Prob. Theory and Math. Stat., pp. 23–32
B. Grigelionis *et al.* (Eds)

BISMUT TYPE DIFFERENTIATION OF SEMIGROUPS

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ABSTRACT

We present a unified approach to Bismut type differentiation formulas for heat semigroups on functions and forms. Both elliptic and hypoelliptic situations are considered. Nonlinear extensions apply to the case of harmonic maps between Riemannian manifolds and solutions to the nonlinear heat equation.

1. INTRODUCTION

Let M be a smooth n-dimensional manifold. On M consider a Stratonovich SDE with smooth coefficients of the type

$$\delta X = A(X)\,\delta Z + A_0(X)\,dt \tag{1.1}$$

where $A_0 \in \Gamma(TM)$ is a vector field and $A\colon M \times \mathbb{R}^r \to TM$, $(x,z) \mapsto A(x)z$ a bundle map over M for some r. The driving process Z is assumed to be an $\mathbb{R}^r$-valued Brownian motion on some filtered probability space satisfying the usual completeness conditions. We write $X_\bullet(x)$ for the solution to (1.1) starting from the point $x \in M$, and denote its maximal lifetime by $\zeta(x)$.

Solutions to (1.1) are diffusions with generator given in Hörmander form as

$$L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2 \tag{1.2}$$

with $A_i = A(\,\cdot\,)e_i \in \Gamma(TM)$ for $i = 1, \ldots, r$. We consider the minimal semigroup to (1.1)

$$(P_t f)(x) = \mathbb{E}\big[\big(f \circ X_t(x)\big)\, 1_{\{t<\zeta(x)\}}\big], \tag{1.3}$$

likewise let

$$u(x) = \mathbb{E}\big[\big(f \circ X_{\tau(x)}(x)\big] \tag{1.4}$$

be the representation of an L-harmonic function u on some domain $D \subset M$, where $\tau(x)$ is the first exit time of X from D when starting at x.

Our aim, if possible, is to find formulas for the differential $d(P_t f)$, respectively du, which do not involve derivatives of f. Such differentiation formulas are sometimes called "Bismut type" formulas. For a survey on applications of differentiation formulas we refer to (Elworthy and Li, 1994).

2. AN INTEGRATION BY PARTS ARGUMENT

In this section we explain a general integration by parts argument at the level of local martingales which will serve as a key ingredient in our approach to differentiation formulas. Suppose that we are given a time-space function F transforming our diffusion into a continuous local martingale $F(\,\cdot\,, X_\cdot(x))$. All processes considered in this paper are supposed to have continuous paths. The function F is assumed to be C^1 in the space variable with a derivative continuous in both variables. We have mainly two cases in mind, namely

(1) $F(\,\cdot\,, X_\cdot(x)) = u(X_\cdot(x))$ where u is some L-harmonic function, and

(2) $F(\,\cdot\,, X_\cdot(x)) = (P_{t-\cdot} f)\big(X_\cdot(x)\big)$ for some $f \in B(M)$, i.e. f is a bounded measurable function on M.

Suppose further that $W_{0,s}\colon T_xM \to T_{X_s(x)}M$ are linear transports on M along $X(x)$ such that, for each $v \in T_xM$,

$$dF(s,\,\cdot\,)_{X_s(x)}\, W_{0,s} v\,, \quad s \geq 0, \tag{2.1}$$

is a local martingale. We assume that $W_{s,0} = W_{0,s}^{-1}$ and $W_{0,0} = \mathrm{id}_{T_xM}$. Note that a possible choice for $W_{0,s}$ is for instance $W_{0,s} = T_xX_s$. This follows from the fact that the derivative of a C^1-family (w.r.t. compact convergence in probability) of local martingales is again a local martingale (Arnaudon and Thalmaier, 1998a).

