\partial s}\Big|_{s=0} X_t(s) = \Theta_{0,t} v.$$

Let

$$\tilde{X}_t(s) := u(T_2 - t, X_t(s)).$$

By Corollary 8.3, the process $(\tilde{X}_t(s))_{0 \le t \le T_2 - T_1}$ is an N-valued $(\tilde{\nabla}(T_2 - t))_{0 \le t \le T_2 - T_1}$-martingale for each $s \in \mathbb{R}$. Moreover,

$$\frac{\partial}{\partial s}\Big|_{s=0} \tilde{X}_t(s) = \mathrm{d}u(T_2 - t, X_t) \frac{\partial}{\partial s}\Big|_{s=0} X_t(s) = \mathrm{d}u(T_2 - t, X_t) \Theta_{0,t} v.$$

Therefore, the result follows immediately from Corollary 9.5. ◻

Remark 9.7 If the local martingale in Proposition 9.6 is a true martingale, we obtain the stochastic representation formula

$$\mathrm{d}u(T_2, x) = E\left[\tilde{\Theta}^{-1}_{0,T_2-T_1}\mathrm{d}u(T_1, X_{T_2-T_1})\Theta_{0,T_2-T_1}\right]. \tag{9.4}$$

Theorem 9.8 *Let M be a connected differentiable manifold equipped with a smooth family $(g(t))_{-\infty<t\leq T}$ of Riemannian metrics satisfying*

$$\frac{\partial g}{\partial t} + \mathrm{Ric}_{g(t)} \geq K > 0 \tag{9.5}$$

(uniformly strict super Ricci flow), and let $(N, \tilde{g})$ be a Riemannian manifold of non-positive sectional curvature. Then every ancient solution $u\colon (-\infty, T] \times M \to N$ of the non-linear heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta^{g(t),\tilde{g}}u$$

whose differential is bounded is constant.

Proof of Theorem 9.8. The curvature conditions imply that $\|\Theta_{0,s}\| \leq e^{-K_1 s/2}$ and $\|\tilde{\Theta}^{-1}_{0,s}\| \leq 1$, so that the local martingale in Proposition 9.6 is bounded and hence a true martingale. The representation formula (9.4) then implies that

$$\|\mathrm{d}u(t, x)\|_{\tilde{g}} \leq e^{-Ks/2} \sup_{y\in M} \|\mathrm{d}u(t-s, y)\|.$$

The claim now follows from letting $s \to \infty$. □

Remark 9.9 More refined representation formulas and Liouville theorems for the non-linear heat equation in the spirit of [17] can be found in our recent preprint [8].

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