Stochastic Riemannian Geometry

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This is a very preliminary draft of an introduction to *Stochastic Differential Geometry*. The first chapter develops notions of *Stochastic Analysis on Manifolds* and is based on a lecture course given at the University of Luxembourg. The text will be continued in parallel to the teaching of courses on *Stochastic Riemannian Geometry* at the University of Luxembourg.

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Preface

The purpose of these notes is to develop fundamental tools of Stochastic Analysis on differentiable manifolds, and to provide a unified and comprehensive introduction to stochastic methods in Riemannian geometry. Right from the beginning, these objectives demand to carry over classical notions from Stochastic Analysis on Euclidean space to general manifolds and to develop the necessary concepts in a coordinate-free manner.

One of the immediate obstacles of Stochastic Analysis on manifolds is related to the fact that, in general, it is not feasible to transfer processes via charts from $\mathbb{R}^n$ to curved spaces, and to deal appropriately with certain classes of manifold-valued processes in terms of local coordinates. Itô's formula for $\mathbb{R}^n$-valued semimartingales shows that concepts like Brownian motions or local martingales are not invariant under coordinate transformations.

It is an elementary observation based on Itô's formula which leads to an intrinsic notion of manifold-valued semimartingales. However it turns out that martingale theory, traditionally based on the linear concept of conditional expectations, requires on manifolds an additional geometric structure such as a linear connection in the tangent bundle.

In the situation of Riemannian manifolds there is a canonical linear connection linked to the Riemannian geometry of the manifold, the so-called Levi-Civita connection, but for various purposes it is desirable to work also with more general linear connections. We develop martingale theory on general manifolds endowed with a linear connection.

Brownian motion on a Riemannian manifold is the special case of a martingale related to the Levi-Civita connection. Brownian motions are associated to the Riemannian metric via the Laplace-Beltrami operator and generalize the class of standard $\mathbb{R}^n$-valued Brownian motions. By definition, Brownian motions are local objects in the sense that for small times their behaviour is controlled by local geometry. However, their large-scale probabilistic behaviour reflects global aspects of topology and geometry of the manifold. Brownian motion picks up global invariants of the manifold, in their behaviour as time goes to infinity, and allows to interpolate between local and global geometry.

CHAPTER 1

Stochastic Analysis on Manifolds

In this chapter we deal with the theory of continuous manifold-valued semimartingales and develop fundamental tools about diffusions, martingales and Brownian motions on differentiable manifolds.

We start with a brief review of basic concepts from differential topology, mainly to fix the notions for further reference. For more details and additional information the reader is referred to standard textbooks (e.g., [**15**] or [**45**]).

DEFINITION 1.0.1 (Topological manifold). A Hausdorff topological space M endowed with a countable basis for the topology is called *n-dimensional topological manifold*, if for every point $x \in M$ there is an open neighbourhood U of x in M and a homeomorphism $h\colon U \to U'$ onto an open subset $U' \subset \mathbb{R}^n$.

DEFINITION 1.0.2 (Chart). A homeomorphism $h\colon U \to U'$ from some open subset $U \subset M$ onto an open subset $U' \subset \mathbb{R}^n$ is called a (n-dimensional) *chart* for M. Charts are denoted by (h, U). A chart for M is said to be a *chart about* $x \in M$ if $x \in U$.

DEFINITION 1.0.3 (Transition map). Let (h, U) and (k, V) be charts for M. The homeomorphism

$$k \circ h^{-1}|h(U \cap V)\colon h(U \cap V) \to k(U \cap V)$$

is called *transition map from* (h, U) *to* (k, V).