LEMMA 1. *For $F(\,\cdot\,, X_\cdot(x))$ as above, let $W_{0,\cdot}$ be a linear transport along $X(x)$ such that* (2.1) *are local martingales. Then, for any $\mathbb{R}^r$-valued process k in $L^2_{loc}(Z)$ and any $v \in T_xM$,*

$$\begin{aligned} N_s := dF(s,\,\cdot\,)_{X_s(x)}\, W_{0,s} \left(v - \int_0^s W_{0,r}^{-1}\, A\big(X_r(x)\big) k_r\, dr \right) \\ + F(s, X_s(x)) \int_0^s \langle k, dZ\rangle \end{aligned} \tag{2.2}$$

is a local martingale for $0 \leq s < \sigma$ where $[0,\sigma[$ denotes the stochastic interval on which $F(\,\cdot\,, X_\cdot(x))$ is defined, e.g. $\sigma = t \wedge \zeta(x)$ or $\tau(x)$, etc.

Proof. We have to show that

$$dF(s,\,\boldsymbol{\cdot}\,)_{X_s(x)}\, W_{0,s} \int_0^s W_{0,r}^{-1}\, A\big(X_r(x)\big)\, k_r\, dr - F(s, X_s(x)) \int_0^s \langle k, dZ\rangle, \quad s \geq 0,$$

is a local martingale. Let $h_s = \int_0^s W_{0,r}^{-1}\, A\big(X_r(x)\big)\, k_r\, dr$. Then

$$dF(s,\,\boldsymbol{\cdot}\,)_{X_s(x)}\, W_{0,s} h_s - \int_0^s dF(r,\,\boldsymbol{\cdot}\,)_{X_r(x)}\, W_{0,r}\, dh_r, \quad s \geq 0,$$

is a local martingale, and the claim follows upon noting that $F(s, X_s(x)) = F(0,x) + \int_0^s dF(r,\,\boldsymbol{\cdot}\,)_{X_r(x)}\, A(X_r(x))\, dZ_r$ which implies that

$$\int_0^s dF(r,\,\boldsymbol{\cdot}\,)_{X_r(x)}\, W_{0,r}\, dh_r - F(s, X_s(x)) \int_0^s \langle k, dZ\rangle, \quad s \geq 0,$$

is a local martingale.

The procedure is now straightforward. Suppose that we are able to choose the process k in (2.2) in such a way that the following two conditions hold:

(a) $\int_0^\sigma W_{0,r}^{-1}\, A\big(X_r(x)\big)\, k_r\, dr = v$ a.s., and

(b) $N^\sigma \equiv (N_{t\wedge\sigma})_{t\geq 0}$ is a uniformly integrable martingale.

Then, evaluating $\mathbb{E}[N_0] = \mathbb{E}[N_\sigma]$ gives a formula for $dF(0,\,\boldsymbol{\cdot}\,)_x v$ in terms of the process $F(\,\boldsymbol{\cdot}\,, X_{\boldsymbol{\cdot}}(x))$. In the elliptic case, both (a) and (b) can easily be achieved (Thalmaier, 1997), by exploiting the fact that there is a right-inverse to the process $A\big(X_{\boldsymbol{\cdot}}(x)\big)$.

3. THE ELLIPTIC CASE

Suppose that our system (1.1) is elliptic, i.e. $A(x)\colon \mathbb{R}^r \to T_x M$ is onto for each $x \in M$. Then there is a Riemannian metric g on M which makes

$$A(x)^*\colon T_x M \to \mathbb{R}^r$$

to an isometric inclusion for each $x \in M$. In particular, the generator (1.2) then writes as

$$L = \tfrac{1}{2}\Delta_M + V \tag{3.1}$$

where Δ_M is the Laplace-Beltrami operator on M with respect to the metric g and V is a first order term, i.e. $V \in \Gamma(TM)$.

