Master in Mathematics
Probabilistic Models in Finance
University of Luxembourg

Sample Exam 2020

Evaluation of European options

Let $(\Omega, (\mathbf{S}_n), (\mathscr{F}_n), \mathbb{P})$ be a financial market which is arbitrage free and complete. We adopt the general notation of the course. In particular, set

$$\mathbf{S}_n = ((1+r)^n, S_n)$$

where r is the interest rate of the market and (S_n) denotes the price of the risky asset. The price of a European option h at time n will be denoted by $\mathcal{P}_n(h)$.
We fix a real number $k \geq 0$. A financial institution issues the European option

$$h = S_N \vee k$$

(Note that such an option allows its owner to profit from rising courses, at the same time limiting the amount of loss when prices fall below k).

1. Show the inequalities

$$\mathcal{P}_n(h) \geq S_n\,, \quad \mathcal{P}_n(h) \geq \frac{k}{(1+r)^{N-n}} \quad \forall\, n \leq N.$$

2. An investor constitutes a portfolio ϕ as follows: at time 0 he buys the option $h = S_N \vee k$, borrows the option $g := S_N \wedge k$ and doesn't modify the composition of the portfolio until time N. Show that the value of this portfolio at time n satisfies the inequality

$$V_n(\phi) \geq \left| S_n - \frac{k}{(1+r)^{N-n}} \right|.$$

   In the sequel, we consider a financial market following the Cox, Ross and Rubinstein model. Suppose that $-1 < a < r < b$.

3. Show that $\mathcal{P}_n(h) = S_n \;\forall\, n$ if and only if $k \leq S_0(1+a)^N$.

4. Show that that there exists a function $u\colon \{0,1,2,\ldots,N\} \times \mathbb{R}_+ \to \mathbb{R}\,,\ (n,s) \mapsto u(n,s)$, such that

$$\mathcal{P}_n(h) = u(n, S_n) \quad \forall\, n \in \{0,1,2,\ldots,N\}\,.$$

   Determine this function u explicitly. Show in addition that, for any fixed n, the function $s \mapsto u(n,s)$ is increasing.

5. Let $\Phi = (\phi^0, \phi)$ be an autofinancing portfolio replicating h.

   (a) Express ϕ in terms of u.
   (b) Which condition on k is necessary and sufficient for $\phi^0 = 0$?
   (c) Show that, following the strategy Φ, no short-selling of the risky asset at any moment is necessary.
   (d) Does the last remark also apply for the hedging of a European call on the risky asset of strike price K?
   (e) Show by example that replicating a sell option on the risky asset of strike price K (European put) in general requires short-selling of this asset. (We may restrict ourselves to the case $N = 1$).