Master in Mathematics

Probabilistic Models in Finance

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Preface

These are very preliminary notes to the Course *Probabilistic Models in Finance* taught at the University of Luxembourg in the Master program in Mathematics during the academic year 2020/2021.

CHAPTER 1

Introduction

1.1. Some financial vocabulary

DEFINITION 1.1 (Option). An *option* is a contract which gives the holder the right (but not the obligation) to purchase (or to sell) a certain quantity of an asset (e.g. shares or stocks in a company, commodities such as gold, oil or electricity, currencies etc) for a prescribed price (called *exercise price*) at a prescribed time in the future (or up to a prescribed time in the future). This prescribed time is called *maturity date*, *date of expiry* or *expiration time*.

Here, an *asset* refers to any financial object whose value is known at present but is liable to change in the future.

DEFINITION 1.2 (Call/Put options). Options to buy are called *call options* or *calls*, while options to sell are called *put options* or *puts*.

According to the *date of expiry*, two kinds of options can be distinguished:

- *American options* which can be exercised by its holder at any time before expiry, versus
- *European options* which can only be exercised by its holder at the expiration time.

DEFINITION 1.3 (Exercise price or strike price). The *exercise price* (or *strike price*) is the prescribed price (fixed in advance) at which the transaction is done in case the option is exercised.

The exercise price has to be distinguished from the price of the option.

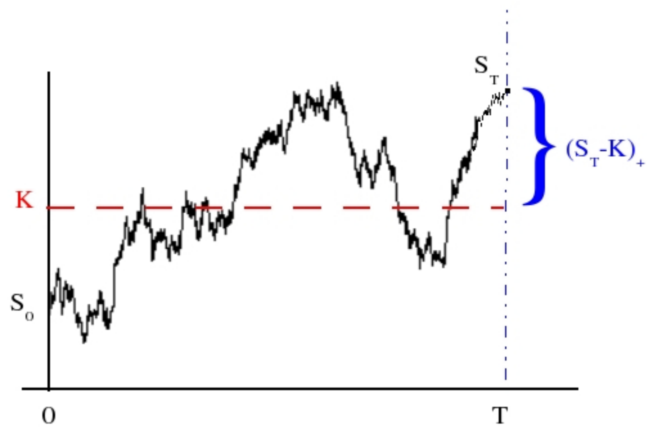
DEFINITION 1.4 (Price or value of an option). The *price or value of an option* is the price which has to be paid (at time 0) to acquire the option.

REMARK 1.5. Options are examples of so-called *financial derivatives*; their value is *derived* from an underlying asset such as stocks, bonds, currencies, or even indexes. In this sense, a *derivative* or *contingent claim* is a security whose value depends on the value of some underlying asset. Financial derivatives include forwards, futures, options, swaps, etc.

EXAMPLE 1.6. Consider a *European call* with date of expiration T on a stock or share of price S_t at time t where $0 \le t \le T$. Here S_t is the spot price (i.e. price of the stock at time t). Let K be the exercise price (strike price) of the call.

- At expiration time T: The value of the call is
$$(S_T - K)_+ = \max(0, S_T - K)$$
and coincides with the *payoff* of the option at maturity T.

- At time 0 the situation is less clear. What is an appropriate price to pay for the option? In other words, how much should the buyer of the option be willing to pay at time 0?



REMARK 1.7. This leads to the following two fundamental questions:

1. *Problem of Pricing*: How to evaluate at time 0 the amount $(S_T - K)_+$ disposable at time T?
2. *Problem of Hedging*: How can the seller (writer) of the option provide the amount $(S_T - K)_+$ (necessary to meet his obligations from the contract) at time T?

It will turn out that both questions are different perspectives of the *same* problem. The following hypothesis will be crucial for all subsequent considerations.

Fundamental hypothesis: $\boxed{\text{Absence of arbitrage}}$ i.e. impossibility of receiving a riskless gain.

REMARK 1.8. This hypothesis is sometimes expressed as NFLVR which stands for “No free lunch with vanishing risk”.

EXAMPLE 1.9. Let C_t be the price of a European call and P_t the price of a European put (at time $t \le T$) with the same date T of expiration and same strike price K, on a stock of price S_t at time t. Here we assume that options can also be traded (bought and sold) at any time t between 0 and maturity T, and hence have a value/price at time t. Note that

$$C_T = (S_T - K)_+ \quad \text{and} \quad P_T = (K - S_T)_+.$$

For simplicity, we suppose that it is possible to borrow and deposit money at a constant interest rate r.

Observation. By absence of arbitrage, we are able deduce the so-called *put-call parity*:

$$\boxed{C_t - P_t = S_t - K\,e^{-r(T-t)}, \quad \forall t \le T.}$$

Why? For instance, suppose that for some $t_0 \in [0, T[$ we have

$$C_{t_0} - P_{t_0} > S_{t_0} - K\,e^{-r(T-t_0)}. \qquad (*)$$

We are going to show that $(*)$ opens up an arbitrage opportunity as follows. We perform the following investment strategy:

At time t_0, we buy one share/stock, buy a put option and sell a call option. The balance of these operations amounts to:

$$\Delta := C_{t_0} - P_{t_0} - S_{t_0}.$$

If $\Delta > 0$, we deposit the amount Δ at an interest rate r in a cash account.
If $\Delta < 0$, we borrow the money at the same rate.

Then, at time T, there are two possibilities (scenarios): either $S_T > K$ or $S_T \leq K$.

1. $\boxed{S_T > K}$ In this case the call is "in the money" and will hence be exercised by its holder (buyer); we have to pay the difference $S_T - K$. In addition, we sell the share at the actual price S_T and close the cash deposit, resp. loan. In total, our balance reads as:
$$(K - S_T) + S_T + e^{r(T-t_0)}(C_{t_0} - P_{t_0} - S_{t_0}) > 0.$$
Recall that $e^{r(T-t_0)}(C_{t_0} - P_{t_0} - S_{t_0}) > -K$ by $(*)$. Hence, in this case, we make a clear profit.
2. $\boxed{S_T \leq K}$ We exercise the put and sell the stock. In total, again we gain:
$$(K - S_T) + S_T + e^{r(T-t_0)}(C_{t_0} - P_{t_0} - S_{t_0}) > 0.$$

Conclusion: In both cases, at time T, we make a positive gain (profit), without investing money at time 0. In other words, we found an arbitrage opportunity.

COMMENTS 1.10.

- *Some historical milestones:*
  - Louis Bachelier (1900): *Théorie de la spéculation.*
  - Black-Scholes (1973): Absence of arbitrage and certain assumption on the evolution of stock prices imply explicit formulas for the pricing of European calls/puts and hedging strategies for the seller.
- *Idea:* Fair price of a call/put is the amount of money initially necessary to construct a strategy which produces exactly the amount $(S_t - K)_+$, resp. $(K - S_T)_+$, at expiration time T.

1.2. A first example

EXAMPLE 1.11. (A single period binary model; toy model) Suppose that the current price in USD (American Dollar) of 100 EUR would be $S_0 = 150$, in other words, to buy 100 EUR would cost 150 USD. Let us consider in this situation a European call with strike price $K = 150$ and expiration time T. What could be a fair price for such an option?

We make the following simplifying *hypothesis*:

$$S_T = \begin{cases} 180, & \text{with probability } p, \\ 90, & \text{with probability } 1 - p. \end{cases}$$

In other words, at time T only two scenarios are possible. Recall that S_T is the cost in USD of 100 EUR at time T. The payoff of our option would thus be

$$\mathrm{PO} = (S_T - K)_+ = \begin{cases} 180 - 150 = 30 \text{ USD}, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

Then, taking expectation,

$$\mathbb{E}[\mathrm{PO}] = 30p.$$

This leads to the question whether $30p$ USD might be a fair price to pay for the option?

Assumption: Suppose for simplicity that interest rates are zero and that currency is bought and sold at the same exchange rate. To make things even more practical, let us suppose that $p = 1/2$.

Question: Is then 15 USD a fair price for the option? Note that fair price means excluding risk-free profit for both the buyer and the seller of the option.

CLAIM 1.12. *If the price is* 15 USD*, one can make a risk-free profit (as buyer).*

Strategy: I buy the option and borrow 33,3 EUR (to convert it straight into 50 USD).

- *At time* 0: I have one option and 35 USD (50 from the conversion of my EUR loan less 15 paid for the option).
- *At time* T: I am in one of the following two scenarios
  1. $S_T = 180$: In this case I exercise the option which allows me to buy 100 EUR for 150 USD. I use 33,3 EUR to pay off my EUR debt, leaving me with 66,67 EUR. This amount I convert back into USD at the *current* exchange rate. This nets
  $$66{,}67 \times 1,8 = 2/3 \times 180 = 120 \text{ USD}.$$
  In total: At time T, I have
  $$35 \text{ USD } - 150 \text{ USD } + 120 \text{ USD } = 5 \text{ USD}$$
  which is a clear profit.
  2. $S_T = 90$: In this case I throw away the (worthless) call and convert my 35 USD into EUR, netting
  $$35/0{,}9 = 38{,}89 \text{ EUR}.$$
  I pay back my debt (= 33,33 EUR), leaving a profit of 5,56 EUR.

Conclusion: Whatever the actual exchange rate is at time T, the proposed strategy allows to make a profit. Thus from the point of view of the seller, the price of the option was too low.

What is the right price?
Let's now take the point of view of the seller: If I am the seller, at time T, I'll need $(S_T - K)_+$ USD to meet the claim against me.

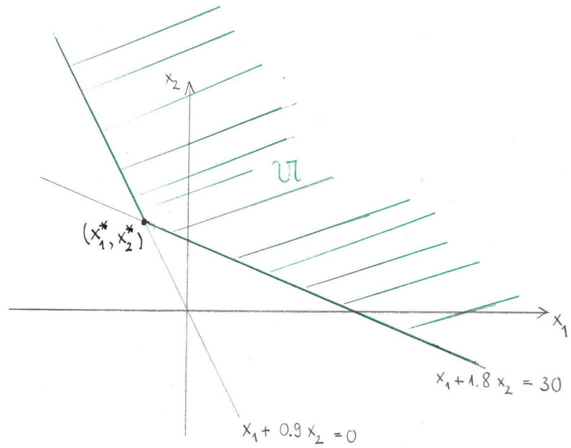
Question: How much money do I need at time 0 (to be held in combination of EUR and USD) to guarantee this at time T?

Suppose that at time 0, my portfolio is constituted of x_1 USD and x_2 EUR (this holding has to worth at least $(S_T - K)_+$ USD at time T).

1. $S_T = 180$:
In this case I need at least 30 USD, i.e.
$$x_1 + \frac{180}{100} x_2 \geq 30. \tag{A}$$

2. $S_T = 90$:
   The payoff of the option is 0; I just need not to be out of the pocket, i.e.

$$x_1 + \frac{90}{100} x_2 \geq 0. \tag{B}$$



Profit is guaranteed (without risk) for the seller if $(x_1, x_2) \in \mathscr{U}$ where $\mathscr{U}$ is the set of (x_1, x_2) satisfying (A) and (B). On the boundary $\partial\mathscr{U}$, there is a positive probability of profit, except at the point of intersection (x_1^*, x_2^*) of the lines

$$x_1 + \frac{180}{100} x_2 = 30 \quad \text{and} \quad x_1 + \frac{90}{100} x_2 = 0.$$

At (x_1^*, x_2^*) the seller is guaranteed to have exactly the money required to meet the claim against him at time T.

$$\left.\begin{aligned} x_1 + \frac{180}{100} x_2 &= 30 \\ x_1 + \frac{90}{100} x_2 &= 0 \end{aligned}\right\} \iff x_1^* = -30 \text{ USD}; \quad x_2^* = \frac{100}{3} \text{ EUR}.$$

To purchase $\frac{100}{3}$ EUR at time 0 requires $\frac{100}{3} \times \frac{150}{100} = 50$ USD. The value of the portfolio at time 0 is then $50 - 30 = 20$ USD.

Conclusion: The seller requires 20 USD at time 0 to construct a portfolio that will be worth the payoff of the option at time T (for any lower price there is a strategy for which the buyer makes a risk-free profit). Thus the fair price is 20 USD.

Important observations:

1. We didn't use the probability p at any point in the calculation!
2. The seller can *hedge* the contingent claim $(S_T - K)_+$ using a portfolio consisting of x_1 USD and x_2 EUR. He *replicates* the claim by this hedging portfolio.

In this picture the price of a call/put should correspond to the amount of money initially necessary to construct a portolio which produces exactly $(S_T - K)_+$, respectively $(K - S_T)_+$, at expiration date.

CHAPTER 2

Financial markets (time-discrete models)

2.1. Time-discrete markets

Definition 2.1 (Discrete model for a financial market). A *discrete model for a financial market* consists of the following data:

1. a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with $\operatorname{card}(\Omega) < \infty$ such that
$$\mathscr{F} = \mathscr{P}(\Omega) = \{A \colon A \subset \Omega\}$$
and $\mathbb{P}(\{\omega\}) > 0$ for each $\omega \in \Omega$, equipped with a filtration $(\mathscr{F}_n)_{n=0,1,\ldots,N}$ of sub-σ-algebras of $\mathscr{F}$:
$$\mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_N.$$
We assume that
$$\mathscr{F}_0 = \{\varnothing, \Omega\} \quad \text{and} \quad \mathscr{F}_N = \mathscr{F} = \mathscr{P}(\Omega).$$
Here $\mathscr{F}_n$ is interpreted as the information available at time n ("σ-algebra of events up to time n") and N as horizon, i.e. the expiry date of the options.
2. a family of non-negative adapted random variables:
$$(S_n^0, S_n^1, \ldots,\ S_n^d), \quad n = 0, 1, \ldots, N,$$
on $(\Omega, (\mathscr{F}_n), \mathbb{P})$, representing the prices of $d+1$ financial assets at time n. In other words, for each n,
$$S_n^0, S_n^1, \ldots, S_n^d$$
are positive $\mathscr{F}_n$-measurable random variables, or equivalently,
$$S_n := (S_n^0,\ S_n^1, \ldots, S_n^d)$$
is an $\mathbb{R}_+^{d+1}$-valued $\mathscr{F}_n$-measurable random variable, the so-called "price vector".

The asset indexed by 0 (i.e. S_n^0) is called *riskless asset*:
$$S_0^0 = 1 \quad \text{and} \quad S_n^0 = (1+r)^n,$$
where r denotes the *interest rate* over one period (for riskless deposits). The assets indexed by $1, \ldots, d$ are called *risky assets*.

The coefficient
$$\beta_n := \frac{1}{S_n^0} = \frac{1}{(1+r)^n}$$
is called *discount factor at time n*. It represents the amount of money (e.g. in EUR) which needs to be invested at time 0 to have 1 EUR available at time n. We call the prices
$$\tilde{S}_n := \beta_n\, S_n$$
discounted prices. We thus have
$$\tilde{S}_n = (\tilde{S}_n^0, \tilde{S}_n^1, \ldots, \tilde{S}_n^d) = (1, \beta_n S_n^1, \ldots, \beta_n S_n^d).$$

2.2. Strategies and portfolios

DEFINITION 2.2 (Trading strategy). A *trading strategy* (or portfolio) is a predictable stochastic process

$$\phi = (\phi_n)_{n=0,1,\dots,N}$$

taking values in $\mathbb{R}^{d+1}$, where the ith component ϕ_n^i of ϕ_n denotes the number of shares of asset i held in the portfolio at time n (for $i \in \{0, 1, \dots, d\}$).