In this case, there is an intrinsic choice for the linear transports $W_{0,\bullet}$ in (2.1). To this end, let $W_{0,s}\colon T_xM \to T_{X_s(x)}M$ be defined by the following covariant equation along $X(x)$:

$$\frac{D}{ds}W_{0,s} = -\frac{1}{2}\,\mathrm{Ric}^M(W_{0,s},\,\cdot\,)^\# + \nabla V(W_{0,s}), \quad W_{0,0} = \mathrm{id}_{T_xM}\,. \tag{3.2}$$

Then the local martingale property of (2.1) is elementary to check: indeed, this can be done either directly by applying Weitzenböck's formula to the Laplacian on 1-forms (see, for instance, (Thalmaier, 1998)), or by using the method of filtering out redundant noise from the derivative process T_xX, cf. (Elworthy and Yor, 1993). Let

$$B_r = \int_0^r //_{0,s}^{-1} A(X_s(x))\,dZ_s \tag{3.3}$$

be the martingale part of the anti-development $\mathcal{A}(X) := \int_0^\bullet //_{0,s}^{-1}\,\delta X_s$ of $X = X(x)$. Obviously, B is a Brownian motion on T_xM and, by definition, $A(X_s(x))\,dZ_s = //_{0,s}\,dB_s$. In this situation, Lemma 1 is easily adapted to give the local martingale property of

$$N_s = dF(s,\,\cdot\,)_{X_s(x)}\,W_{0,s}\left(v - \int_0^s W_{0,r}^{-1}\,//_{0,r}k_r\,dr\right) + F(s, X_s(x))\int_0^s \langle k, dB\rangle$$

Taking $h_s = v - \int_0^s W_{0,r}^{-1}\,//_{0,r}k_r\,dr$, we get the following typical applications in the elliptic case, see (Thalmaier, 1997) for details. In the sequel, let $\mathbb{H}(\mathbb{R}_+, T_xM)$ denote the Cameron-Martin space of paths $\gamma\colon \mathbb{R}_+ \to T_xM$ of finite energy.

THEOREM 2 (Differentiation formula for heat semigroups). *Let $f\colon M \to \mathbb{R}$ be bounded measurable, $x \in M$ and $v \in T_xM$. Then, for any bounded adapted process h with paths in $\mathbb{H}(\mathbb{R}_+, T_xM)$ such that $\big(\int_0^{\tau(x)\wedge t}|\dot h_s|^2\,ds\big)^{1/2} \in L^1$, and the property that $h_0 = v$, $h_s = 0$ for all $s \geq \tau(x)\wedge t$, the following formula holds:*

$$d(P_tf)_xv = -\mathbb{E}\left[f\big(X_t(x)\big)\,1_{\{t<\zeta(x)\}}\int_0^{\tau(x)\wedge t}\big\langle W_{0,s}(\dot h_s), //_{0,s}\,dB_s\big\rangle\right] \tag{3.4}$$

where $\tau(x)$ is the first exit time of $X(x)$ from some relatively compact open neighbourhood D of x.

THEOREM 3 (Differentiation formula for harmonic functions). *Let $D \subset M$ be a nonvoid relatively compact open subset with smooth boundary $\partial D \neq \emptyset$, and*

$\tau(x) = \inf\{t \geq 0 : X_t(x) \notin D\}$ the first exit time of X from D when started at $x \in D$. Let $u \in C(\bar{D})$ be L-harmonic on D. Then

$$(du)_x v = -\mathbb{E}\left[u\big(X_{\tau(x)}(x)\big) \int_0^{\tau(x)} \big\langle W_s(\dot{h}_s), //_{0,s}\, dB_s \big\rangle \right] \tag{3.5}$$

for any bounded adapted process h with paths in $\mathbb{H}(\mathbb{R}_+, T_x M)$ such that $h_0 = v$, $h_s \equiv 0$ for $s \geq \tau(x)$, and the property that $\big(\int_0^{\tau(x)} |\dot{h}_s|^2\, ds\big)^{1/2} \in L^{1+\varepsilon}$ for some $\varepsilon > 0$.