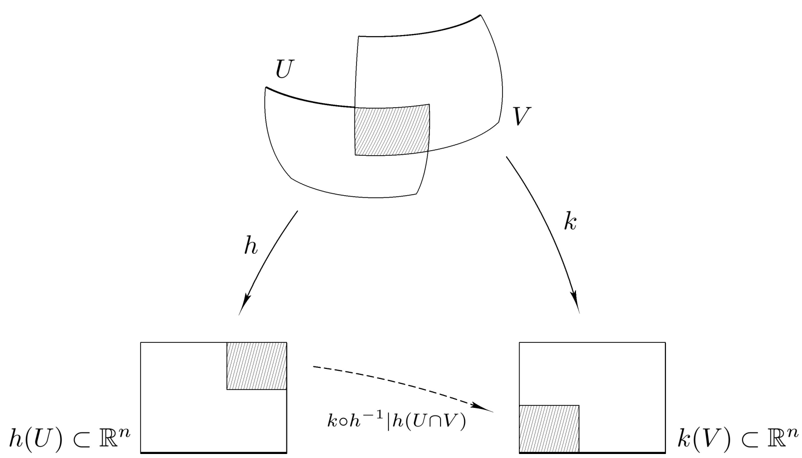


Figure 1.0.1. Transition map from (h, U) to (k, V)

DEFINITION 1.0.4 (Atlas, differentiable structure). A family $(h_i, U_i)_{i \in I}$ of n-dimensional charts for M is called *atlas for M* if the U_i cover M. An atlas is said to be *differentiable* if all its transition maps are differentiable (i.e., C^∞). A maximal differentiable atlas for M is called a *n-dimensional differentiable structure for M*.

If $\mathfrak{A}$ denotes a differentiable atlas for M and if (h,U), (k,V) are two additional n-dimensional charts with smooth transition maps to all charts of $\mathfrak{A}$, then they also change smoothly between each other. In particular,

$$\mathscr{D}(\mathcal{A}) := \big\{(h,U)\ n\text{-dimensional chart for } M \mid (h,U) \text{ changes smoothly with } \mathcal{A}\big\}$$

defines an n-dimensional differentiable structure for M.

DEFINITION 1.0.5 (Differentiable manifold). A *n-dimensional differentiable manifold* is a pair $(M,\mathscr{D})$ where M is a topological Hausdorff space with a countable basis for the topology and $\mathscr{D}$ an n-dimensional differentiable structure for M.

In the sequel we deal with *differentiable* manifolds only; the addition "differentiable" or "smooth" is mostly omitted. Furthermore, the differentiable structure $\mathscr{D}$ of a manifold $(M,\mathscr{D})$ is suppressed in the notation; one writes simply M and refers to the charts of $\mathscr{D}$ also as *charts of M*. By convention, the empty topological space is assumed to be a manifold of arbitrary (also negative) dimension; the (well-defined) dimension of non-empty manifolds is denoted $\dim M$.

EXAMPLE 1.0.6. The direct product $M\times N$ of two manifolds M and N is canonically a manifold of dimension $\dim M + \dim N$ (products of charts define a suitable atlas and the required differentiable structure).

DEFINITION 1.0.7 (Submanifold). Let M be an n-dimensional manifold and $0 \le k \le n$. A subspace $M_0 \subset M$ is said to be a *k-dimensional* (or *$(n-k)$-codimensional*) *submanifold of M*, if about every point in M_0 there exists a chart (h,U) for M such that $h(U\cap M_0) = h(U)\cap(\mathbb{R}^k\times\{0\})$. The subspace M_0 then itself is a k-dimensional manifold in the obvious way.

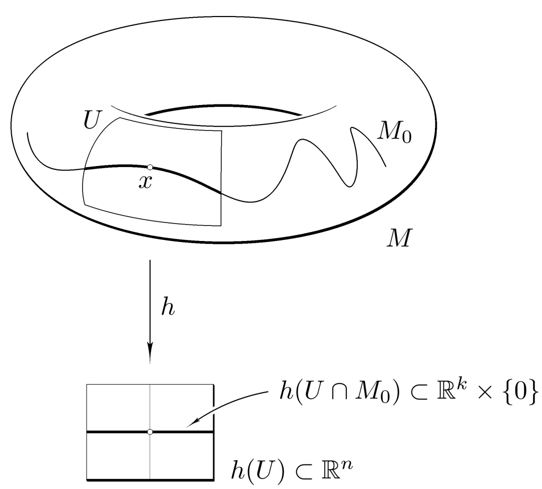


Figure 1.0.2. Submanifold of M

DEFINITION 1.0.8 (Differentiable map). Let $f\colon M \to N$ be a continuous map between manifolds and $x \in M$. The map f is said to be *differentiable at x*, if for one (and then every) chart (h,U) at x and for one (and then every) chart (k,V) at $f(x)$, the "pushed down" mapping $k\circ f\circ h^{-1}$ (defined on $h(f^{-1}(V)\cap U)$) is differentiable at $h(x)$.