Recall that ϕ *predictable* means the following:

$$\begin{cases} \phi_n \text{ is } \mathscr{F}_{n-1}\text{-measurable}, & \text{for each } 1 \le n \le N, \\ \phi_0 \text{ is } \mathscr{F}_0\text{-measurable}. \end{cases}$$

This reflects the fact that the composition of the portfolio at time n is decided with respect to the information available at time $n-1$ and is kept until time n when new quotations are available.

DEFINITION 2.3 (Value of the portfolio). The *value of the portfolio at time n* (with respect to the strategy $\phi = (\phi_n)_{n=0,\dots,N}$) is given by

$$V_n(\phi) := \langle \phi_n, S_n \rangle_{\mathbb{R}^{d+1}} = \sum_{k=0}^{d} \phi_n^k \, S_n^k, \quad n = 0, 1, \dots, N.$$

The *discounted value of the portfolio* is:

$$\tilde{V}_n(\phi) = \beta_n \, V_n(\phi) = \langle \phi_n, \tilde{S}_n \rangle_{\mathbb{R}^{d+1}}$$

where $\beta_n = 1/S_n^0$ and $\tilde{S}_n = (1, \beta_n S_n^1, \dots, \beta_n S_n^d)$.

DEFINITION 2.4 (Self-financing strategy). A strategy $\phi = (\phi_n)_{n=0,1,\dots,N}$ is called *self-financing*, if

$$\langle \phi_n, S_n \rangle = \langle \phi_{n+1}, S_n \rangle \quad \forall \, n \in \{0, 1, \dots, N-1\}.$$

Interpretation: The investor rearranges his positions between time n and time $n+1$ (by passing from ϕ_n to ϕ_{n+1}) such that this readjustment preserves the total value of the portfolio.

REMARK 2.5. A strategy is self-financing if the variations of the value of the portfolio are only due to changes in the stock prices, i.e., for each $n = 0, \dots, N-1$ we have

$$\begin{aligned} & \langle \phi_n, S_n \rangle = \langle \phi_{n+1}, S_n \rangle \\ & \iff \langle \phi_{n+1}, S_{n+1} - S_n \rangle = \langle \phi_{n+1}, S_{n+1} - S_n \rangle \\ & \iff V_{n+1}(\phi) - V_n(\phi) = \langle \phi_{n+1}, S_{n+1} - S_n \rangle. \end{aligned}$$

PROPOSITION 2.6. *The following conditions are equivalent:*

i) *The strategy $\phi = (\phi_n)_{n=0,1,\dots,N}$ is self-financing.*
ii) *For each $n \in \{1, \dots, N\}$, we have*

$$V_n(\phi) - V_{n-1}(\phi) = \langle \phi_n, \Delta S_n \rangle, \text{ where } \Delta S_n := S_n - S_{n-1}.$$

iii) *For each $n \in \{1, \dots, N\}$, we have*

$$\tilde{V}_n(\phi) - \tilde{V}_{n-1}(\phi) = \langle \phi_n, \Delta \tilde{S}_n \rangle, \text{ where } \Delta \tilde{S}_n := \tilde{S}_n - \tilde{S}_{n-1}.$$

iv) *For each* $n \in \{1, \dots, N\}$, *we have*

$$V_n(\phi) = V_0(\phi) + \sum_{k=1}^{n} \langle \phi_k, \Delta S_k \rangle.$$

v) *For each* $n \in \{1, \dots, N\}$, *we have*

$$\tilde{V}_n(\phi) = V_0(\phi) + \sum_{k=1}^{n} \langle \phi_k, \Delta \tilde{S}_k \rangle.$$

PROOF. By definition, the strategy (ϕ_n) is self-financing if and only if

$$\begin{aligned}
&\langle \phi_{n-1}, S_{n-1} \rangle = \langle \phi_n, S_{n-1} \rangle, \quad \forall n \\
&\quad \Longleftrightarrow V_n(\phi) - V_{n-1}(\phi) = \langle \phi_n, \Delta S_n \rangle, \quad \forall n \\
&\quad \Longleftrightarrow \langle \phi_{n-1}, \tilde{S}_{n-1} \rangle = \langle \phi_n, \tilde{S}_{n-1} \rangle, \quad \forall n \\
&\quad \Longleftrightarrow \tilde{V}_n(\phi) - \tilde{V}_{n-1}(\phi) = \langle \phi_n, \Delta \tilde{S}_n \rangle, \quad \forall n.
\end{aligned}$$

Hence the conditions i), ii) and iii) are equivalent.

Next we verify ii) $\Longleftrightarrow$ iv). Suppose that $V_k(\phi) - V_{k-1}(\phi) = \langle \phi_k, \Delta S_k \rangle$ for each $k \in \{1, \dots, N\}$. Then, by summation,

$$\begin{aligned}
V_n(\phi) - V_0(\phi) &= \sum_{k=1}^{n} V_k(\phi) - V_{k-1}(\phi) \\
&= \sum_{k=1}^{n} \langle \phi_k, \Delta S_k \rangle.
\end{aligned}$$

Conversely, if for all $n \in \{1, \dots, N\}$,

$$V_n(\phi) = V_0(\phi) + \sum_{k=1}^{n} \langle \phi_k, \Delta S_k \rangle,$$

then

$$V_n(\phi) - V_{n-1}(\phi) = \langle \phi_n, \Delta S_n \rangle.$$

This shows the equivalence of ii) and iv). The equivalence iii) $\Longleftrightarrow$ v) is obtained in the same way using discounted values. $\square$

REMARK 2.7. For a self-financing strategy, the discounted value of the portfolio is expressed in terms of the initial wealth and the trading strategy $\phi_n^1, \dots, \phi_n^d$, $1 \le n \le N$. Since $\tilde{S}_k^0 = 1$ (thus $\Delta \tilde{S}_k^0 = 0$), the 0-th component of the trading strategy does not enter into the formula for $\tilde{V}_n(\phi)$.

More precisely, we have the following observation.

PROPOSITION 2.8. *For any predictable process, taking values in* $\mathbb{R}^d$,

$$(\phi_n^1, \dots, \phi_n^d), \quad 0 \le n \le N,$$

and any $\mathscr{F}_0$*-measurable random variable* V_0, *there exists a unique real-valued predictable process* $(\phi_n^0)_{0 \le n \le N}$ *such that*

$$(\phi_n^0, \phi_n^1, \dots, \phi_n^d)_{0 \le n \le N}$$

is a self-financing trading strategy with $V_0(\phi) = V_0$.

PROOF. Indeed, the self-financing condition determines the component ϕ_n^0 as follows:

$$\begin{array}{l}\tilde{V}_n(\phi) = \langle \phi_n, \tilde{S}_n \rangle_{\mathbb{R}^{d+1}} = \phi_n^0 \underbrace{\tilde{S}_n^0}_{=1} + \phi_n^1 \tilde{S}_n^1 + \ldots + \phi_n^d \tilde{S}_n^d, \\ \quad \| \\ \tilde{V}_n(\phi) = V_0 + \sum_{k=1}^{n} \langle \phi_k, \Delta \tilde{S}_k \rangle = V_0 + \sum_{k=1}^{n} \left(\phi_k^1 \Delta \tilde{S}_k^1 + \ldots + \phi_k^d \Delta \tilde{S}_k^d \right),\end{array}$$

from where ϕ_n^0 can be calculated. It remains to check the predictability of ϕ_n^0. Indeed, we can write

$$\begin{aligned}\phi_n^0 &= V_0 + \sum_{k=1}^{n} \left(\phi_k^1 \Delta \tilde{S}_k^1 + \ldots + \phi_k^d \Delta \tilde{S}_k^d \right) - \left(\phi_n^1 \tilde{S}_n^1 + \ldots + \phi_n^d \tilde{S}_n^d \right) \\ &= V_0 + \sum_{k=1}^{n-1} \left(\phi_k^1 \Delta \tilde{S}_k^1 + \ldots + \phi_k^d \Delta \tilde{S}_k^d \right) - \left(\phi_n^1 \tilde{S}_{n-1}^1 + \ldots + \phi_n^d \tilde{S}_{n-1}^d \right),\end{aligned}$$

from where ϕ_n^0 is seen to be $\mathscr{F}_{n-1}$-measurable. □

Interpretation:
$\phi_n^0 < 0$: the investor borrows the amount $|\phi_n^0|$ in the riskless asset at time n.
$\phi_n^k < 0$ for $k = 1, \ldots, d$: the investor is "short" a number $|\phi_n^k|$ of asset k, i.e. short-selling and borrowing are allowed in the investment.

The idea that the investor should be able to pay back his debts (in riskless or risky assets) at any time leads to the following definition.

DEFINITION 2.9. A trading strategy $\phi = (\phi_n)_{n=0,1,\ldots,N}$ is called *admissible* if

a) ϕ is self-financing, and
b) $V_n(\phi) \geq 0, \quad \forall n = 0, 1, \ldots, N$.

DEFINITION 2.10. A trading strategy $\phi = (\phi_n)_{n=0,1,\ldots,N}$ is called *arbitrage strategy* if

a) ϕ is self-financing, and
b) $V_0(\phi) = 0$, $V_N(\phi) \geq 0$ and $\mathbb{P}(V_N(\phi) > 0) > 0$ (in other words, $V_0(\phi) = 0$, but $V_N(\phi) \gneqq 0$).

DEFINITION 2.11. A financial market is called *arbitrage-free* ("viable") if there is no arbitrage opportunity on this market, i.e.

$$V_0(\phi) = 0 \quad \text{and} \quad V_N(\phi) \geq 0 \ \Rightarrow \ V_N(\phi) = 0.$$

LEMMA 2.12. *If there exists an arbitrage strategy ϕ, then there exists also an admissible arbitrage strategy ϕ^*, i.e. an arbitrage strategy such that*

$$V_0(\phi^*) = 0, \ V_n(\phi^*) \geq 0, \ \forall n = 1, \ldots, N \ \textit{and} \ \mathbb{P}\left\{V_N(\phi^*) > 0\right\} > 0.$$

PROOF. Let

$$\phi = (\phi^0, \underbrace{\phi^1, \ldots, \phi^d}_{=:\varphi}) = (\phi_n^0, \underbrace{\phi_n^1, \ldots, \phi_n^d}_{=:\varphi_n})_{0 \leq n \leq N}$$

be an arbitrage strategy. Let

$$u(n) := \mathbb{P}\left\{V_n(\phi) \geq 0\right\} \quad \text{and} \quad v(n) := \mathbb{P}\left\{V_n(\phi) > 0\right\}.$$

Define

$$p := \min\{n \mid u(n) = 1 \text{ and } v(n) > 0\}.$$

From the definition of p we derive the following two properties:

(a) Since $v(0) = 0$, $u(N) = 1$ and $v(N) > 0$, we have

$$1 \leq p \leq N.$$

(b) For each $n < p$, either $v(n) = 0$ or $u(n) < 1$ holds.

(I) *Claim.* There exists an $\mathscr{F}_{p-1}$-measurable random variable $\eta = \left(\eta^1, \dots, \eta^d\right)$, taking values in $\mathbb{R}^d$, such that

$$\sum_{i=1}^{d} \eta^i \Delta \tilde{S}_p^i \geq 0 \quad \text{and} \quad \mathbb{P}\left\{\sum_{i=1}^{d} \eta^i \Delta \tilde{S}_p^i > 0\right\} > 0. \tag{$*$}$$

Proof of the claim. Indeed, we either have $v(p-1) = 0$ or $u(p-1) < 1$.

i) Suppose that $v(p-1) = 0$. Let $\eta := \varphi_p$. Then,

$$\begin{aligned}\sum_{i=1}^{d} \eta^i \Delta \tilde{S}_p^i &= \langle \phi_p, \Delta \tilde{S}_p \rangle_{\mathbb{R}^{d+1}} \\ &= \tilde{V}_p(\phi) - \underbrace{\tilde{V}_{p-1}(\phi)}_{\leq 0} \geq \tilde{V}_p(\phi), \quad \text{since } v(p-1) = 0.\end{aligned}$$

Using $u(p) = 1$ and $v(p) > 0$ we obtain $(*)$.

ii) Suppose that $u(p-1) < 1$. Let $\eta := \varphi_p 1_{\{V_{p-1}(\phi) < 0\}}$, then

$$\begin{aligned}\sum_{i=1}^{d} \eta^i \Delta \tilde{S}_p^i &= 1_{\{V_{p-1}(\phi)<0\}} \langle \phi_p, \Delta \tilde{S}_p \rangle_{\mathbb{R}^{d+1}} \\ &= 1_{\{V_{p-1}(\phi)<0\}} \left(\tilde{V}_p(\phi) - \tilde{V}_{p-1}(\phi)\right) \\ &\geq 1_{\{V_{p-1}(\phi)<0\}} \left(-\tilde{V}_{p-1}(\phi)\right), \quad \text{since } u(p) = 1, \\ &\geq 0.\end{aligned}$$

Using $u(p-1) < 1$ we obtain $(*)$.

(II) Let η be the $\mathbb{R}^d$-valued random variable such that $(*)$ holds. Define a predictable sequence $(\varphi_n^*)_{n=0,1,\dots,N}$ of $\mathbb{R}^d$-valued random variables by

$$\varphi_n^* = \begin{cases} \eta, & \text{for } n = p, \\ 0, & \text{for } n \neq p. \end{cases}$$

Let

$$\phi^* = (\phi_n^*)_{n=0,1,\dots,N}$$

be the self-financing strategy (with initial value $V_0(\phi^*) = 0$) associated to (φ_n^*) according to Proposition 2.8. Then

$$\tilde{V}_n(\phi^*) = \sum_{k=1}^{n} \langle \phi_k^*, \Delta \tilde{S}_k \rangle = \sum_{k=1}^{n} \sum_{i=1}^{d} \left(\phi_k^*\right)^i \Delta \tilde{S}_k^i = \sum_{k=1}^{n} \sum_{i=1}^{d} \left(\varphi_k^*\right)^i \Delta \tilde{S}_k^i.$$

Hence, by the definition of φ_k^*, we get

$$\tilde{V}_n(\phi^*) = \begin{cases} 0, & n < p, \\ \sum_{i=1}^{d} \eta^i \Delta \tilde{S}_p^i, & n \geq p. \end{cases}$$

By $(*)$ of part 1), we thus have $\tilde{V}_n(\phi^*) \geq 0$ for each $n \in \{0, 1, \ldots, N\}$ and

$$\mathbb{P}\left\{\tilde{V}_N(\phi^*) > 0\right\} > 0. \qquad \square$$

CHAPTER 3

Martingales and arbitrage

3.1. Conditional expectation and martingales

Let X be a square-integrable real-valued random variable on a discrete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e. X is $\mathscr{F}$-measurable and $\mathbb{E}[|X|^2] < \infty$.

We denote the Hilbert space of all such random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ by $L^2(\Omega, \mathscr{F}, \mathbb{P})$, i.e.

$$L^2(\Omega, \mathscr{F}, \mathbb{P}) = \{X \text{ random variable such that } \mathbb{E}[|X|^2] < \infty\}.$$

Note that the inner product and the norm on $L^2(\Omega, \mathscr{F}, \mathbb{P})$ are given by

$$\langle X, Y \rangle = \mathbb{E}[XY], \quad \text{respectively } \|X\| = \sqrt{\mathbb{E}[X^2]}.$$

Let $\mathscr{A} \subset \mathscr{F}$ be a sub-σ-algebra. We want to define the conditional expectation $\mathbb{E}[X|\mathscr{A}]$. By definition, $\mathbb{E}[X|\mathscr{A}]$ should be $\mathscr{A}$-measurable. In addition, $\mathbb{E}[X|\mathscr{A}]$ should be a "good" approximation of X.