For possible choices of the process h see (Thalmaier and Wang, 1998) where formula (3.5) is used to prove local gradient estimates of harmonic functions.

4. NONLINEAR GENERALIZATIONS

The arguments of the previous section are easily extended to nonlinear situations, e.g. to harmonic maps $u\colon M \to N$ between Riemannian manifolds, and more generally, to solutions of the nonlinear heat equation

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u \tag{4.1}$$

for maps between M and N where $\Delta u = \operatorname{trace} \nabla du$ is the tension of u. In these cases

(1) $F_s \equiv F(s, X_s(x)) = u(X_s(x))$, respectively

(2) $F_s \equiv F(s, X_s(x)) = u_{t-s}(X_s(x))$,

define ∇-martingales on N for Brownian motions X on M.

DEFINITION 4 (Arnaudon and Thalmaier, 1998b). *For a continuous semimartingale Y taking values in a manifold N, endowed with a torsionfree connection ∇, the geodesic transport (also called deformed or damped transport; Dohrn-Guerra transport) $\Theta_{0,t}\colon T_{Y_0}N \to T_{Y_t}N$ on N along Y is defined by the following covariant equation along Y:*

$$d\,(//_{0,\cdot}^{-1}\Theta_{0,\cdot}) = -\tfrac{1}{2}\, //_{0,\cdot}^{-1} R(\Theta_{0,\cdot}, dY)dY, \quad \Theta_{0,0} = \mathrm{id}, \tag{4.2}$$

where $//_{0,t}\colon T_{Y_0}N \to T_{Y_t}N$ is parallel transport on N along Y and R denotes the curvature tensor to ∇. The $T_{Y_0}N$-valued process

$$\mathcal{A}_{\mathrm{def}}(Y) = \int_0^{\cdot} \Theta_{0,s}^{-1}\, \delta Y_s \tag{4.3}$$

is called deformed anti-development of Y.

Now let $\Theta_{0,\bullet}$ be the geodesic transport on N along the martingale $F(\,\cdot\,,X_\bullet(x))$. Exploiting the fact that the Stratonovich integral in (4.3) can be replaced by an Itô integral, we get

$$\mathcal{A}_{\mathrm{def}}\big(F(\,\cdot\,,X_\bullet(x))\big) = \int_0^\bullet \Theta_{0,s}^{-1}\, dF(s,\,\cdot\,)_{X_s(x)}\, A\big(X_s(x)\big)\, dZ_s.$$

Further, by taking into account that (Arnaudon and Thalmaier, 1998b)

$$m_s := \Theta_{0,s}^{-1}\, dF(s,\,\cdot\,)_{X_s(x)}\, W_{0,s}$$

is local martingale in $T_x^*M \otimes T_{F(0,x)}N$, we obtain the following generalization of Lemma 1 to the nonlinear case.

LEMMA 5. *Let $F(\,\cdot\,,X_\bullet(x))$, $x \in M$, be a family of ∇-martingales on N, as above. Then, for any predictable $\mathbb{R}^r$-valued process k in $L^2_{\mathrm{loc}}(Z)$,*

$$\Theta_{0,\bullet}^{-1}\, dF(\,\cdot\,,X_\bullet(x))\, W_{0,\bullet} \int_0^\bullet W_{0,s}^{-1}\, /\!/_{0,s} k_s\, ds - \mathcal{A}_{\mathrm{def}}\big(F(\,\cdot\,,X_\bullet(x))\big) \int_0^\bullet \langle k, dZ\rangle$$

is a local martingale in $T_{F(0,x)}N$.

Following the lines of Section 3, Lemma 5 leads to explicit differentiation formulas in the nonlinear case, cf. (Arnaudon and Thalmaier, 1998b) for technical details.