Analogously, f is said to be *k-times differentiable at x* (where $k\in\mathbb{N}$), resp., *infinitely often differentiable at x*, if the same property holds true for the pushed down mapping at $h(x)$. In the case when f is k-times, resp. infinitely often differentiable at any point $x \in M$, we write $f \in C^k(M;N)$ and $f \in C^\infty(M;N)$, respectively. The expression "f is *differentiable*" or "*smooth*" always means $f \in C^\infty(M;N)$, i.e. "infinitely often differentiable".

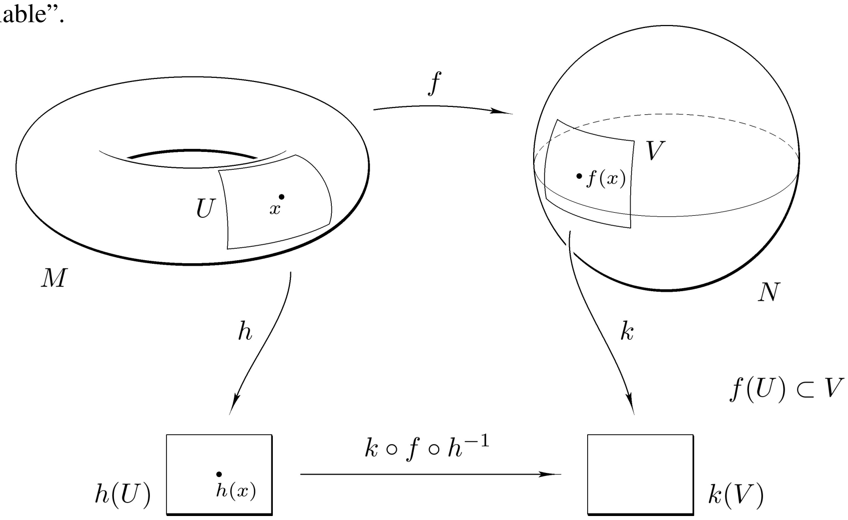


Figure 1.0.3. Maps in local coordinates

Finally, f is said to be a *diffeomorphism*, if f is bijective and both f as well as f^{-1} are differentiable.

The space of real-valued differentiable functions on M is denoted $C^\infty(M)$. Differentiable functions on M of compact support are called *test functions* on M; the space of test functions on M is denoted $C_c^\infty(M)$.

Note that the derivative of the pushed down map at $h(x)$, expressed in terms of the Jacobian $J_{h(x)}(k \circ f \circ h^{-1})$, depends on the specific choice of charts, whereas the rank of the derivative at $h(x)$, denoted $\mathrm{rank}_x f$, is independent of coordinates.

A useful fact (inverse function theorem) is that a differentiable map $f\colon M \to N$ between manifolds of equal dimension n is a local diffeomorphism at x (i.e., a diffeomorphism of an open neighbourhood of x onto some open neighbourhood of $f(x)$) if and only if $\mathrm{rank}_x f = n$.

For the construction of a chart independent version of the differential (as a canonical linearization of the differentiable map $f\colon M \to N$ locally at x) it is suitable to approximate the manifold M at x itself by a linear object, i.e. the tangent space $T_x M$.