Hence, we have $X \in L^2(\Omega, \mathscr{F}, \mathbb{P})$, and we observe that

$$\underbrace{L^2(\Omega, \mathscr{A}, \mathbb{P}|\mathscr{A})}_{=:\ L^2(\mathbb{P}|\mathscr{A})} \subset \underbrace{L^2(\Omega, \mathscr{F}, \mathbb{P})}_{=:\ L^2(\mathbb{P})}$$

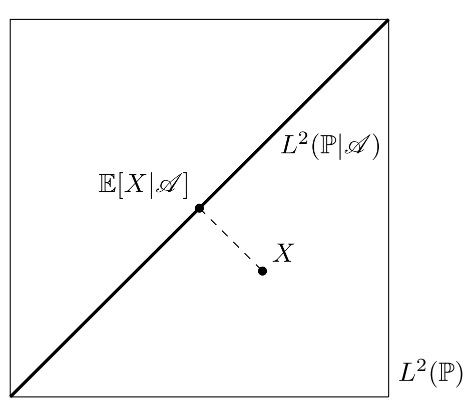
is a closed subspace.

Definition 3.1. We define $\mathbb{E}[X|\mathscr{A}]$ as the orthogonal projection of X onto $L^2(\mathbb{P}|\mathscr{A})$. Thus, by definition

$$X - \mathbb{E}[X|\mathscr{A}] \perp 1_A \quad \forall\, A \in \mathscr{A}.$$

This characterizes $\mathbb{E}[X|\mathscr{A}]$ as the best $\mathscr{A}$-measurable approximation of X in the quadratic mean sense:

$$\mathbb{E}\left[|X - \mathbb{E}[X|\mathscr{A}]|^2\right] = \min_{Y \in\ L^2(\mathbb{P}|\mathscr{A})} \mathbb{E}\left[|X - Y|^2\right].$$



NOTE. For $A \in \mathscr{A}$ we have

$$X - \mathbb{E}[X|\mathscr{A}] \perp 1_A \Longleftrightarrow \mathbb{E}[1_A X] = \mathbb{E}\left[1_A \mathbb{E}[X|\mathscr{A}]\right].$$

Hence, the property

$$\mathbb{E}[1_A X] = \mathbb{E}\left[1_A \mathbb{E}[X|\mathscr{A}]\right] \text{ for all } A \in \mathscr{A}$$

characterizes $\mathbb{E}[X|\mathscr{A}]$.

PROPOSITION 3.2. *The conditional expectation has the following properties:*

(i) (Positivity) *If* $X \geq 0$, *then* $\mathbb{E}[X|\mathscr{A}] \geq 0$.
(ii) (Linearity) $\mathbb{E}[\alpha X + \beta Y|\mathscr{A}] = \alpha \mathbb{E}[X|\mathscr{A}] + \beta \mathbb{E}[Y|\mathscr{A}]$, $\forall \alpha, \beta \in \mathbb{R}$.
(iii) (Contractivity) $\left|\mathbb{E}[X|\mathscr{A}]\right| \leq \mathbb{E}[|X|\,|\mathscr{A}]$.
(iv) (Taking out what is known) $\mathbb{E}[YX|\mathscr{A}] = Y\mathbb{E}[X|\mathscr{A}]$ *if* Y *is* $\mathscr{A}$*-measurable.*
(v) (Tower property) *If* $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \mathscr{F}$, *then* $\mathbb{E}\left[\mathbb{E}[X|\mathscr{A}_2]\,|\mathscr{A}_1\right] = \mathbb{E}[X|\mathscr{A}_1]$.
(vi) (Conditioning with respect to trivial σ-fields)
If $\mathscr{A} = \{\emptyset, \Omega\}$, *then* $\mathbb{E}[X|\mathscr{A}] = \mathbb{E}[X]$.

Recall that a discrete financial market is called arbitrage-free if for any self-financing strategy ϕ such that $V_0(\phi) = 0$ and $V_N(\phi) \geq 0$ it follows that $V_N(\phi) = 0$. The following characterization of arbitrage-free financial markets will be the basis of all further considerations.

THEOREM 3.3 (Fundamental theorem of asset pricing). *A discrete financial market is arbitrage-free if and only if there exists a probability measure* $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$ *such that under* $\mathbb{P}^*$, *the discounted prices* $(\tilde{S}_n)_{n=0,\ldots,N}$ *are martingales.*

RECALL. $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ (in symbols, $\mathbb{P}^* \sim \mathbb{P}$)

$$\begin{aligned} &\Longleftrightarrow \quad \mathbb{P}^* \ll \mathbb{P} \text{ and } \mathbb{P} \ll \mathbb{P}^*, \\ &\Longleftrightarrow \quad \forall A \in \mathscr{F},\ \mathbb{P}^*(A) = 0 \text{ if and only if } \mathbb{P}(A) = 0, \\ &\Longleftrightarrow \quad \mathbb{P}^*\left(\{\omega\}\right) > 0 \text{ for each } \omega \in \Omega. \end{aligned}$$

Before starting to prove Theorem 3.3 we do some preparations and collect the required concepts.

DEFINITION 3.4 (Martingale). Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability equipped with a filtration $(\mathscr{F}_n)_{n=0,1,\ldots,N}$ of sub-σ-algebras

$$\mathscr{F}_0 \subset \mathscr{F}_1 \subset \ldots \subset \mathscr{F}_N.$$

For simplicity we shall assume that $\operatorname{card}(\Omega) < \infty$, $\mathscr{F} = \mathscr{P}(\Omega)$ and that $\mathbb{P}\left(\{\omega\}\right) > 0$ for each $\omega \in \Omega$.

An adapted sequence $(M_n)_{n=0,1,\ldots,N}$ of real random variables is called

- *martingale* if $\mathbb{E}\left[M_{n+1}\,|\mathscr{F}_n\right] = M_n, \quad \forall\, n = 0, 1, \ldots, N-1$.
- *supermartingale* if $\mathbb{E}\left[M_{n+1}\,|\mathscr{F}_n\right] \leq M_n, \quad \forall\, n = 0, 1, \ldots, N-1$.
- *submartingale* if $\mathbb{E}\left[M_{n+1}\,|\mathscr{F}_n\right] \geq M_n, \quad \forall\, n = 0, 1, \ldots, N-1$.

An adapted sequence $(M_n)_{n=0,1,\ldots,N}$ of $\mathbb{R}^d$-valued random variables is a martingale, resp. a super-/submartingale, if each component $(M_n^i)_{n=0,1,\ldots,N}$ is a real-valued martingale, resp. super-/submartingale, $i = 1, \ldots, d$.

NOTE.

i) An adapted process $(M_n)_{0\le n\le N}$ is a martingale if and only if

$$\mathbb{E}\left[M_n \,|\mathscr{F}_m\right] = \; M_m, \quad \text{for all } 0 \le m \le n \le N.$$

Thus, in particular,

$$\begin{aligned}\mathbb{E}\left[M_n\right] &= \mathbb{E}\left[M_m\right], \quad \forall\, n, m \in \{0, 1, \dots, N\},\\ \mathbb{E}\left[M_n\right] &= \mathbb{E}\left[M_0\right].\end{aligned}$$

ii) Similar properties hold for super/sub-martingale.

DEFINITION 3.5. Let $(M_n)_{n=0,1,\dots,N}$ be adapted and $(H_n)_{n=0,1,\dots,N}$ be predictable, i.e. M_n is $\mathscr{F}_n$-measurable and H_n is $\mathscr{F}_{n-1}$-measurable. Set

$$\Delta M_n := M_n - M_{n-1}.$$

The sequence $(X_n)_{n=0,1,\dots,N}$ defined by

$$\begin{cases} X_0 := H_0 M_0, \\ \Delta X_n = H_n \Delta M_n, \quad \text{for } n \ge 1, \end{cases}$$

is called *transform of* (M_n) *by* (H_n) (or (H_n)-*transform of* (M_n)) and denoted by

$$X_n = (H * M)_n, \quad n = 0, 1, \dots, N.$$

The sequence $(H * M)_{n=0,1,\dots,N}$ is thus given by

$$\begin{cases} (H * M)_0 = H_0 M_0, \\ (H * M)_n = H_0 M_0 + \sum\limits_{k=1}^{n} H_k \Delta M_k, \quad \text{for } n \ge 1. \end{cases}$$

If (M_n) is a martingale, then $H * M \mathrel{\widehat{=}} (H * M)_n$ is called *martingale transform of* (M_n) *by* (H_n).

PROPOSITION 3.6 (Martingale transforms). *On $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_n))$ let $(M_n)_{0\le n\le N}$ be an adapted sequence and $(H_n)_{0\le n\le N}$ a predictable sequence. If (M_n) is a martingale, then $(H * M)_n$ is a martingale as well.*

In particular, (M_n) is a martingale if and only if $(H * M)_n$ is a martingale for all predictable sequences (H_n).

PROOF. Let (M_n) be a martingale and (H_n) be predictable. Let

$$X_n = (H * M)_n = H_0 M_0 + \sum_{k=1}^{n} H_k \Delta M_k.$$

Then $(X_n)_{0\le n\le N}$ is clearly adapted. For all $n \ge 0$, we have

$$\begin{aligned}\mathbb{E}[X_{n+1} - X_n|\mathscr{F}_n] &= \mathbb{E}[H_{n+1}\Delta M_{n+1}|\mathscr{F}_n], \quad \text{as } \Delta X_{n+1} = H_{n+1}\Delta M_{n+1},\\ &= H_{n+1}\,\mathbb{E}[\Delta M_{n+1}|\mathscr{F}_n], \quad \text{as } H_{n+1} \text{ is } \mathscr{F}_n\text{-measurable},\\ &= H_{n+1}\,\mathbb{E}[M_{n+1} - M_n|\mathscr{F}_n]\\ &= 0, \quad \text{as } \mathbb{E}[M_{n+1} - M_n|\mathscr{F}_n] = 0.\end{aligned}$$

This shows that $\mathbb{E}[X_{n+1} - X_n|\mathscr{F}_n] = 0$, hence $\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$. □

Proposition 3.7 (Characterization of martingales). *An adapted sequence (M_n) of real valued random variables is a martingale if and only if*

$$\mathbb{E}\left[\sum_{n=1}^{N} H_n \Delta M_n\right] = 0,$$

for all predictable sequences (H_n).

Proof. Let (M_n) be a martingale and (H_n) be predictable. Then $(H * M)_n$ is a martingale and this implies:

$$\mathbb{E}\left[(H * M)_N\right] = \mathbb{E}\left[(H * M)_0\right].$$

Thus

$$0 = \mathbb{E}\left[(H * M)_N - (H * M)_0\right] = \mathbb{E}\left[\sum_{n=1}^{N} H_n \Delta M_n\right].$$

Conversely, suppose that

$$\mathbb{E}\left[\sum_{n=1}^{N} H_n \Delta M_n\right] = 0$$

for any predictable sequence (H_n). Let $A \in \mathscr{F}_m$, $m \in \{0, 1, \dots, N-1\}$.

We want to show that

$$\mathbb{E}\left[1_A (M_{m+1} - M_m)\right] = 0.$$

Consider the predictable sequence (H_n) defined by

$$H_n = \begin{cases} 0, & n \neq m+1, \\ 1_A, & n = m+1. \end{cases}$$

Then

$$\begin{aligned} 0 = \mathbb{E}\left[\sum_{n=1}^{N} H_n \Delta M_n\right] &= \mathbb{E}\left[1_A \Delta M_{m+1}\right] \\ &= \mathbb{E}\left[1_A (M_{m+1} - M_m)\right]. \end{aligned}$$

Thus,

$$\mathbb{E}[1_A M_{m+1}] = \mathbb{E}[1_A M_m] \quad \forall A \in \mathscr{F}_m,$$

which means that

$$\mathbb{E}[M_{m+1} | \mathscr{F}_m = M_m.$$

□

Remark 3.8. Let (M_n) be an adapted sequence of random variables taking values in $\mathbb{R}^d$ and (H_n) be a predictable sequence of random variables taking values in $\mathbb{R}^d$ as well. Define

$$(H * M)_n = \sum_{i=1}^{d} (H^i * M^i)_n, \quad n = 0, 1, \dots, N.$$

Then the following conditions are equivalent:

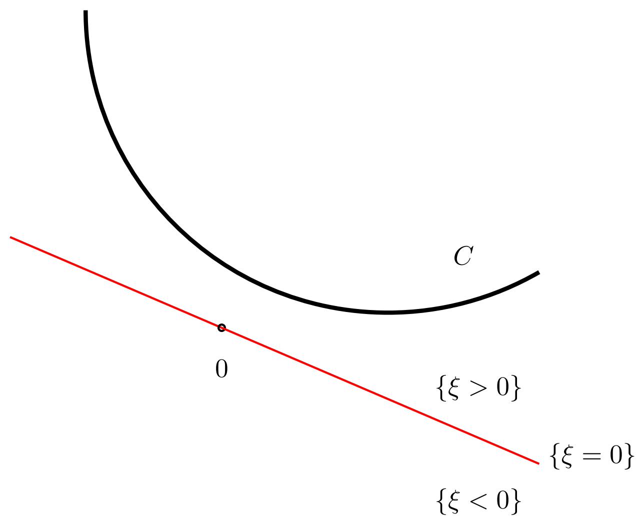
(i) (M_n) is a martingale;
(ii) $(H * M)_n$ is a martingale for any predictable sequence (H_n);
(iii) $\mathbb{E}\left[\sum_{n=1}^{N} \langle H_n, \Delta M_n \rangle_{\mathbb{R}^d}\right] = 0$ for any predictable sequence (H_n);
(iv) $\mathbb{E}\big[(H * M)_N\big] = \mathbb{E}\big[(H * M)_0\big]$ for any predictable sequence (H_n).

3.2. Some tools from Convex Analysis

THEOREM 3.9 (Theorem of Hahn-Banach: separation of convex sets). *Let $(E, \langle\, ,\, \rangle)$ be a finite-dimensional Euclidean vector space and let C be a closed convex subset of E such that $0 \notin C$. Then there exists a linear form $\xi = \langle \xi, \cdot \rangle$ on E and a constant $\alpha > 0$ such that*

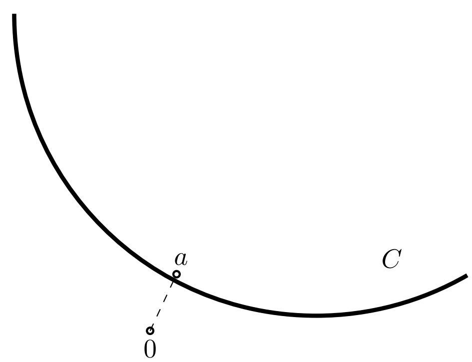
$$\forall x \in C, \quad \xi(x) \equiv \langle \xi, x \rangle \geq \alpha.$$

In particular, $\{\xi = 0\} \cap C = \emptyset$.