THEOREM 6 (A nonlinear differentiation formula). *Let $u\colon [0,t] \times M \to N$ be a smooth solution of Eq.* (4.1) *and $v \in T_xM$. Then*

$$(du_t)_x v = -\mathbb{E}\Big[\mathcal{A}_{\mathrm{def}}\big(u(t-\,\cdot\,,X_\bullet(x))\big)_{\tau(x)\wedge t} \int_0^{\tau(x)\wedge t} \big\langle W_{0,s}\,\dot h_s\,,/\!/_{0,s}\, dB_s\big\rangle\Big]$$

for any bounded adapted process h with sample paths in the Cameron-Martin space $\mathbb{H}([0,t],T_xM)$ such that $\big(\int_0^{\tau(x)\wedge t} |\dot h_s|^2\, ds\big)^{1/2} \in L^{1+\varepsilon}$ for some $\varepsilon > 0$, and the property that $h_0 = v$, $h_s = 0$ for all $s \geq \tau(x) \wedge t$; here $\tau(x)$ is again the first exit time of $X(x)$ from some relatively compact neighbourhood D of x.

In the same way, formulas for the differential of harmonic maps $u\colon M \to N$ between Riemannian manifolds can be derived:

$$(du)_x v = -\mathbb{E}\Big[\mathcal{A}_{\mathrm{def}}\big(u(X_\bullet(x))\big)_{\tau(x)} \int_0^{\tau(x)} \big\langle W_{0,s}\,\dot h_s\,,/\!/_{0,s}\, dB_s\big\rangle\Big], \tag{4.4}$$

with assumptions analogous to Theorem 6. Note that the stochastic integral in (4.4) depends only on the local geometry of M about the point x.

Formula (4.4) has been used in (Arnaudon and Thalmaier, 1998b) to prove local gradient estimates for harmonic maps of bounded dilatation, recovering theorems of Goldberg-Ishihara-Petridis and Shen as special cases.

5. EXTENSIONS TO DIFFERENTIAL FORMS

In this section we sketch extensions from functions to differential forms. For more general results in this direction, e.g. for sections in vector bundles, see (Driver and Thalmaier, 1999).

On a Riemannian manifold M, endowed with the Levi-Civita connection ∇, consider the exterior algebra bundle

$$E = \Lambda T^*M \equiv \bigoplus_{p\geq 0} \Lambda^p T^*M.$$

Let Δ denote the de Rham-Hodge Laplacian on $\Gamma(E)$ with sign convention

$$\Delta = -(dd^* + d^*d) \tag{5.1}$$

where d and d^* are the exterior differential, respectively codifferential. Note that for $a \in \Gamma(E)$,

$$da = c^+\nabla a, \quad d^*a = -c^-\nabla a \tag{5.2}$$

where $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ is the induced covariant derivative on E and

$$\begin{aligned} c^+ &= \wedge : T^*M \otimes \Lambda T^*M \to \Lambda T^*M, \quad s\otimes v \mapsto s\wedge v \\ c^- &= \llcorner : T^*M \otimes \Lambda T^*M \to \Lambda T^*M, \quad s\otimes v \mapsto s \llcorner v \end{aligned}$$

denote wedge product and contraction in $E = \Lambda T^*M$, respectively. On ΛTM the operations $c^\pm$ are defined analogously. Recall that, by definition,

$$s \llcorner v = \sum_{k=1}^{p} (-1)^{k+1} \langle s, v_k\rangle\, v_1 \wedge \ldots \widehat{v}_k \wedge \ldots v_p \quad \text{for } v = v_1 \wedge \ldots \wedge v_p.$$

Further, by Weitzenböck's formula,

$$\Delta a = \Box a - \mathcal{R}a \tag{5.3}$$

where $\Box a = \operatorname{trace} \nabla^2 a$ is the "rough Laplacian" on $\Gamma(E)$ and $\mathcal{R} \in \Gamma(\operatorname{Hom} E)$ the Weitzenböck curvature term.