DEFINITION 1.0.9 (Tangent space; geometric definition). Let M be a manifold and $x \in M$. Let

$$K_x M := \big\{\, \alpha\colon]{-\varepsilon}, \varepsilon[\to M \text{ differentiable} \,\big|\, \varepsilon > 0, \alpha(0) = x \big\}$$

denote the set of differentiable curves α through x. Two curves $\alpha, \beta \in K_x M$ are called *tangentially equivalent*, written $\alpha \sim \beta$, if $(h \circ \alpha)^{\cdot}(0) = (h \circ \beta)^{\cdot}(0)$ for one (and then any) chart (h, U) at x. The quotient $(T_x M)_{\mathrm{geom}} := K_x M/_\sim$ is called the (*geometric*) *tangent space of M in x*, and the classes $[\alpha] \in (T_x M)_{\mathrm{geom}}$ are called the (*geometric*) *tangent vectors of M* in the point x.

Note that by definition $\alpha \sim \alpha_h$ for $\alpha \in K_x M$ and (h, U) a chart about x, where $\alpha_h(t) := h^{-1}\big(h(x) + t\,(h \circ \alpha)^{\cdot}(0)\big)$ for t sufficiently small.

The fact that $(T_xM)_{\mathrm{geom}}$ is a finite-dimensional real vector space is not obvious from the given definition. It becomes however evident by adopting a slightly different point of view. First of all, two real differentiable functions defined locally about x are called *equivalent*, if they coincide on some neighbourhood of x. The resulting equivalence classes are called *germs* of differentiable functions at x. The set $\mathcal{E}_xM$ of these germs inherits the structure of a real algebra in a natural way. In the notation it is usually not distinguished between a germ $\varphi \in \mathcal{E}_xM$ and its representative (a differentiable function defined locally about x).

The scalar multiplication $\varphi a = \varphi(x)a$ for $\varphi \in \mathcal{E}_xM$, $a \in \mathbb{R}$, gives $\mathbb{R}$ the structure of an $\mathcal{E}_xM$-module. An $\mathbb{R}$-*derivation of* $\mathcal{E}_xM$ *in* $\mathbb{R}$ is an $\mathbb{R}$-linear map $v\colon \mathcal{E}_xM \to \mathbb{R}$ satisfying the product rule

$$v(\varphi\psi) = \varphi\, v(\psi) + \psi\, v(\varphi) \quad \text{for } \varphi, \psi \in \mathcal{E}_xM.$$

The set $\mathrm{Der}_{\mathbb{R}}(\mathcal{E}_xM, \mathbb{R})$ of $\mathbb{R}$-derivations of $\mathcal{E}_xM$ in $\mathbb{R}$ forms naturally an $\mathcal{E}_xM$-module, and in particular a real vector space.

DEFINITION 1.0.10 (Tangent space; algebraic definition). Let M be a manifold and $x \in M$. The real vector space

$$(T_xM)_{\mathrm{alg}} := \mathrm{Der}_{\mathbb{R}}(\mathcal{E}_xM, \mathbb{R})$$

is called the (*algebraic*) *tangent space of* M *at* x, and $\mathbb{R}$-derivations $v \in (T_xM)_{\mathrm{alg}}$ are called (*algebraic*) *tangential vectors of* M at the point x.

REMARK 1.0.11. Let M be manifold. For any $x \in M$ the spaces $(T_xM)_{\mathrm{geom}}$ and $(T_xM)_{\mathrm{alg}}$ are canonically identified; more precisely the following maps are inverse to each other:

$$\begin{aligned}
&(T_xM)_{\mathrm{geom}} \to (T_xM)_{\mathrm{alg}}\,, \quad [\alpha] \mapsto \big(\mathcal{E}_xM \to \mathbb{R}\,,\ \varphi \mapsto (\varphi \circ \alpha)^{\cdot}(0)\big)\,,\\
&(T_xM)_{\mathrm{alg}} \to (T_xM)_{\mathrm{geom}}\,, \quad v \mapsto \Big[\,]{-\varepsilon}, \varepsilon[\, \to M\,,\ t \mapsto h^{-1}\big(h(x) + t\, v(h)\big)\Big]
\end{aligned}$$

where (h, U) is a chart for M at x and $v(h) := \big(v(h^1), \ldots, v(h^n)\big) \in \mathbb{R}^n$.