PROOF. There exists a (unique) point $a \in C$ of minimal norm, i.e.

$$\|a\|^2 \leq \|x\|^2, \quad \forall x \in C.$$



Let $x \in C$ be arbitrary and $0 < \vartheta \leq 1$. Then, by the convexity of C, also

$$\vartheta x + (1 - \vartheta) a = a + \vartheta (x - a) \in C.$$

Hence, for any $\vartheta \in]0, 1]$,

$$\|a\|^2 \leq \|a + \vartheta (x - a)\|^2.$$

In other words,

$$\|a\|^2 \leq \|a\|^2 + \vartheta^2 \|x - a\|^2 + 2\vartheta \langle a, x - a \rangle \quad \forall \vartheta \in]0, 1],$$

and thus

$$2\langle a, x-a\rangle + \vartheta\|x-a\|^2 \geq 0.$$

Hence

$$\langle a, x\rangle \geq \|a\|^2 =: \alpha > 0 \quad \forall x \in C,$$

and we may take $\xi = \langle a, \cdot\rangle$. □

Corollary 3.10. *Let $V \subset E$ be a linear subspace and let $K \subset E$ be a convex and compact subset such that $V \cap K = \emptyset$. Then there exists linear form $\xi = \langle \xi, \cdot\rangle$ on E such that*

$$\forall x \in K,\ \xi(x) > 0 \quad \textit{and} \quad \forall x \in V,\ \xi(x) = 0$$

(i.e. the linear subspace V is contained in the hyperplane $\{\xi = 0\}$ which does not intersect K).

Proof. Let

$$C := K + V$$

which is obviously closed (V is closed as linear subspace of a finite-dimensional vector space), convex and $0 \notin C$ (since $0 \in C$ would mean that $0 = k + v$ for some $k \in K$, $v \in V$ and hence $k = -v \in V \cap K$, but $V \cap K = \emptyset$). Thus, by Hahn-Banach, there exists a linear form ξ on E such that

$$\forall\, x \in C, \quad \xi(x) > 0.$$

Now, let $y \in K$ and $z \in V$. Then $y - \lambda z \in C$ for any $\lambda \in \mathbb{R}$. But then, for any $\lambda \in \mathbb{R}$,

$$\xi(y - \lambda z) > 0, \quad \text{i.e.} \quad \xi(y) - \lambda \xi(z) > 0.$$

This clearly implies that

$$\xi(z) = 0 \quad \text{and} \quad \xi(y) > 0.$$

□

3.3. Fundamental theorem of asset pricing

Theorem 3.11. *A (discrete) financial market is arbitrage-free if and only if there exists a probability measure $\mathbb{P}^*$ with $\mathbb{P}^*(\{\omega\}) > 0$, $\forall \omega \in \Omega$ such that the discounted asset prices $\left(\tilde{S}^i_n\right)_{n=0,1,\ldots,N}$ are $\mathbb{P}^*$-martingales ($i = 1, \ldots, d$).*

Proof.

1) Suppose that there exists a probability measure $\mathbb{P}^* \in \operatorname{Prob}(\Omega)$ with $\operatorname{supp} \mathbb{P}^* = \Omega$ such that $(\tilde{S}^i_n)_{n=0,1,\ldots,N}$ are $\mathbb{P}^*$-martingales ($i = 1, \ldots, d$). Let $\phi = (\phi_n)$ be a self-financing strategy such that $V_0(\phi) = 0$ and $V_N(\phi) \geq 0$. Then

$$\begin{aligned}
\tilde{V}_n(\phi) &= V_0(\phi) + \sum_{k=1}^{n} \left\langle \phi_k, \Delta \tilde{S}_k \right\rangle \\
&= \sum_{k=1}^{n} \sum_{i=0}^{d} \phi^i_k \Delta \tilde{S}^i_k \\
&= \sum_{i=0}^{d} \left(\phi^i * \tilde{S}^i\right)_n = (\phi * \tilde{S})_n.
\end{aligned}$$

But under $\mathbb{P}^*$, the discounted price vector $(\tilde{S}_n)_{n=0,1,\ldots,N}$ is a $\mathbb{P}^*$-martingale, and hence $(\tilde{V}_n(\phi))_{n=0,1,\ldots,N}$ as martingale transform is a martingale as well. Thus, in particular,

$$\mathbb{E}^*\big[\tilde{V}_N(\phi)\big] = \mathbb{E}^*\big[\tilde{V}_0(\phi)\big] = 0$$

where $\mathbb{E}^*[\ldots] = \int \ldots d\mathbb{P}^*$ denotes the expectation with respect to $\mathbb{P}^*$. Hence if $\tilde{V}_N(\phi) \geq 0$ then

$$\tilde{V}_N(\phi) = 0.$$

2) Conversely, suppose now that the market is arbitrage-free. Let

- $E = \{X \mid X \text{ random variable on } \Omega\} \mathrel{\widehat{=}} \{X \colon \Omega \to \mathbb{R}\} \mathrel{\widehat{=}} \mathbb{R}^{\mathrm{card}(\Omega)}$ and let $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product on E:

$$\langle X, Y \rangle = \sum_{\omega \in \Omega} X(\omega) Y(\omega).$$

- $V = \big\{\tilde{V}_N(\phi) \mid \phi \in \mathscr{S}_0 \big\}$ with

$$\mathscr{S}_0 = \{\phi \,|\, \phi \text{ self-financing strategy such that } V_0(\phi) = 0\}\,.$$

  Thus V is the "space of contingent claims (derivatives) attainable at price 0". Obviously $V \subset E$ is a linear subspace.

- $K = \left\{X \mid X \text{ random variable on } \Omega,\ X \geq 0 \text{ and } \sum_{\omega \in \Omega} X(\omega) = 1\right\}$

  which is obviously a convex and compact subset of K.

Note that $V \cap K = \emptyset$ by the assumption of arbitrage-freeness.

Now, by Hahn-Banach, there exists a random variable α on Ω, such that

$$\sum_{\omega \in \Omega} \alpha\,(\omega)\, X\,(\omega) > 0, \quad \forall\, X \in K, \quad \text{and} \tag{3.1}$$

$$\sum_{\omega \in \Omega} \alpha\,(\omega)\, \tilde{V}_N(\phi)\,(\omega) = 0, \quad \forall\, \phi \in \mathscr{S}_0. \tag{3.2}$$

Relation (3.1) implies that $\alpha\,(\omega) > 0$ for all $\omega \in \Omega$ (for instance, one may take $X = \mathbb{1}_{\{\omega\}} \in K$). We define now

$$\mathbb{P}^* = \frac{\alpha}{\|\alpha\|} \quad \text{where} \quad \|\alpha\| = \sum_{\omega \in \Omega} \alpha\,(\omega)\,.$$

Then $\mathbb{P}^*$ is a probability measure on Ω and $\mathbb{P}^*\left(\{\omega\}\right) = \frac{\alpha(\omega)}{\|\alpha\|} > 0,\ \forall\ \omega \in \Omega$. Equation (3.2) writes as

$$\mathbb{E}^*\big[\tilde{V}_N(\phi)\big] = 0, \quad \forall\, \phi \in \mathscr{S}_0. \tag{3.3}$$

We have to show that $(\tilde{S}_n)$ is a martingale under $\mathbb{P}^*$. Equivalently, we have to show that, for all predictable sequence $(\varphi_n)_{0 \leq n \leq N}$ with values in $\mathbb{R}^d$,

$$\mathbb{E}^*\left[\sum_{n=1}^N \langle \varphi_n, \Delta\tilde{S}_n \rangle\right] = 0.$$

Thus let $(\varphi_n)_{0 \leq n \leq N}$ be a predictable sequence with values in $\mathbb{R}^d$ and let $\phi_n = \big(\phi_n^0, \varphi_n\big)$ be the self-financing strategy in $\mathscr{S}_0$ associated to (φ_n). Then

$$\tilde{V}_N(\phi) = \tilde{V}_0(\phi) + \sum_{n=1}^N \langle \varphi_n, \Delta\tilde{S}_n \rangle.$$

Since $\tilde{V}_0(\phi) = V_0(\phi) = 0$, taking expectation with respect to $\mathbb{P}^*$ on both sides and using Eq. (3.3), we obtain

$$\mathbb{E}^* \left[\sum_{n=1}^{N} \langle \varphi_n, \Delta \tilde{S}_n \rangle \right] = 0$$

as wanted. $\square$

CHAPTER 4

Option pricing and hedging

4.1. Examples of options

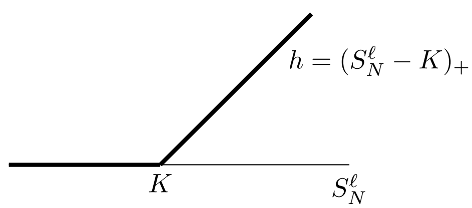
Definition 4.1 (European option). A *European option* (or *contingent claim*) with maturity N is an $\mathscr{F}_N$-measurable non-negative random variable h. The random variable $\tilde{h} = \beta_N h$ is called the *discounted option.*

In other words, European options are characterized by their payoff (or value) h at maturity. We may distinguish call and put options.

Example 4.2. Let $(S_n^\ell)_{n=0,\dots,N}$ be the price process of an underlying asset labeled ℓ.

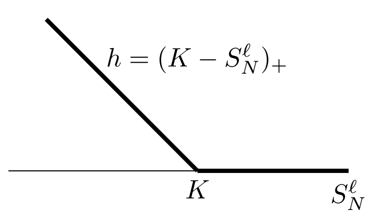
1) A *European call* with *strike price* K and *maturity* (date of expiration) N on the underlying S^ℓ is given by the following payoff at maturity:

$$h = (S_N^\ell - K)_+$$



2) A *European put* with *strike price* K and *maturity* N on the same underlying is given by the following payoff at maturity:

$$h = (K - S_N^\ell)_+$$



3) Let $a, b > 0$ such that $a < S_0^\ell < b$. The European option

$$h = S_N^\ell \vee a \wedge b$$

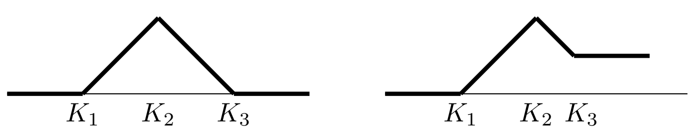
is called "guaranteed placement". It has the property that there is a limitation of the losses at level a and a limitation of the profits at level b.

4) A European option of the type

$$h = \left|S_N^\ell - K\right| = \left(S_N^\ell - K\right)_+ + \left(K - S_N^\ell\right)_+$$

is called a *straddle*. It is replicated by holding a European call and a European put with the same strike price and maturity. A straddle can lead to substantial profit if a large move in the stock price is expected, but if one does not know in which direction the move will be.

5) *Spread trading*
   (a) *Bullish (resp. Bearish) vertical spread.* Buy one European call (resp. put) and sell a second one with the same maturity, but a higher (resp. lower) strike price.
   Note that the bullish vertical spread can be constructed with either calls or puts. When it is done with calls, then as indicated you buy a call and simultaneously sell a call at a higher strike. Since the call with the lower strike price will always be worth more than the call with the higher strike price, a bull vertical will always be established for a debit.
   Similarly, if a bearish vertical spread is constructed with puts is done by buying a put and selling a lower strike put. This construction will also always be done for a debit, since the higher strike put will always be more expensive than the lower strike put.
   (b) *Strip (resp. strap).* Buy one (resp. two) European calls and two (resp. one) European puts with the same strike price.
   (c) *Strangle.* Buy a European call and a European put with same maturity but different strike prices.
   (d) *Butterfly spread.* Buy one European call with strike price K_1, sell two European puts with strike price K_2, and buy one European call with strike price K_3 (where $K_1 \leq K_2 \leq K_3$).



if $2K_2 = K_1 + K_3$ if $2K_2 > K_1 + K_3$

Note that up to now all options were of the type $h = F(S_N^\ell)$.

6) One may replace S_N^ℓ by some mean value of the type

$$S_I^\ell = \frac{1}{|I|} \sum_{n \in I} S_n^\ell \quad \text{for some given subset } I \subset \{0, 1, \dots, N\}.$$

This gives the notion of an *Asiatic option.* For example:

$h := (S_I^\ell - K)_+$ is an *Asiatic Call* with strike K over I,
$h := (K - S_I^\ell)_+$ is an *Asiatic Put* with strike K over I.

7) *Lookback calls.* These are options of the type

$$h = S_N^\ell - \min_{0 \leq n \leq N} S_n^\ell.$$

8) *Barrier options.* These are options which are activated (resp. deactivated) if the asset price crosses a prescribed barrier.
   a) *Knock-ins.* The barrier is up-and-in (resp. down-and-in) if the option is only active if the barrier is hit from below (resp. above).
   b) *Knock-outs.* The barrier is up-and-out (resp. down-and-out) if the option is worthless if the barrier is hit from below (resp. above).

For example, if $S_0^\ell > c$, a "down-and-in" call is given by

$$h = (S_N^\ell - K)_+ \, \mathbb{1}_{\left\{ \min\limits_{0 \le n \le N} S_n^\ell \le c \right\}}.$$

4.2. Prices excluding arbitrage

Definition 4.3 (Pricing rule). A *pricing rule* for European options h of maturity N is a mapping

$$\pi : \; h \mapsto \pi(h),$$

where

$$\pi(h) = (\pi_n(h))_{n=0,\dots,N}$$

is an adapted sequence of random variables such that $\pi_n(h) \ge 0$ and $\pi_N(h) = h$.

Interpretation $\pi_n(h)$ refers to the price (value) of the option h at time $n \le N$. Options are traded at the stock market and thus have a price at any time $n \le N$, which equilibrates offer and demand. In this way, options can be considered as additional risky assets on the financial market.

Definition 4.4 (Arbitrage-freeness of a pricing rule). A pricing rule for European options of maturity N is *arbitrage-free* if for any finite set of European options $h^1, \dots, h^\ell$ the financial market

$$\big(\Omega, (\mathscr{F}_n), \mathbb{P}; S^0, S^1, \dots, S^d, \pi(h^1), \dots, \pi(h^\ell)\big)$$

is arbitrage-free.

Definition 4.5 (Prices excluding arbitrage). Let h be a European option in an arbitrage-free financial market $(\Omega, (\mathscr{F}_n), \mathbb{P}; (S_n))$. If there exists an adapted sequence $(H_n)_{\{0 \le n \le N\}}$ of positive random variables such that

i) $H_N = h$,
ii) the financial market $(\Omega, (\mathscr{F}_n), \mathbb{P}; (S_n), (H_n))$ is still arbitrage-free,

then H_0 is said to be a *price excluding arbitrage* (p.e.a) for the option h, and (H_n) a *sequence of prices excluding arbitrage.*

Proposition 4.6. *In an arbitrage-free market, let be given*

- *a European call* $C := (S_N^\ell - K)_+$, *and*
- *a European put* $P := (K - S_N^\ell)_+$

with same strike price $K > 0$. *Suppose that* (C_n) *is a sequence of p.e.a for the call* C *and* (P_n) *a sequence of p.e.a for the put* P, *in the sense that the augmented market*

$$(\Omega, (\mathscr{F}_n), \mathbb{P}; (S_n), (C_n), (P_n))$$

is still arbitrage-free. Then the so-called "call-put parity" holds:

$$C_n - P_n = S_n^\ell - \frac{K}{(1+r)^{N-n}}, \quad \forall\, n = 0, \dots, N. \tag{$*$}$$

PROOF. The market carries now $d+3$ assets numbered $0, 1, \ldots, d, c, p$. We write

$$\bar{S}_n = (S_n, C_n, P_n), \quad n = 0, 1, \ldots, N.$$

By assumption, this augmented market is still arbitrage-free. We have to show the following claim: *If* $(*)$ *fails, then there exists an arbitrage strategy.*

For simplicity, let us assume that $(*)$ already fails for $n = 0$.