For $x \in M$ fixed, the Weitzenböck term $\mathcal{R}$ can be used to define a process Q with values in $\operatorname{Aut}(E_x)$ via the pathwise equation

$$\frac{d}{dt}Q_t = -\frac{1}{2} Q_t\, \mathcal{R}_{//_{0,t}}, \quad Q_0 = \mathrm{id}_{E_x}, \tag{5.4}$$

where $\mathcal{R}_{//_{0,t}} = //_{0,t}^{-1} \circ \mathcal{R}_{X_t(x)} \circ //_{0,t}$.

For the remainder of this section, let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the natural pairing between ΛT_x^*M and ΛT_xM, and let $Q_t^* \in \mathrm{Aut}(\Lambda T_xM)$ be determined by $\langle Q_t v, w\rangle = \langle v, Q_t^* w\rangle$ for $v \in \Lambda T_x^*M$, $w \in \Lambda T_xM$.

The following lemma is crucial for derivative formulas in our situation, see (Driver and Thalmaier, 1999) for generalizations in various directions.

LEMMA 7. *Let a be a solution to the heat equation on differential forms, i.e. on $\Gamma(E)$,*

$$\frac{d}{dt}a = \frac{1}{2}\Delta a$$

where Δ is the de Rham-Hodge Laplacian given by (5.1)*. Then*

$$\begin{aligned} N_s = &\big\langle Q_s\, /\!/_{0,s}^{-1} da_{t-s}(X_s(x)), \ell(s)\big\rangle \\ &-\Big\langle Q_s\, /\!/_{0,s}^{-1} a_{t-s}(X_s(x)), \int_0^s (Q_r^*)^{-1}\big(dB_r \llcorner Q_r^*\, \dot\ell(r)\big)\Big\rangle \end{aligned}$$

and

$$\begin{aligned} N_s^* = &\big\langle Q_s\, /\!/_{0,s}^{-1} d^* a_{t-s}(X_s(x)), \ell(s)\big\rangle \\ &+\Big\langle Q_s\, /\!/_{0,s}^{-1} a_{t-s}(X_s(x)), \int_0^s (Q_r^*)^{-1}\big(dB_r \wedge Q_r^*\, \dot\ell(r)\big)\Big\rangle \end{aligned}$$

are local martingales for any adapted process ℓ with paths in the Cameron-Martin space $\mathbb{H}([0,t], \Lambda T_xM)$. Here $X(x)$ is a Brownian motion on M starting from x and B its anti-development taking values in T_xM.

The following theorem is a typical application of Lemma 7, see also (Elworthy and Li, 1998). For simplicity, we assume M to be compact. For L^2-sections a of E, let the semigroup $P_t a = e^{t/2\bar\Delta}a$ be defined by means of the spectral theorem, where $\bar\Delta$ denotes the closure of Δ. Then

$$(P_t a)_x = \mathbb{E}\left[Q_t\, /\!/_{0,t}^{-1} a(X_t(x))\right].$$

THEOREM 8. *Let M be a compact manifold, $a \in L^2$-$\Gamma(\Lambda T^*M)$ be a bounded L^2-section and $v \in \Lambda T_xM$ for some $x \in M$. Then*

$$\begin{aligned} (dP_t a)_x v &= -\mathbb{E}\left[\Big\langle Q_t\, /\!/_{0,t}^{-1} a(X_t(x)), \int_0^t (Q_s^*)^{-1}\big(dB_s \llcorner Q_s^*\, \dot\ell(s)\big)\Big\rangle\right] \\ (d^* P_t a)_x v &= \ \mathbb{E}\left[\Big\langle Q_t\, /\!/_{t,a}^{-1}(X_t(x)), \int_0^t (Q_s^*)^{-1}\big(dB_s \wedge Q_s^*\, \dot\ell(s)\big)\Big\rangle\right] \end{aligned}$$

for any adapted process ℓ with sample paths in $\mathbb{H}([0,t], \Lambda T_xM)$ such that $\ell(0) = v$, $\ell(t) = 0$, and the property that $\big(\int_0^t |\dot\ell(s)|^2 ds\big)^{1/2} \in L^1$.