DEFINITION 1.0.12 (Tangent space). Let M be a manifold and $x \in M$. The real vector space $T_xM := (T_xM)_{\mathrm{alg}} \equiv (T_xM)_{\mathrm{geom}}$ is called the *tangent space of* M *at* x, its elements (considered either as derivations or represented by curves) are the *tangent vectors of* M at the point x.

EXAMPLE 1.0.13. Any n-dimensional real vector space V is a n-dimensional manifold in a canonical way. Furthermore, for $x \in V$, we have $T_xV \cong V$ canonically (as real vector spaces). Indeed, if $h\colon V \xrightarrow{\sim} \mathbb{R}^n$ is an isomorphism of vector spaces (and hence a global chart), then the homomorphisms

$$\begin{aligned}
&(T_xV)_{\mathrm{alg}} \to V\,, \quad v \mapsto h^{-1}\big(v(h^1), \ldots, v(h^n)\big)\\
&V \to (T_xV)_{\mathrm{alg}}\,, \quad v \mapsto \big(\mathcal{E}_xV \to \mathbb{R}\,,\ \varphi \mapsto \tfrac{d}{dt}\varphi(x + tv)\big|_{t=0}\big)
\end{aligned}$$

are inverse to each other and independent of the particular choice of h.

DEFINITION 1.0.14 (Differential). Let $f\colon M \to N$ be a differentiable map between manifolds and $x \in M$. The *differential of f at x*

$$df_x \equiv f_{*x}\colon T_xM \to T_{f(x)}N$$

is respectively geometrically or algebraically explained as

$$(df_x)_{\mathrm{geom}}\colon (T_xM)_{\mathrm{geom}} \to (T_{f(x)}N)_{\mathrm{geom}}\,, \quad [\alpha] \mapsto [f \circ \alpha]$$

$$(df_x)_{\mathrm{alg}}\colon (T_xM)_{\mathrm{alg}} \to (T_{f(x)}N)_{\mathrm{alg}}\,, \quad v \mapsto \big(\mathcal{E}_{f(x)}N \to \mathbb{R}\,,\, \varphi \mapsto v(\varphi\circ f)\big).$$

Both mappings are well-defined and consistent with respect to the canonical identification of geometric and algebraic tangent space.

REMARK 1.0.15 (Functorality of the differential). We have $d(\mathrm{id}_M)_x = \mathrm{id}_{T_xM}$ for $x \in M$. Further, for any differentiable maps $f\colon M \to N$ and $g\colon N \to L$ between manifolds, the chain rule $d(g\circ f)_x = dg_{f(x)} \circ df_x$ holds. In particular, if f is a local diffeomorphism at x, then $df_x\colon T_xM \to T_{f(x)}N$ is a linear isomorphism.

The definitions of tangent spaces and differentials are obviously of local nature; for instance, let $U \subset M$ be open and $x \in U$, then $T_xM \cong T_xU$ in a canonical (and trivial) way, namely via $(d\iota)_x$ where $\iota\colon U \hookrightarrow M$ denotes the inclusion, and one identifies the tangent spaces T_xM and T_xU.

EXAMPLE 1.0.16 (Basis for T_xM). Let M be an n-dimensional manifold and (h,U) be a chart about $x \in M$. Then

$$dh_x\colon T_xM \xrightarrow{\sim} T_{h(x)}\mathbb{R}^n \cong \mathbb{R}^n, \quad v \mapsto \big(v(h^1),\dots,v(h^n)\big),$$

is an isomorphism of real vector spaces, in particular, $\dim_{\mathbb{R}} T_xM = n$. Thus, by means of

$$\big(\tfrac{\partial}{\partial h^i}\big)_x := (dh_x)^{-1}(e_i) = d(h^{-1})_{h(x)}(e_i), \quad i = 1,\dots,n,$$

an $\mathbb{R}$-basis for T_xM is given; note that here $\partial_{i,x} \equiv \big(\frac{\partial}{\partial h^i}\big)_x \in T_xM$ represents the derivation $\varphi \mapsto \frac{\partial}{\partial x^i}(\varphi\circ h^{-1})\big(h(x)\big)$.