1. Suppose that

$$C_0 - P_0 > S_0^\ell - \frac{K}{(1+r)^N} \tag{$**$}$$

Let

$$\varphi = (\varphi_n)_{0 \le n \le N} = \left(\varphi_n^k : \ k \in \{1, 2, \ldots, d, c, p\}\right)_{0 \le n \le N}$$

be the predictable sequence taking values in $\mathbb{R}^{d+2}$ which is defined as follows:

$$\left.\begin{array}{l} \varphi_n^\ell = 1, \quad \varphi_n^c = -1, \quad \varphi_n^p = 1, \\ \varphi_n^k = 0, \quad \text{if } k \notin \{\ell, c, p\} \end{array}\right\} \quad \text{for } n = 0, 1, \ldots, N.$$

There exists a unique self-financing strategy ϕ which coincides with φ for the risky assets such that $V_0(\phi) = 0$.

Then, for each $n \le N$,

$$V_n(\phi) = S_n^\ell - C_n + P_n + \phi_n^0 S_n^0, \tag{1}$$

and in particular,

$$S_0^\ell - C_0 + P_0 + \phi_0^0 = 0.$$

Recall that ϕ self-financing means that

$$\langle \phi_n, \bar{S}_n \rangle = \langle \phi_{n+1}, \bar{S}_n \rangle, \quad n \le N - 1,$$

where $\bar{S}_n = (S_n, C_n, P_n)$. Hence, for each $n \le N-1$,

$$V_n(\phi) = \left\langle \phi_{n+1}, \bar{S}_n \right\rangle = S_n^\ell - C_n + P_n + \phi_{n+1}^0 S_n^0. \tag{2}$$

Combining equations (1) and (2) we conclude that ϕ_n^0 is independent of n, i.e.

$$\phi_n^0 = \phi_0^0 = C_0 - P_0 - S_0^\ell. \tag{3}$$

Finally, by substituting (3) into (1), we obtain:

$$V_n(\phi) = S_n^\ell - C_n + P_n + \left(C_0 - P_0 - S_0^\ell\right)(1+r)^n,$$

and hence

$$V_N(\phi) = K + \left(C_0 - P_0 - S_0^\ell\right)(1+r)^N > 0, \quad \text{by hypothesis } (**).$$

Thus ϕ is an arbitrage strategy, in contradiction to arbitrage-freeness of the market.

2. If

$$C_0 - P_0 < S_0^\ell - \frac{K}{(1+r)^N},$$

then a similar strategy (buy a call, sell a put and borrow the stock) allows arbitrage. $\square$

REMARK 4.7 (Practical interpretation of the strategy above).

- At time 0: Buy one stock, one put and sell one call. This gives in total the balance:
$$\Delta := C_0 - P_0 - S_0^\ell.$$
If $\Delta > 0$, put the sum in the cash account (at interest rate r),
if $\Delta < 0$, a loan is taken (at the same interest rate r).
- Nothing is touched until maturity N. The actual balance of the cash account at maturity is then
$$\left(C_0 - P_0 - S_0^\ell\right)(1+r)^N .$$

Two scenarios are possible:

a) $S_N^\ell > K$: The buyer of the call will exercise the option; we hand him over the stock (we own) at the price K; the put is worthless.
Then the value of the portfolio is
$$K + \left(C_0 - P_0 - S_0^\ell\right)(1+r)^N > 0.$$

b) $S_N^\ell \leq K$: The buyer of the call lost his money (the call is worthless), we sell the stock at the price K (possible by exercising the put).
Then, the value of the portfolio is
$$K + \left(C_0 - P_0 - S_0^\ell\right)(1+r)^N > 0.$$

4.3. Pricing in complete markets

DEFINITION 4.8 (Attainability, replicating potfolios). Assume an arbitrage-free market be given. A European option h (or contingent claim h) is called *attainable* if there exists a self-financing strategy ϕ such that
$$h = V_N(\phi)$$
(i.e. the value of the portfolio $V_N(\phi) = \langle \phi_N, S_N \rangle$ at maturity N matches exactly that of the option). In this case we also say that the option h can be *replicated*; the strategy ϕ is called a *replicating portfolio* or *hedging portfolio*.

REMARK 4.9. Note that for a replicating portfolio we have
$$\left.\begin{aligned} h &= V_0(\phi) + \sum_{k=1}^N \langle \phi_k, \Delta S_k \rangle \\ \tilde{h} &= V_0(\phi) + \sum_{k=1}^N \langle \phi_k, \Delta \tilde{S}_k \rangle \end{aligned}\right\} \quad \text{since } \phi \text{ is self-financing.}$$

DEFINITION 4.10 (Complete market). A financial market is called *complete* if every European option can be replicated.

NOTATION 4.11. Let $\mathscr{P}$ be the set of probability measures of full support on Ω under which $(\tilde{S}_n)_{n=0,\ldots,N}$ is a (vector-valued) martingale.

Recall that $\mathscr{P} \neq \emptyset$ since by assumption the market is arbitrage-free.

PROPOSITION 4.12. *Let h be a European option, ϕ a replicating portfolio and let $\mathbb{P}^* \in \mathscr{P}$. Then*
$$\tilde{V}_n(\phi) = \mathbb{E}^*\left[\tilde{h} | \mathscr{F}_n\right], \quad \textit{for all } n \leq N,$$

i.e.

$$V_n(\phi) = \frac{1}{(1+r)^{N-n}} \ \mathbb{E}^* \left[h|\mathscr{F}_n\right], \quad n \le N.$$

PROOF. Under $\mathbb{P}^*$, the price process $(\tilde{S}_n)$ is a martingale, and $\left(\tilde{V}_n(\phi)\right)$ is the martingale transform of $(\tilde{S}_n)$ by (ϕ_n). Thus, $\tilde{V}_n(\phi)$ is a martingale under $\mathbb{P}^*$ with terminal value $\tilde{V}_N(\phi) = \tilde{h}$. Hence,

$$\tilde{V}_n(\phi) = \mathbb{E}\left[\tilde{V}_N(\phi)|\mathscr{F}_n\right] = \mathbb{E}^*\left[\tilde{h}|\mathscr{F}_n\right]. \qquad \square$$

REMARK 4.13. In an arbitrage-free market any self-financing strategy replicating h is admissible. Indeed, we have $V_N(\phi) = h \ge 0$. For $\mathbb{P}^* \in \mathscr{P}$,

$$\begin{aligned}\tilde{V}_n(\phi) &= \mathbb{E}^*\left[\tilde{h}|\mathscr{F}_n\right] \\ &= \beta_N \, \mathbb{E}^*[h|\mathscr{F}_n] \ge 0, \quad \forall \ n \le N.\end{aligned}$$

Thus $V_n(\phi) \ge 0$ for each $n \le N$.

Conclusion. We found that

$$\tilde{V}_n(\phi) = \mathbb{E}^*[\tilde{h}|\mathscr{F}_n],$$

or equivalently,

$$V_n(\phi) = \frac{1}{(1+r)^{N-n}} \ \mathbb{E}^* \left[h|\mathscr{F}_n\right].$$

We make the following observations: A priori, neither the measure $\mathbb{P}^* \in \mathscr{P}$ is unique, nor is the replicating portfolio ϕ. On the other hand, $V_n(\phi)$ does not depend on $\mathbb{P}^*$, and the right-hand side

$$\frac{1}{(1+r)^{N-n}} \ \mathbb{E}^*[h|\mathscr{F}_n]$$

does not depend on ϕ. Hence,

$$\pi_n(h) := V_n(\phi) = \frac{1}{(1+r)^{N-n}} \mathbb{E}^*[h|\mathscr{F}_n],$$

depends only on h and n ("*price of the option h at time n*"). In particular,

$$\pi\left(h\right) := \pi_0\left(h\right) = V_0\left(\phi\right) = \frac{1}{(1+r)^{N}} \mathbb{E}^*[h].$$

PROPOSITION 4.14. *Let h be a European option (in an arbitrage-free market) which can be replicated. Then, $\pi(h)$ is the only price of h excluding arbitrage.*

PROOF. We have to show that $\pi(h)$ is a price excluding arbitrage (p.e.a.) and that it is the only price excluding arbitrage.

1) *Existence:* $\pi(h)$ is a p.e.a. Indeed, let ϕ be a portfolio replicating h and $\pi(h) = V_0(\phi)$. Then, in particular $h = V_N(\phi)$. Define $H_n := V_n(\phi)$ and choose $\mathbb{P}^* \in \mathscr{P}$. Then, both $(\tilde{S}_n)$ and $(\tilde{H}_n)$ are $\mathbb{P}^*$-martingales. In other words, $(\tilde{S}_n, \tilde{H}_n)$ is a $\mathbb{P}^*$-martingale, which means that the augmented market

$$\left(\Omega, \left(\mathscr{F}_n\right), \mathbb{P}; \bar{S}_n = \left(S_n, H_n\right)\right)$$

is arbitrage-free. In particular, H_0 is a p.e.a.

2) *Uniqueness:* Let (h_n) be an adapted sequence of non-negative random variables satisfying $h_N = h$ and let $\pi := h_0$. We must show: If $\pi \neq \pi(h)$, then there exists an arbitrage strategy in the market $\big(\Omega, (\mathscr{F}_n), \mathbb{P}; \bar{S}_n = (S_n, h_n)\big)$.
Note that now we have $\bar{S}_n = \big(S_n^0, S_n^1, \ldots, S_n^d, h_n\big)$ where $S_n^1, \ldots, S_n^d$ are the risky assets and h_n is the supplementary risky asset of price h_n at time n.

A. Suppose, for instance, $\pi > \pi(h)$.
Let ϕ be a strategy replicating h. Consider $\bar{\varphi}_n := \big(\phi_n^1, \ldots, \phi_n^d, -1\big)$ and let $\bar{\phi}_n$ be the self-financing portfolio which coincides with $\bar{\varphi}$ on the $d+1$ risky assets such that $V_0\big(\bar{\phi}\big) = 0$.
Then
$$\begin{aligned} V_n\big(\bar{\phi}\big) &= \bar{\phi}_n^0 S_n^0 + \bar{\phi}_n^1 S_n^1 + \ldots + \bar{\phi}_n^d S_n^d + \bar{\phi}_n^{d+1} h_n \\ &= \bar{\phi}_n^0 S_n^0 + \phi_n^1 S_n^1 + \ldots + \phi_n^d S_n^d - h_n, \end{aligned}$$
and hence
$$V_n\big(\bar{\phi}\big) = \big(\bar{\phi}_n^0 - \phi_n^0\big) S_n^0 + V_n(\phi) - h_n. \tag{1}$$
For $n = 0$, we have $V_0(\bar{\phi}) = 0$, and thus $\bar{\phi}_0^0 - \phi_0^0 = h_0 - V_0(\phi) = \pi - \pi(h)$.
Since $\bar{\phi}$ is self-financing, we have $\langle \bar{\phi}_n, \bar{S}_n \rangle_{\mathbb{R}^{d+2}} = \langle \bar{\phi}_{n+1}, \bar{S}_n \rangle_{\mathbb{R}^{d+2}}$, and thus
$$\begin{aligned} V_n\big(\bar{\phi}\big) &= \bar{\phi}_{n+1}^0 S_n^0 + \phi_{n+1}^1 S_n^1 + \ldots + \phi_{n+1}^d S_n^d - h_n \\ &= \big(\bar{\phi}_{n+1}^0 - \phi_{n+1}^0\big) S_n^0 + \langle \phi_{n+1}, S_n \rangle_{\mathbb{R}^{d+1}} - h_n. \end{aligned}$$
This shows
$$V_n(\bar{\phi}) = (\bar{\phi}_{n+1}^0 - \phi_{n+1}^0) S_n^0 + V_n(\phi) - h_n. \tag{2}$$
Combining (1) and (2) we find that
$$\bar{\phi}_{n+1}^0 - \phi_{n+1}^0 = \bar{\phi}_n^0 - \phi_n^0 = \ldots = \bar{\phi}_0^0 - \phi_0^0 = \pi - \pi(h).$$
Hence,
$$V_n\big(\bar{\phi}\big) = (\pi - \pi(h))\,(1+r)^n + V_n(\phi) - h_n,$$
and thus
$$V_N\big(\bar{\phi}\big) = (\pi - \pi(h))\,(1+r)^N > 0,$$
i.e. $\bar{\phi}$ is an arbitrage strategy.

B. The case $\pi < \pi(h)$ is treated analogously.

We thus conclude that indeed $\pi = \pi(h)$. □

Remark 4.15 (Interpretation of the strategy in case A). The speculator, attracted by the over-evaluation of the option h, borrows the option at time zero, and sells it immediately at the market for the price π. From this sum, he takes out the quantity $\pi(h)$ which allows him to buy a portfolio replicating h. The remaining sum $\pi - \pi(h)$ he puts in the riskless money account.

Example 4.16 (*Call-Put Parity; revisited*). In an arbitrage-free complete market consider the options
$$\left.\begin{array}{ll} C := (S_N^\ell - K)_+ & \text{European call} \\ P := (K - S_N^\ell)_+ & \text{European put} \end{array}\right\} \text{with same strike price and same maturity.}$$

Let C_n be a p.e.a. of C at time n and P_n be a p.e.a. of P at time n. Then

$$C_n = \frac{1}{(1+r)^{N-n}}\mathbb{E}^*\ [C|\mathscr{F}_n] \quad \text{and}$$
$$P_n = \frac{1}{(1+r)^{N-n}}\mathbb{E}^*\ [P|\mathscr{F}_n].$$

Hence,

$$\begin{aligned}
C_n - P_n &= \frac{1}{(1+r)^{N-n}}\mathbb{E}^*\ [C-P|\mathscr{F}_n] \\
&= \frac{1}{(1+r)^{N-n}}\mathbb{E}^*\ [S_N^\ell|\mathscr{F}_n] - \frac{1}{(1+r)^{N-n}}\mathbb{E}^*\ [K|\mathscr{F}_n] \\
&= \frac{1}{(1+r)^{N-n}}\frac{1}{\beta_n}\mathbb{E}^*\ [\tilde{S}_N^\ell|\mathscr{F}_n] - \frac{K}{(1+r)^{N-n}} \\
&= (1+r)^n\tilde{S}_n^\ell - \frac{K}{(1+r)^{N-n}} \\
&= S_n^\ell - \frac{K}{(1+r)^{N-n}}, \quad n = 0,1,\dots,N.
\end{aligned}$$

We conclude this section by summarizing the obtained results.

Summary. Given a European option h in a complete and arbitrage-free market

$$(\Omega, (\mathscr{F}_n), \mathbb{P}; (S_n))$$

where $S_n = \left(S_n^0, S_n^1, \dots, S_n^d\right)$, $n = 0,1,\dots,N$. By definition, h is a non-negative $\mathscr{F}_N$-measurable random variable.

By completeness of the market there exists a replicating portfolio ϕ for h, this is a self-financing portfolio $\phi = (\phi_n)$ such that $h = V_N(\phi)$.