6. FORMULAS IN THE HYPOELLIPTIC CASE

In the rest of this survey, we give a generalization of the results in Section 3. We consider again our system (1.1) but now with the assumption that the generator $L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2$ is only hypoelliptic, i.e.

$$\operatorname{Lie}\bigl(A_1, \ldots, A_r, [A_0, A] : A \in \mathcal{L}\bigr)(x) = T_xM, \quad \text{for all } x \in M, \tag{H1}$$

where $\mathcal{L} := \operatorname{Lie}\bigl(A_0, A_1, \ldots, A_r\bigr)$. In other words: the ideal in $\mathcal{L}$ generated by $(A_1, \ldots, A_r)$ is assumed to be full at each $x \in M$. We want to extend Theorem 1 and Theorem 2 to the hypoelliptic situation.

Under hypothesis (H1), the "Malliavin Covariance"

$$C_t(x) = \sum_{i=1}^{r} \int_0^t (X_{s*}^{-1} A_i)_x \otimes (X_{s*}^{-1} A_i)_x \, ds \tag{6.1}$$

defines a positive symmetric bilinear form on T_x^*M for $x \in M$ such that $t < \zeta(x)$. We read $C_t(x)$: $T_x^*M \to T_xM$, $C_t(x)^{-1}$: $T_xM \to T_x^*M$ and write

$$C_t(x) = \int_0^t (X_{s*}^{-1} A)_x \, (X_{s*}^{-1} A)_x^* \, ds \tag{6.2}$$

where $(X_{s*}^{-1}A)_x$: $\mathbb{R}^r \to T_xM$ is defined by $z \mapsto \sum_{i=1}^{r} (X_{s*}^{-1}A_i)_x \, z^i$. Recall that by Lemma 1, for any $\mathbb{R}^r$-valued k process in $L^2_{\text{loc}}(Z)$,

$$N_s := dF(s, \,\cdot\,) \, X_{s*} \left[v - \int_0^s X_{r*}^{-1} \, A\bigl(X_r(x)\bigr) k_r \, dr \right] + F(s, X_s(x)) \int_0^s \langle k, dZ \rangle$$

is a local martingale on $[0, t \wedge \zeta(x)[$.

Remark 9. For $x \in M$ and $v \in T_xM$ one may consider the system

$$\dot{h}_s = -(X_{s*}^{-1}A)_x \, k_s, \quad h_0 = v.$$

Note that if we are able to find a predictable k with values in $\mathbb{R}^r$ such that

(1) $h_\sigma = 0$ where $\sigma = \tau(x)$ or $\sigma = \tau(x) \wedge t$, and in addition,

(2) $\bigl(\int_0^\sigma |k_s|^2 ds\bigr)^{1/2} \in L^{1+\varepsilon}$ for some $\varepsilon > 0$,

then the results of Section 3 in the elliptic case carry over almost verbatim. However, it can be shown (Arnaudon and Thalmaier, 1998c) that, in general, this is not possible in the hypoelliptic situation. A counterexample in three dimensions (a two-dimensional Brownian motion and its Lévy area as a third coordinate) has been communicated to us by Jean Picard.

The following result in the hypoelliptic case is taken from (Arnaudon and Thalmaier, 1998c). A similar result can be given for L-harmonic functions.

THEOREM 10. *Let M be a smooth manifold and $f\colon M \to \mathbb{R}$ be bounded measurable. Assume that* (H1) *holds. Let $v \in T_xM$ and $t > 0$. Then, for P_tf defined by* (1.3)*, there is a formula of the type:*

$$d(P_tf)_xv = \mathbb{E}\Big[f\big(X_t(x)\big)\,1_{\{t<\zeta(x)\}}\Phi_tv\Big] \tag{6.3}$$

where Φ_t is a T_x^*M-valued random variable, L^p-integrable for $1 \le p < \infty$ and local in the following sense: For any relatively compact neighbourhood D of x in M there is a choice for Φ_t which is $\mathcal{F}_\sigma$-measurable where $\sigma = t \wedge \tau(x)$ and $\tau(x)$ is the first exit time of X from D when starting at x.