By arbitrage-freeness of the market there is $\mathbb{P}^* \in \operatorname{Prob}(\Omega)$ of full support such that $(\tilde{S}_n)$ is a $\mathbb{P}^*$-martingale. Then

$$(\tilde{V}_n(\phi))_n, \quad n = 0,1,\dots,N$$

is a $\mathbb{P}^*$-martingale as well, as martingale transform of $(\tilde{S}_n)$ by (ϕ_n). Thus, in particular,

$$\tilde{V}_n(\phi) = \mathbb{E}^*[\tilde{V}_N|\mathscr{F}_n] = \mathbb{E}^*[\tilde{h}|\mathscr{F}_n], \quad \text{where } \tilde{h} = \frac{h}{S_N^0} = \frac{h}{(1+r)^N},$$

i.e.

$$V_n(\phi) = \frac{1}{(1+r)^{N-n}}\ \mathbb{E}^*\left[h|\mathscr{F}_n\right], \quad \text{and}$$
$$V_0(\phi) = \frac{1}{(1+r)^N}\ \mathbb{E}\left[h\right].$$

Note that $\pi_n(h) = V_n(\phi)$ is the only sequence of p.e.a. Indeed, let (h_n) be another sequence of p.e.a. Then

$$(\Omega, (\mathscr{F}_n), \mathbb{P}; (S_n), (h_n))$$

is arbitrage-free, and hence there exists $\mathbb{P}' \in \operatorname{Prob}(\Omega)$ with $\operatorname{supp}\mathbb{P}' = \Omega$ such that $(\tilde{S}_n)$ and $(\tilde{h}_n)$ are $\mathbb{P}'$-martingales. Thus $(\tilde{V}_n(\phi))$ is a $\mathbb{P}'$-martingale as well. Hence

$$\tilde{h}_n = \mathbb{E}'[\tilde{h}_N|\mathscr{F}_n] = \mathbb{E}'[\tilde{V}_N(\phi)|\mathscr{F}_n] = \tilde{V}_n(\phi),$$

and thus $h_n = V_n(\phi)$ for each n.

We call

$$\pi_n(h) = \frac{1}{(1+r)^{N-n}} \mathbb{E}^*[h|\mathscr{F}_n] \quad \text{“\textit{price of the option h at time n}”},$$

and

$$\pi_0(h) = \frac{1}{(1+r)^{N}} \mathbb{E}^*[h] \quad \text{“\textit{price of the option h}”}.$$

4.4. Pricing in incomplete markets

PROPOSITION 4.17 (Prices excluding arbitrage of European options in not necessarily complete markets). *Given an arbitrage-free market. Let h be a European option which is not necessarily replicable. Denote by $\Pi(h) \subset \mathbb{R}_+$ the set of prices of h which exclude arbitrage. Then*

$$\Pi(h) = \left\{\mathbb{E}^*[\tilde{h}] : \mathbb{P}^* \in \mathscr{P}\right\}$$

where $\mathscr{P} \mathrel{\widehat{=}}$ probability measures on Ω of full support under which $(\tilde{S}_n)$ is a martingale.

Notation. The set $\mathscr{P}$ is also called set of “*equivalent martingale measures*” or “*risk-neutral measures*”

PROOF.

1. Let $\pi \in \Pi(h)$. Then, by definition, there is an adapted sequence (h_n) such that
$$h_N = h, \quad h_0 = \pi,$$
and such that
$$(\Omega, (\mathscr{F}_n), \mathbb{P}; (S_n, h_n))$$
is arbitrage-free. Hence, there exists $\mathbb{P}^* \in \mathrm{Prob}(\Omega)$ with $\operatorname{supp} \mathbb{P}^* = \Omega$ such that $(\tilde{S}_n)$ and $(\tilde{h}_n)$ are $\mathbb{P}^*$-martingales. In particular, $\mathbb{P}^* \in \mathscr{P}$ and $\pi = h_0 = \mathbb{E}^*[\tilde{h}]$, i.e.,
$$\pi \in \left\{\mathbb{E}^*[\tilde{h}] : \mathbb{P}^* \in P\right\}.$$
2. Conversely, let $\pi = \mathbb{E}^*[\tilde{h}]$ where $\mathbb{P}^* \in \mathscr{P}$. We have to show that π is a p.e.a. for the option h. Let $\tilde{h}_n := \mathbb{E}^*[\tilde{h}|\mathscr{F}_n]$ and $h_n = \tilde{h}_n (1+r)^n$. Then $(\tilde{S}_n)$ and $(\tilde{h}_n)$ are both $\mathbb{P}^*$-martingales, and thus $(\Omega, (\mathscr{F}_n), \mathbb{P}; (S_n, h_n))$ is still arbitrage-free. In addition, $h_0 = \pi$ and $h_N = h$, hence $\pi \in \Pi(h)$. □

REMARK 4.18. Let h be a European option in an arbitrage-free market,

$$\Pi(h) = \left\{\mathbb{E}^*[\tilde{h}] : \mathbb{P}^* \in \mathscr{P}\right\}.$$

- If h can be replicated, then $|\Pi(h)| = 1$.
- If h cannot be replicated, then $\Pi(h) \subset R_+$ is an open interval (see Föllmer-Schied *Stochastic Finance* for a proof).

THEOREM 4.19. *An arbitrage-free market is complete if and only if there exists a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which discounted prices are martingales.*

PROOF. "$\Rightarrow$": Assume that the market is arbitrage-free and complete. Thus any European option h (i.e. non-negative $\mathscr{F}_N$-measurable random variable h) can be replicated in the sense that there is an admissible strategy ϕ such that $h = V_N(\Phi)$. Since ϕ is self-financing, we have

$$\tilde{V}_N(\phi) = V_0(\phi) + \sum_{n=1}^{N} \langle \phi_n, \Delta \tilde{S}_n \rangle.$$

Suppose that $\mathbb{P}^1$ and $\mathbb{P}^2$ are two probability measures equivalent to $\mathbb{P}$ such that $(\tilde{S}_n)_{n=0,1,\ldots,N}$ is a martingale both under $\mathbb{P}^1$ and $\mathbb{P}^2$. Then also $(\tilde{V}_n(\phi))_{n=0,1,\ldots,N}$ is a martingale both under $\mathbb{P}^1$ and $\mathbb{P}^2$. Thus

$$\mathbb{E}^1[\beta_N h] = \mathbb{E}^1[V_0(\phi)] = V_0(\phi),$$

$$\mathbb{E}^2[\beta_N h] = \mathbb{E}^2[V_0(\phi)] = V_0(\phi).$$

This implies $\beta_N \mathbb{E}^1[h] = \beta_N \mathbb{E}^2[h]$, and thus $\mathbb{E}^1[h] = \mathbb{E}^2[h]$ for any $\mathscr{F}_N$-measurable $h \geq 0$. With $h = \mathbb{1}_A$ where $A \in \mathscr{F}_N$, we get

$$\mathbb{P}^1(A) = \mathbb{P}^2(A), \quad \text{for all } A \in \mathscr{F}_N = \mathscr{F}.$$

In other words, we have $\mathbb{P}^1 = \mathbb{P}^2$.

"$\Leftarrow$": Suppose that the market is arbitrage-free and incomplete. Let $\mathbb{P}^* \in \mathscr{P}$. There exists an $\mathscr{F}_N$-measurable random variable $h \leq 0$ without replicating portfolio. Let

$$\begin{aligned} V &:= \left\{ \tilde{V}_N(\phi) \mid \phi \text{ is self-financing} \right\} \\ &= \left\{ c + \sum_{n=1}^{N} \langle \varphi_n, \Delta \tilde{S}_n \rangle \;\middle|\; \begin{array}{l} c \in \mathbb{R} \text{ and } \varphi_n = (\varphi_n^1, \ldots, \varphi_n^d)_{n=1,\ldots,N} \\ \text{a predictable sequence taking values in } \mathbb{R}^d \end{array} \right\}. \end{aligned}$$

Then $V \subset E = \{X \mid X \text{ random variable on } \Omega\} = \mathbb{R}^{|\Omega|}$. We consider on E the scalar product

$$\langle X, Y \rangle := \mathbb{E}^* [XY].$$

Note that $V \subsetneqq E$, since $\tilde{h} \in E \setminus V$.

Thus $\exists\, 0 \neq X \in E$ such that $X \perp V$ (in particular, $\mathbb{E}^*[X] = 0$). Let

$$\mathbb{P}^{**} := \frac{k + X}{k} \mathbb{P}^* = \left(1 + \frac{X}{k}\right) \mathbb{P}^*$$

where $k \in \mathbb{R}_+$ sufficiently large such that $k + X > 0$. Then $\mathbb{P}^{**}$ is a probability measure since

$$\mathbb{P}^{**}(\Omega) = \int_\Omega \left(1 + \frac{X}{k}\right) d\mathbb{P}^* = \mathbb{P}^*(\Omega) + \frac{1}{k}\mathbb{E}^*[X] = \mathbb{P}^*(\Omega) = 1.$$

Thus $\mathbb{P}^{**} \in \operatorname{Prob}(\Omega)$ and $\mathbb{P}^{**} \sim \mathbb{P}^*$, but $\mathbb{P}^{**} \neq \mathbb{P}^*$.

Let (φ_n) be an arbitrary predictable sequence taking values in $\mathbb{R}^d$. Then

$$\begin{aligned}\mathbb{E}^{**}\left[\sum_{n=1}^{N}\left\langle\varphi_n,\Delta\tilde{S}_n\right\rangle\right] &= \mathbb{E}^{*}\left[\left(1+\frac{X}{k}\right)\sum_{n=1}^{N}\left\langle\varphi_n,\Delta\tilde{S}_n\right\rangle\right] \\ &= \mathbb{E}^{*}\left[\sum_{n=1}^{N}\left\langle\varphi_n,\Delta\tilde{S}_n\right\rangle\right]+\frac{1}{k}\mathbb{E}^{*}\left[X\sum_{n=1}^{N}\left\langle\varphi_n,\Delta\tilde{S}_n\right\rangle\right]=0.\end{aligned}$$

(Note that the first term of the right-hand side vanishes since $(\tilde{S}_n)$ is a martingale under $\mathbb{P}^*$; the second vanishes since $X \perp V$). Thus $(\tilde{S}_n)$ is also a martingale under $\mathbb{P}^{**}$. Hence, both $\mathbb{P}^*$ and $\mathbb{P}^{**}$ are probability measures in $\mathscr{P}$, but $\mathbb{P}^* \neq \mathbb{P}^{**}$. $\square$

CHAPTER 5

The Cox-Ross-Rubinstein model

The Cox-Ross-Rubinstein (CRR) model is a discrete-time version of the Black-Scholes model. Only one risky asset $(S_n)_{0\leqslant n\leqslant N}$ is given in addition to the riskless asset $(S_n^0)_{0\leqslant n\leqslant N}$. More precisely, we have

$$S_n^0 = (1+r)^n, \quad r \geqslant 0,$$
$$S_{n+1} = \left\{\begin{matrix} S_n(1+a), & \text{or} \\ S_n(1+b) & \end{matrix}\right\} \quad \text{where } -1 < a < b.$$

In other words, the quotients

$$\left(\frac{S_{n+1}}{S_n}\right)_{n=0,1,\dots,N-1}$$

form a sequence of random variables taking values in $\{1+a,\, 1+b\}$.

5.1. Modeling of the Cox-Ross-Rubinstein model

Consider

$$\Omega := \{1+a,\, 1+b\}^N = \big\{\omega = (\omega_1,\dots,\omega_N) \mid \omega_i \in \{1+a,\, 1+b\}\big\}.$$

Let $\mathbb{P}$ be a probability measure on Ω such that $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$, and consider the following random variables on $(\Omega, \mathscr{F}, \mathbb{P})$:

$$\xi_n \colon \Omega \to \{1+a,\, 1+b\}, \quad \xi_n(\omega) = \omega_n.$$

Then

$$\begin{cases} S_0 = s_0 \text{ (with } s_0 > 0), \text{ and} \\ S_{n+1} = S_n\, \xi_{n+1}, \quad n = 0,1,\dots,N-1; \end{cases}$$

i.e.

$$\begin{aligned} S_0 &= s_0, \\ S_1 &= s_0\, \xi_1, \\ &\vdots \\ S_n &= S_{n-1}\, \xi_n = S_{n-2}\, \xi_{n-1}\, \xi_n = \ldots = s_0\, \xi_1\, \xi_2 \cdot \ldots \cdot \xi_n. \end{aligned}$$

The filtration is given by

$$\mathscr{F}_n = \sigma\{X_0,\dots,X_n\} = \sigma\{\xi_1,\dots,\xi_n\}$$

where $\mathscr{F}_0 = \{\emptyset, \Omega\}$.

PROPOSITION 5.1. *In the CRR model, the sequence of discounted prices*

$$(\tilde{S}_n)_{n=0,1,\dots N}$$

is a martingale under $\mathbb{P}$ *if and only if*

$$\mathbb{E}\big[\xi_{n+1} \,|\, \mathscr{F}_n\big] = 1 + r, \quad 0 \leqslant n \leqslant N-1.$$

PROOF. We have the following equivalences:

$$\begin{aligned}
\mathbb{E}\big[\tilde{S}_{n+1} \,|\, \mathscr{F}_n\big] &= \tilde{S}_n \\
&\Longleftrightarrow \frac{1}{\tilde{S}_n}\mathbb{E}\left[\tilde{S}_{n+1} \,|\, \mathscr{F}_n\right] = 1 \\
&\Longleftrightarrow \mathbb{E}\left[\frac{\tilde{S}_{n+1}}{\tilde{S}_n} \,|\, \mathscr{F}_n\right] = 1 \\
&\Longleftrightarrow \mathbb{E}\left[\frac{\beta_{n+1} S_{n+1}}{\beta_n S_n} \,|\mathscr{F}_n\right] = 1 \\
&\Longleftrightarrow \mathbb{E}\left[\frac{1}{1+r}\xi_{n+1} \,|\mathscr{F}_n\right] = 1, \quad \text{as } \frac{\beta_{n+1}}{\beta_n} = \frac{(1+r)^n}{(1+r)^{n+1}} = \frac{1}{1+r}, \\
&\Longleftrightarrow \mathbb{E}\left[\xi_{n+1} \,|\mathscr{F}_n\right] = 1 + r.
\end{aligned}$$

□

COROLLARY 5.2. *If the market is arbitrage-free, then* $r \in \,]a, b[$.

PROOF. Arbitrage-freeness implies existence of a measure $\mathbb{P}^* \in \mathrm{Prob}(\Omega)$ such that $\mathbb{P}^* \sim \mathbb{P}$ (i.e. $\operatorname{supp} \mathbb{P}^* = \Omega$) under which $(\tilde{S}_n)$ is a martingale. By Proposition 5.1, we then have for each n,

$$\mathbb{E}^*\left[\xi_{n+1} \,|\, \mathscr{F}_n\right] = 1 + r, \quad \text{and hence } \mathbb{E}\left[\xi_{n+1}\right] = 1 + r.$$

On the other hand, ξ_{n+1} takes values in $\{1+a,\, 1+b\}$ and takes each of these values with probability > 0. This implies $1 + r \in \,]1 + a,\, 1 + b[$ and hence $r \in \,]a, b[$. □

REMARK 5.3. In the case $r \notin \,]a, b[$ it is easy to construct arbitrage. Indeed, suppose that

$r \leq a$: Borrow the amount S_0 at time 0 to buy one unit of the risky asset. At time N, sell the risky asset and pay back the loan.

$$\begin{aligned}
\text{Realized profit} &= S_N - S_0(1+r)^N \\
&\geqslant S_N - S_0(1+a)^N \quad (\text{since } S_0\,(1+r)^N \leqslant S_0(1+a)^N) \\
&\geqslant 0 \quad (\text{and} > 0 \text{ with probability} > 0).
\end{aligned}$$

$r \geq b$: Analoguous argument: Short-sell the risky asset at time 0.

PROPOSITION 5.4. *Consider a CRR model with* $r \in \,]a, b[$. *A probability measure* $\mathbb{P}$ *on* Ω *is a martingale measure (in the sense that the sequence of discounted prices* $(\tilde{S}_n)$ *is a* $\mathbb{P}$*-martingale) if and only if the random variables* $\xi_1, \dots, \xi_N$ *are i.i.d. such that*

$$\mathbb{P}\left\{\xi_n = 1 + a\right\} = p, \quad \textit{where } p = \frac{b-r}{b-a}.$$

PROOF.
"$\Leftarrow$": Let $p = \frac{b-r}{b-a}$ so that

$$r = ap + b(1-p).$$

Assume that the ξ_n are independent and that

$$\mathbb{P}\{\xi_n = 1+a\} = p = 1 - \mathbb{P}\{\xi_n = 1+b\}.$$

Then, for $0 \le n \le N-1$,

$$\begin{aligned}\mathbb{E}[\xi_{n+1} \mid \mathscr{F}_n] &= \mathbb{E}[\xi_{n+1}], \quad \text{since } \xi_{n+1} \perp\!\!\!\perp \mathscr{F}_n,\\ &= (1+a)\,p + (1+b)\,(1-p)\\ &= ap + b\,(1-p) + p + (1-p)\\ &= 1+r.\end{aligned}$$

Hence $\tilde{S}_n$ is a $\mathbb{P}$-martingale.

"$\Rightarrow$": Let $\tilde{S}_n$ be a martingale under $\mathbb{P}$, thus

$$\mathbb{E}[\xi_{n+1} \mid \mathscr{F}_n] = 1+r, \quad \forall n \leqslant N-1.$$

Since

$$\xi_{n+1} = (1+a)\mathbb{1}_{\{\xi_{n+1}=1+a\}} + (1+b)\mathbb{1}_{\{\xi_{n+1}=1+b\}},$$

this implies, for each $n \leqslant N-1$,

$$(1+a)\underbrace{\mathbb{E}\left[\mathbb{1}_{\{\xi_{n+1}=1+a\}} \mid \mathscr{F}_n\right]}_{=:\,A} + (1+b)\underbrace{\mathbb{E}\left[\mathbb{1}_{\{\xi_{n+1}=1+b\}} \mid \mathscr{F}_n\right]}_{=:\,B} = 1+r.$$

In other words, we have

$$\begin{cases} aA + bB = r,\\ \quad A + B = 1,\end{cases}$$

from where we obtain

$$\begin{cases} A = \dfrac{b-r}{b-a} = p,\\ B = \dfrac{r-a}{b-a} = 1-p.\end{cases}$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{\{\xi_{n+1}=1+a\}} \mid \mathscr{F}_n\right] = p &\implies \mathbb{P}\{\xi_{n+1} = 1+a\} = p,\\ \mathbb{E}\left[\mathbb{1}_{\{\xi_{n+1}=1+b\}} \mid \mathscr{F}_n\right] = 1-p &\implies \mathbb{P}\{\xi_{n+1} = 1+b\} = 1-p.\end{aligned}$$

This implies in particular,

$$\xi_{n+1} \perp\!\!\!\perp \mathscr{F}_n.$$

(Indeed, for $F \in \mathscr{F}_n$, we have

$$\begin{aligned}\mathbb{P}\big(\{\xi_{n+1} = 1+a)\} \cap F\big) &= \mathbb{E}[\mathbb{1}_{\{\xi_{n+1}=1+a)\}}\,\mathbb{1}_F]\\ &= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\xi_{n+1}=1+a)\}}\,\mathbb{1}_F|\mathscr{F}_n]\right]\\ &= \mathbb{E}\big[\mathbb{E}[\mathbb{1}_{\{\xi_{n+1}=1+a)\}}|\mathscr{F}_n]\,\mathbb{1}_F\big]\\ &= \mathbb{E}[p\mathbb{1}_F] = p\,\mathbb{P}(F)\\ &= \mathbb{P}\big(\{\xi_{n+1} = 1+a)\}\big)\mathbb{P}(F);\end{aligned}$$

same argument applies for the event $\{\xi_{n+1} = 1+b)\}$).

By induction, we obtain, $\forall\, x_i \in \{1+a,\, 1+b\}$,

$$\mathbb{P}\left\{\xi_1 = x_1, \ldots, \xi_n = x_n\right\} = \prod_{i=1}^{n} p_i$$

where

$$p_i = \begin{cases} p & \text{if } x_i = 1+a, \\ 1-p & \text{if } x_i = 1+b. \end{cases}$$

In other words, the random variables $\xi_1, \ldots, \xi_N$ are i.i.d. and

$$\mathbb{P}\left\{\xi_n = 1+a\right\} = p = 1 - \mathbb{P}\left\{\xi_n = 1+b\right\}. \qquad \square$$

Corollary 5.5. *Given a CRR market with $r \in \,]a,b[$. Let $p := \frac{b-r}{b-a} \in \,]0,1[$ and denote by μ the probability measure on $\{1+a,\, 1+b\}$ with*

$$\mu\left(\{1+a\}\right) = p \quad \textit{and} \quad \mu\left(\{1+b\}\right) = 1-p.$$

Then

$$\mathbb{P}^* := \underbrace{\mu \otimes \ldots \otimes \mu}_{N \text{ times}}$$

is the ***only*** *martingale measure on $\Omega = \{1+a,\, 1+b\}^N$. Moreover $\mathbb{P}^*$ has full support on Ω. The market is hence* **arbitrage-free** *and* **complete**.

5.2. Pricing of puts and calls in the CRR model

Let $C = (S_N - K)_+$ and $P = (K - S_N)_+$ where K is the strike price. Denote by C_n, resp P_n, the p.e.a. at time n. Then

$$C_n = \frac{1}{(1+r)^{N-n}}\, \mathbb{E}^*\left[(S_N - K)_+ \,|\mathscr{F}_n\right],$$

$$P_n = \frac{1}{(1+r)^{N-n}}\, \mathbb{E}^*\left[(K - S_N)_+ \,|\mathscr{F}_n\right], \quad 0 \leqslant n \leqslant N.$$

In particular,

$$C_n - P_n = S_n - \frac{K}{(1+r)^{N-n}}, \quad 0 \leqslant n \leqslant N. \qquad \text{(Call-put parity)}$$

Goal: We want to write $C_n = c\,(n, S_n)$ for some explicit function c. Note that

$$C_n = \frac{1}{(1+r)^{N-n}}\, \mathbb{E}^*\left[\left(S_n \prod_{i=n+1}^{N} \xi_i - K\right)_+ \middle|\, \mathscr{F}_n\right],$$

since $S_N = S_n \prod_{i=n+1}^{N} \xi_i$.

Lemma 5.6. *Let X, Y be random variables on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $\mathscr{A} \subset \mathscr{F}$ be a sub-σ-algebra. Suppose that X is $\mathscr{A}$-measurable and $Y \perp\!\!\!\perp \mathscr{A}$. Then, for any measurable $\phi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$,*

$$\mathbb{E}\left[\phi\,(X, Y)\,|\mathscr{A}\,\right] = \varphi\,(X) \quad \textit{a.s.}$$

where $\varphi\,(x) = \mathbb{E}\left[\phi\,(x, Y)\right]$.

REMARK 5.7. We have $C_n = c(n, S_n)$ where

$$c(n,s) := \frac{1}{(1+r)^{N-n}}\ \mathbb{E}^*\left[\left(s\prod_{i=n+1}^{N}\xi_i - K\right)_+\right].$$

LEMMA 5.8. *Let $\xi_1, \dots, \xi_\ell$ be i.i.d. random variables of common law μ (where $\mu\{1+a\} = p \in \,]0,1[$ and $\mu\{1+b\} = 1-p$) and let $\psi\colon \mathbb{R}^\ell \to \mathbb{R}_+$ be measurable and symmetric, i.e. $\psi(s_1, \dots, s_\ell) = \psi(s_{\pi(1)}, \dots, s_{\pi(\ell)})$ for any permutation π of $(1, \dots, \ell)$ and $(s_1, \dots, s_\ell) \in \mathbb{R}^\ell$. Denote*

$$s^{(j)} = \underbrace{(s, s, \dots, s)}_{j\ \text{times}} \in \mathbb{R}^j, \quad s \in \mathbb{R}.$$

Then, the following formula holds:

$$\mathbb{E}\big[\psi(\xi_1, \dots, \xi_\ell)\big] = \sum_{j=0}^{\ell}\binom{\ell}{j}p^j(1-p)^{\ell-j}\,\psi\big((1+a)^{(j)},\,(1+b)^{(\ell-j)}\big).$$

PROOF. Let $I(\omega) := \#\big\{i \in \{1, \dots, \ell\} \mid \xi_i(\omega) = 1+a\ \big\}$.

i) The random variable I follows a binomial law $\mathscr{B}(\ell; p)$ with parameters ℓ and p. Indeed, we may write $I = \sum_{i=1}^{\ell} U_i$ where $U_i = 1_{\{\xi_i = 1+a\}}$. It suffices to observe that $(U_i)_{1\leqslant i\leqslant \ell}$ are independent Bernoulli variables with

$$\mathbb{P}\{U_i = 1\} = p = 1 - \mathbb{P}\{U_i = 0\}.$$

ii) Since ψ is symmetric, the random variable $\psi(\xi_1, \dots, \xi_\ell)$ is a function of I alone,

$$\psi\big(\xi_1, \dots, \xi_\ell\big) = \psi\big((1+a)^{(I)},\,(1+b)^{(\ell-I)}\big).$$

Thus,

$$\begin{aligned}\mathbb{E}\left[\psi(\xi_1, \dots, \xi_\ell)\right] &= \sum_{j=0}^{\ell}\psi\big((1+a)^{(j)},\,(1+b)^{(\ell-j)}\big)\mathbb{P}\{I = j\}\\ &= \sum_{j=0}^{\ell}\binom{\ell}{j}p^j\,(1-p)^{\ell-j}\,\psi\big((1+a)^{(j)},\,(1+b)^{(\ell-j)}\big).\end{aligned}$$ □

COROLLARY 5.9. *Applying Lemma 5.8 to the function*

$$\psi_s(t_{n+1}, t_{n+2}, \dots, t_N) = (s\,t_{n+1}\,t_{n+2}\cdot\ldots\cdot t_N - K)_+$$

gives the explicit formula:

$$C_n = c(n, S_n)$$

with

$$c(n,s) = \frac{1}{(1+r)^{N-n}}\sum_{j=0}^{N-n}\binom{N-n}{j}p^j(1-p)^{N-n-j}\Big(s(1+a)^j(1+b)^{N-n-j} - K\Big)_+.$$

5.3. Replicating portfolio for a European call

Let $\phi_n = \left(\varphi_n^0, \varphi_n\right)$ be a replicating portfolio for C. Then

$$C_n = c(n, S_n) = V_n(\phi) = \varphi_n^0 (1+r)^n + \varphi_n S_n,$$

and hence

$$c(n, \xi_n\, S_{n-1}) = \varphi_n^0 (1+r)^n + \varphi_n\, \xi_n\, S_{n-1}.$$

Note that ξ_n takes as possible values only $1+a$ and $1+b$. This leads to the following two equations

$$\begin{cases} c(n, (1+a)S_{n-1}) 1_{\{\xi_n = 1+a\}} = \left(\varphi_n^0 (1+r)^n + (1+a)\varphi_n S_{n-1}\right) 1_{\{\xi_n=1+a\}} \\ c\left(n, (1+b)\, S_{n-1}\right) 1_{\{\xi_n=1+b\}} = \left(\varphi_n^0\, (1+r)^n + (1+b)\, \varphi_n S_{n-1}\right) 1_{\{\xi_n=1+b\}}. \end{cases}$$

Under the unique martingale measure $\mathbb{P}^*$, we take conditional expectations $\mathbb{E}^*[\ldots \,|\, \mathscr{F}_{n-1}]$ (note that $\phi_n = \left(\varphi_n^0, \varphi_n\right)$ is $\mathscr{F}_{n-1}$-measurable, and that $\xi_n \perp\!\!\!\perp \mathscr{F}_{n-1}$). This gives

$$(*) \qquad \begin{cases} c\left(n, (1+a)\, S_{n-1}\right) p = \left(\varphi_n^0\, (1+r)^n + (1+a)\, \varphi_n S_{n-1}\right) p, \\ c\left(n, (1+b)\, S_{n-1}\right) (1-p) = \left(\varphi_n^0\, (1+r)^n + (1+b)\, \varphi_n S_{n-1}\right) (1-p)\,, \end{cases}$$

where $p = \mathbb{P}^* \left\{\xi_n = 1+a\right\} = 1 - \mathbb{P}^* \left\{\xi_n = 1+b\right\} \in \,]0,1[$.

It is hence enough to solve system $(*)$ for the two unknowns φ_n^0, φ_n. One obtains

$$\begin{cases} \varphi_n^0 = \dfrac{(1+b)\, c(n, (1+a)S_{n-1}) - (1+a)\, c(n, (1+b)S_{n-1})}{(b-a)(1+r)^n} =: \Delta^0(n, S_{n-1}) \\ \varphi_n = \dfrac{c(n, (1+b)\, S_{n-1}) - c(n, (1+a)\, S_{n-1})}{(b-a)\, S_{n-1}} =: \Delta(n, S_{n-1}). \end{cases}$$

Conclusion: The hedging portfolio $\phi_n = (\varphi_n^0, \varphi_n)$ is unique and given by

$$\begin{cases} \varphi_n^0 = \Delta^0(n, S_{n-1}), \\ \varphi_n = \Delta(n, S_{n-1}). \end{cases}$$

Remark 5.10. The function $s \mapsto \left(S-K\right)_+$ is increasing, hence $c(n, \cdot)$ is increasing for each n, and thus $\varphi_n \geqslant 0$ for each n. This shows that the hedging strategy does not require short-selling of the risky asset at any instant.

Remark 5.11. The calculations above immediately extend to European options of the type

$$h = H(S_N).$$

The unique price (excluding arbitrage) of h at time n then writes as

$$\mathscr{P}_n(h) = c(n, S_n)$$

where

$$c(n,s) = \frac{1}{(1+r)^{N-n}} \sum_{j=0}^{N-n} \binom{N-n}{j} p^j (1-p)^{N-n-j} H(s(1+a)^j (1+b)^{N-n-j}\Big).$$

As above, in case the function $s \mapsto H(s)$ is monotonically increasing, the unique hedging portfolio for h doesn't require short-selling at any time.