We briefly sketch some arguments underlying Theorem 10. For simplicity, we assume that M is compact and f is C^1. Our sketch of proof does not show that Φ_t is indeed local: to see this, the given arguments have to be reformulated in terms of local martingales, cf. (Arnaudon and Thalmaier, 1998c).

Let a be a predictable process with values in $T_x^*M \otimes \mathbb{R}^r$ and $\lambda \in T_xM \equiv \mathbb{R}^n$ (locally about 0) such that for $t > 0$,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t |a_s\lambda|^2\,ds\right)\right] < \infty.$$

Let $dZ^\lambda = dZ + a\lambda\,dt$ and consider the Girsanov exponential $G_{\textstyle\cdot}^\lambda$ defined by

$$G_t^\lambda = \exp\left(-\int_0^t \langle a\lambda, dZ\rangle - \frac{1}{2}\int_0^t |a\lambda|^2 ds\right).$$

We write X^λ for the flow to our SDE driven by the perturbed Brownian motion Z^λ, analogously $C_{\textstyle\cdot}^\lambda(x)$ etc. By Girsanov's theorem, we find that

$$H(\lambda) = \sum_\ell \mathbb{E}\left[f\big(X_t^\lambda(x)\big)\cdot G_t^\lambda \cdot \big(C_t^\lambda(x)^{-1}\big)_{k\ell}\,v^\ell\right]$$

is independent of λ for each k. Thus $\frac{\partial}{\partial\lambda_k}\big|_{\lambda=0}H(\lambda) = 0$, from where we conclude that

$$\sum_{i,k,\ell}\mathbb{E}\Big[\big(D_if\big)\big(X_t(x)\big)\Big(X_{t*}\int_0^t (X_{s*}^{-1}A)_x\,a_s\,ds\Big)_{ik}\big(C_t(x)^{-1}\big)_{k\ell}\,v^\ell\Big]$$
$$= -\sum_{k,\ell}\mathbb{E}\left[f\big(X_t(x)\big)\,\frac{\partial}{\partial\lambda_k}\bigg|_{\lambda=0}\big(G_t^\lambda\,\big(C_t^\lambda(x)^{-1}\big)_{k\ell}\,v^\ell\big)\right].$$

Recall that $(X_{s*}^{-1}A)_x \in \mathbb{R}^r \otimes T_xM$. Now the idea is to set

$$a_s = a_s^n = (X_{s*}^{-1}A)_x^*\, 1_{\{s\le\tau_n\}} \in T_x^*M \otimes \mathbb{R}^r$$

where $\tau_n \nearrow t$ is a sequence of stopping times such that each $a_{\bullet}^n$ satisfies the integrability condition. This gives

$$\mathbb{E}\big[(df)_{X_t(x)}\, X_{t*}\, C_{\tau_n}(x)\, C_t(x)^{-1} v\big] = \mathbb{E}\left[\big(f\circ X_t(x)\big)\cdot \Phi_t^n\, v\right].$$

Finally, taking the limit as $n\to\infty$, we get

$$d(P_tf)_x v = \mathbb{E}\big[(df)_{X_t(x)}\, X_{t*} v\big] = \mathbb{E}\left[\big(f\circ X_t(x)\big)\cdot \Phi_t\, v\right]$$

where

$$\begin{aligned}\Phi_t\, v = &\left(\int_0^t (X_{s*}^{-1}A)_x\, dZ_s\right) C_t^{-1}(x)\, v\\ &+\sum_{k,\ell}\left(C_t(x)^{-1}\left(\frac{\partial}{\partial\lambda_k}\bigg|_{\lambda=0} C_t^{\lambda}(x)\right) C_t(x)^{-1}\right)_{k\ell} v^{\ell}.\end{aligned}$$

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