5.4. From CRR to Black-Scholes

The idea is the following: We construct a sequence of CRR markets. To this end, we fix three (strictly positive) constants: T, R, σ, and ssume that, for each $N \in \{1, 2, 3, \dots\}$, a CRR market is given by the following data:

(1) The assets are quoted at the instants

$$0,\ \frac{T}{N},\ 2\frac{T}{N},\ \dots,\ N\frac{T}{N} = T \quad (N+1 \text{ instants}).$$

(2) The interest rate r_N on each interval from $n\frac{T}{N}$ to $(n+1)\frac{T}{N}$ is

$$r_N = R\frac{T}{N}.$$

(3) Let a_N and b_N be given by

$$1 + a_N = (1 + r_N)\, e^{-\sigma/\sqrt{N}}, \quad \text{and}$$
$$1 + b_N = (1 + r_N)\, e^{\sigma/\sqrt{N}}.$$

(4) Let $\left(\xi_1^N, \dots, \xi_N^N\right)$ be N i.i.d. random variables, taking values in $\{1 + a_N,\, 1 + b_N\}$, such that for each n,

$$\mathbb{E}_N^*\left[\xi_n^N\right] = 1 + r_N.$$

Note that this condition determines the law of ξ_n^N. Indeed:

$$\mathbb{E}_N^*\left[\xi_n^N\right] = (1 + a_N)\,\mathbb{P}_N^*\left\{\xi_n^N = 1 + a_N\right\} + (1 + b_N)\,\mathbb{P}_N^*\left\{\xi_n^N = 1 + b_N\right\},$$

hence

$$p_N := \mathbb{P}_N^*\left\{\xi_n^N = 1 + a_N\right\} = \frac{b_N - r_N}{b_N - a_N}.$$

(5) The price of the risky asset is given by

$$S_n^N = S_{n-1}^N \xi_n^N = \dots = \xi_1^N \cdot \dots \cdot \xi_n^N \quad (\text{we assume that } S_0^N = 1).$$

Thus for each $N \in \{1, 2, 3, \dots\}$ a CRR market is specified. This leads to the following questions:

(i) Is there convergence as $N \to \infty$ to a "market in continuous time" with horizon T?

(ii) We observe that

$$\lim_{N\to\infty} (1 + r_N)^N = \lim_{N\to\infty} \left(1 + \frac{RT}{N}\right)^N = e^{RT}$$

(iii) What is the interpretation of σ?

LEMMA 5.12. *For each $N \geq 1$, let $(X_n^N)_{1\leqslant n\leqslant N}$ be a finite sequence of N i.i.d. random variables, each taking values in*

$$\left\{-\sigma/\sqrt{N},\ \sigma/\sqrt{N}\right\}.$$

Let $\mu_N = \mathbb{E}[X_n^N]$ and suppose that $N\mu_N \to \mu$, as $N \to \infty$. Then the sequence

$$Y_N = X_1^N + \dots + X_N^N$$

converges in distribution to $N(\mu, \sigma^2)$ (normal distribution of mean μ and variance σ^2).

PROOF. Let φ_N be the characteristic function of Y_N, i.e.

$$\varphi_N(t) = \mathbb{E}\big[e^{itY_N}\big] = \big(\mathbb{E}\big[e^{itX_1^N}\big]\big)^N.$$

Recall the *Theorem of Paul Lévy*:

$$X_n \to X \quad \text{in distribution} \quad (\text{i.e. } \mathbb{E}\left[f\left(X_n\right)\right] \to \mathbb{E}\left[f(X)\right] \text{ for any } f \in C_b(\mathbb{R}))$$
$$\iff \quad \mathbb{E}\left[e^{itX_N}\right] \to \mathbb{E}\left[e^{itX}\right], \quad \forall t \in \mathbb{R}.$$

Hence, it suffices to show

$$\lim_{N\to\infty} \varphi_N(t) = e^{it\mu - t^2\sigma^2/2}, \quad \forall t \in \mathbb{R}.$$

Note that

$$e^{itx} = 1 + itx - \frac{(tx)^2}{2} + R_t(x),$$

where $|R_t(x)| \leqslant \frac{|t|^3|x|^3}{6}$. Hence,

$$e^{itX_1^N} = 1 + itX_1^N - \frac{(tX_1^N)^2}{2} + R_t(X_1^N),$$

where $|R_t(X_1^N)| \leqslant \frac{|t\sigma|^3}{6N\sqrt{N}}$. We obtain

$$\mathbb{E}\big[e^{itX_1^N}\big] = 1 + it\mu_N - \frac{t^2}{2}\frac{\sigma^2}{N} + \mathbb{E}\left[R_t(X_1^N)\right].$$

Now let t be fixed. Then

$$\left|\mathbb{E}[R_t(X_1^N)]\right| \leqslant \mathbb{E}\left|R_t(X_1^N)\right| \leqslant \frac{|t|^3\sigma^3}{6N\sqrt{N}} = \mathrm{o}\left(\frac{1}{N}\right).$$

Recall that

$$\mu_N = \frac{\mu}{N} + \mathrm{o}\left(\frac{1}{N}\right),$$

hence

$$\mathbb{E}[e^{itX_1^N}] = 1 + it\frac{\mu}{N} - \frac{t^2}{2}\frac{\sigma^2}{N} + \mathrm{o}\left(\frac{1}{N}\right).$$

This shows that, as wanted,

$$\varphi_N(t) = \left(\mathbb{E}[e^{itX_1^N}]\right)^N = \left(1 + \frac{it\mu - t^2\sigma^2/2}{N} + \mathrm{o}\left(\frac{1}{N}\right)\right)^N \to\ e^{it\mu - t^2\sigma^2/2},$$

as $N \to \infty$. $\square$

PROPOSITION 5.13. *Let $Y_N = \log \tilde{S}_N^N$, where $S_N^N = \xi_1^N \cdot \ldots \cdot \xi_N^N$. Then Y_N converges in distribution to $N(-\sigma^2/2, \sigma^2)$, i.e. a Gaussian variable of mean $-\sigma^2/2$ and variance σ^2.*

PROOF. We have

$$\log \tilde{S}_N^N = \log \frac{\xi_1^N \cdot \ldots \cdot \xi_N^N}{(1+r_N)^N} = X_1^N + \ldots + X_N^N,$$

where

$$X_n^N = \log \frac{\xi_n^N}{1+r_N}.$$

We make the following observations:

(1) Each X_n^N takes its values in the 2-points set
$$\left\{\log\frac{1+a_N}{1+r_N}, \log\frac{1+b_N}{1+r_N}\right\} = \left\{\frac{-\sigma}{\sqrt{N}}, \frac{\sigma}{\sqrt{N}}\right\}.$$
(2) The variables $\left(X_1^N, \dots, X_N^N\right)$ are i.i.d.
(3) Let $\mu_N = \mathbb{E}_N^*[X_1^N]$. Then it suffices to show that
$$N\mu_N \to \mu, \quad \text{as } N \to \infty,$$
where $\mu = -\frac{\sigma^2}{2}$. Indeed we have:
$$\begin{aligned}
\mu_N &= \mathbb{E}_N^*\left[\log\frac{\xi_1^N}{1+r_N}\right] \\
&= \log\left(\frac{1+a_N}{1+r_N}\right)\mathbb{P}_N^*\left\{\xi_1^N = 1+a_N\right\} + \log\left(\frac{1+b_N}{1+r_N}\right)\mathbb{P}_N^*\left\{\xi_1^N = 1+b_N\right\} \\
&= \frac{-\sigma}{\sqrt{N}}\frac{b_N - r_N}{b_N - a_N} + \frac{\sigma}{\sqrt{N}}\frac{r_N - a_N}{b_N - a_N} \\
&= \frac{\sigma\left(r_N - b_N\right) + \sigma\left(r_N - a_N\right)}{\sqrt{N}\left(b_N - a_N\right)} \\
&= \frac{\sigma\left(2r_N - \left(b_N + a_N\right)\right)}{\sqrt{N}\left(b_N - a_N\right)}.
\end{aligned}$$

Now note that
$$\begin{aligned}
1 + a_N &= (1+r_N)\exp\left(\frac{-\sigma}{\sqrt{N}}\right) = (1+r_N)\left[1 - \frac{\sigma}{\sqrt{N}} + \frac{\sigma^2}{2N} + \mathrm{o}\left(\frac{1}{N}\right)\right] \\
1 + b_N &= (1+r_N)\exp\left(\frac{\sigma}{\sqrt{N}}\right) = (1+r_N)\left[1 + \frac{\sigma}{\sqrt{N}} + \frac{\sigma^2}{2N} + \mathrm{o}\left(\frac{1}{N}\right)\right].
\end{aligned}$$

Hence we have
$$b_N - a_N = \frac{2\sigma}{\sqrt{N}} + \mathrm{o}\left(\frac{1}{N}\right),$$

and
$$\begin{aligned}
2r_N - (a_N + b_N) &= 2r_N - \left((1+a_N) + (1+b_N) - 2\right) \\
&= 2r_N - (1+r_N)\left[2 + \frac{\sigma^2}{N} + \mathrm{o}\left(\frac{1}{N}\right)\right] + 2 \\
&= -\frac{\sigma^2}{N} + \mathrm{o}\left(\frac{1}{N}\right).
\end{aligned}$$

Thus,
$$\lim_{N\to\infty} N\mu_N = \lim_{N\to\infty} N\,\frac{\sigma\left(-\frac{\sigma^2}{N} + \mathrm{o}\left(\frac{1}{N}\right)\right)}{\sqrt{N}\left(\frac{2\sigma}{\sqrt{N}} + \mathrm{o}\left(\frac{1}{N}\right)\right)} = -\frac{\sigma^2}{2}.$$

The claim now follows from the lemma. □

We want to calculate the price of a call and a put in "continuous time" by passing to the limit of the corresponding formulas in the CRR model.

Let $\mathbb{P}_N^*$ be the probability measure under which

$$(\xi_1^N, \ldots, \xi_N^N) \text{ is i.i.d. and } \mathbb{P}_N^* \{\xi_n = 1 + a_N\} = \frac{b_N - r_N}{b_N - a_N}.$$

Let

$$P^{(N)} = (K - S_N^N)_+$$

the put with strike price $K > 0$ and $P_0^{(N)}$ its p.e.a. at time 0. We know that

$$\begin{aligned}
P_0^{(N)} &= \frac{1}{(1+r_N)^N} \mathbb{E}_N^* \left[\left(K - S_N^N\right)_+ \right] \\
&= \mathbb{E}_N^* \left[\left(\frac{K}{(1+r_N)^N} - \tilde{S}_N^N \right)_+ \right] \\
&= \mathbb{E}_N^* \left[\left(\frac{K}{(1+r_N)^N} - e^{Y_N} \right)_+ \right].
\end{aligned}$$

Let $\psi(y) = (Ke^{-RT} - e^y)_+$. Then we have

$$\begin{aligned}
\left| P_0^{(N)} - \mathbb{E}_N^* \left[\psi\left(Y_N\right)\right] \right| &= \left| \mathbb{E}_N^* \left[\left(\frac{K}{(1+r_N)^N} - e^{Y_N} \right)_+ - \left(Ke^{-RT} - e^{Y_N}\right)_+ \right] \right| \\
&\leqslant \mathbb{E}_N^* \left| \left(\frac{K}{(1+r_N)^N} - e^{Y_N} \right)_+ - \left(Ke^{-RT} - e^{Y_N}\right)_+ \right| \\
&\underset{|a_+ - b_+| \leqslant |a-b|}{\leqslant} K \left| \frac{1}{(1+r_N)^N} - e^{-RT} \right| \to 0, \quad \text{as } N \to \infty.
\end{aligned}$$

Hence, denoting

$$g_{\mu,\sigma^2}(t)\, dt = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2} dt \mathrel{\widehat{=}} N(\mu, \sigma^2),$$

we obtain as limiting price:

$$\begin{aligned}
\lim_{N\to\infty} P_0^{(N)} &= \lim_{N\to\infty} \mathbb{E}_N^* \left[\psi(Y_N)\right] \\
&= \int_{\mathbb{R}} \psi(t)\, g_{-\sigma^2/2,\, \sigma^2}(t)\, dt \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} \left(Ke^{-RT} - e^t\right)_+ \exp\left(- \frac{(t+\sigma^2/2)^2}{2\sigma^2} \right) dt \\
&\underset{t = -\frac{\sigma^2}{2} + \sigma y}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(Ke^{-RT} - e^{-\frac{\sigma^2}{2} + \sigma y} \right)_+ e^{-y^2/2} dy.
\end{aligned}$$

Next we drop the condition $S_0^N = 1$. We assume that $S_0^N = s_0 > 0$ is a constant independent of N. Note that this just means replacing S_N^N by $s_0 S_N^N = S_N^{N,\text{new}}$. Thus

$$P_0^{(N)} = \frac{1}{(1+r_N)^N} \mathbb{E}_N^* \left[\left(K - s_0 S_N^N\right)_+ \right] = \frac{s_0}{(1+r_N)^N} \mathbb{E}_N^* \left[\left(\frac{K}{s_0} - S_N^N \right)_+ \right],$$

and hence

$$P_0 = \lim_{N\to\infty} P_0^{(N)} = \frac{s_0}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\frac{K}{s_0} e^{-RT} - e^{-\sigma^2/2+\sigma y} \right)_+ e^{-y^2/2}\, dy$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(K\, e^{-RT} - s_0\, e^{-\sigma^2/2+\sigma y} \right)_+ e^{-y^2/2}\, dy.$$

Now let

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy,$$
$$d_1 = \frac{1}{\sigma}\left(\log \frac{s_0}{K} + RT\right) + \frac{\sigma}{2},$$
$$d_2 = \frac{1}{\sigma}\left(\log \frac{s_0}{K} + RT\right) - \frac{\sigma}{2} = d_1 - \sigma.$$

Then,

$$Ke^{-RT} - s_0 e^{-\frac{\sigma^2}{2}+\sigma y} \geqslant 0 \iff e^{-\frac{\sigma^2}{2}+\sigma y} \leqslant \frac{K}{s_0} e^{-RT}$$
$$\iff \sigma y \leqslant \log \frac{K}{s_0} - RT + \frac{\sigma^2}{2}$$
$$\iff y \leqslant -d_2,$$

and hence

$$P_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} \left(Ke^{-RT} - s_0 e^{-\frac{\sigma^2}{2}+\sigma y} \right) e^{-y^2/2} dy$$
$$= Ke^{-RT} N(-d_2) - s_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-\frac{(y-\sigma)^2}{2}} dy.$$

In other words,

$$P_0 = Ke^{-RT} N(-d_2) - s_0 N(-d_1).$$

To calculate the limiting price

$$C_0 = \lim_{N\to\infty} C_0^{(N)}$$

of the corresponding call (same strike price K and same maturity T), we use the Call-Put parity:

$$C_0^{(N)} - P_0^{(N)} = S_0 - \frac{K}{(1+r_N)^N}.$$

Passing to the limit, we obtain

$$C_0 - P_0 = S_0 - Ke^{-RT},$$

and hence

$$C_0 = Ke^{-RT} N(-d_2) - S_0 N(-d_1) + S_0 - Ke^{-RT}$$
$$= S_0\big(1 - N(-d_1)\big) - Ke^{-RT}\big(1 - N(-d_2)\big)$$
$$= S_0 N(d_1) - Ke^{-RT} N(d_2).$$

Remark 5.14. The formulas

$$\begin{cases} C_0 = S_0 N(d_1) - Ke^{-RT} N(d_2) \\ P_0 = -S_0 N(-d_1) + Ke^{-RT} N(-d_2) \end{cases}$$

are the famous "*Black-Scholes formulas*" for a European call, resp. European put.

REMARK 5.15. The only parameter not directly observable on the market is σ: for large N we have

$$\sigma^2 \approx \operatorname{var}\big(\log \tilde{S}_N^N\big).$$

The parameter σ has the interpretation of a price *volatility*.