THEOREM 1.0.17 (Differentials in coordinates). *Let M be an n-dimensional manifold, N an n-dimensional manifold, $f\colon M \to N$ a differentiable map and $x \in M$. Choosing charts (h,U) for M about x and (k,V) for N about $f(x)$, the following diagram commutes:*

$$\begin{tikzcd}
\big(\frac{\partial}{\partial h^i}\big)_x \arrow[d, mapsto] & T_xM \arrow[r, "(df)_x"] \arrow[d, "(dh)_x"] & T_{f(x)}N \arrow[d, "(dk)_{f(x)}"'] & \big(\frac{\partial}{\partial k^j}\big)_{f(x)} \arrow[d, mapsto] \\
e_i & \mathbb{R}^n \arrow[r, dashed, "J_{h(x)}(k\circ f\circ h^{-1})"'] & \mathbb{R}^n & e_j
\end{tikzcd}$$

where $J_{h(x)}(k\circ f\circ h^{-1}) \in \mathrm{M}(n\times n;\mathbb{R})$ is the Jacobian of $k\circ f\circ h^{-1}$ at $h(x)$.

PROOF. Any $v \in T_xM$ can be written as $v = \sum_i v^i\big(\frac{\partial}{\partial h^i}\big)_x$ where $v^i = v(h^i)$. Upon Definition 1.0.14, the differential $(df)_xv \in T_{f(x)}N$ is represented by the derivation $\varphi \mapsto v(\varphi\circ f)$, so that $(df)_xv = \sum_j v(k^j\circ f)\,\big(\frac{\partial}{\partial k^j}\big)_{f(x)}$. Thus, if $v = \sum_i v^i\big(\frac{\partial}{\partial h^i}\big)_x$, then

$$v(k^j\circ f) = \sum_i v^i\big(\tfrac{\partial}{\partial h^i}\big)_x(k^j\circ f) = \sum_i \tfrac{\partial(k^j\circ f\circ h^{-1})}{\partial x^i}\big(h(x)\big)\,v^i,$$

which shows the claim. $\square$

The examples above show that $\mathrm{rank}_x f = \mathrm{rank}(df_x)$ for a differentiable map $f\colon M \to N$ between manifolds and $x \in M$. In particular, if df_x is an isomorphism, then necessarily $\dim M = \dim N$ and f is a local diffeomorphism at x by the local inverse theorem.

DEFINITION 1.0.18 (Immersion, embedding). A map $f\colon M \to N$ between manifolds is called an *immersion*, if f is differentiable and the linear map $df_x\colon T_xM \to T_{f(x)}N$ is injective for any $x \in M$. A map $f\colon M \to N$ is called an *embedding*, if $f(M) \subset N$ is a submanifold and $f\colon M \to f(M)$ a diffeomorphism.

Any embedding is obviously an immersion. Immersions however are not injective in general; even an injective immersion is not necessarily an embedding.

DEFINITION 1.0.19 (Velocity of a curve). Let $\alpha\colon I \to M$ be a curve in M, defined on an open real interval $I =]a, b[$, and let $d\alpha_t\colon \mathbb{R} \cong T_t I \to T_{\alpha(t)} M$ be the differential of α at $t \in I$. The vector $\dot\alpha(t) := d\alpha_t(1) \in T_{\alpha(t)}M$ is called *velocity of* α *at* t; algebraically it is the derivation $\varphi \mapsto (\varphi \circ \alpha)^{\cdot}(t)$, geometrically $\dot\alpha(t)$ is represented by $s \mapsto \alpha(t+s)$. Obviously any equivalence class $[\alpha] \in (T_x M)_{\mathrm{geom}}$ can be written as $\dot\alpha(0)$.

DEFINITION 1.0.20 (Locally trivial fibration, fiber bundle). Let E, M and F be manifolds. A differentiable map $\pi\colon E \to M$ is called a *locally trivial fibration with typical fiber* F (or a *fiber bundle*), if about any point of M there exists an open neighborhood U and a diffeomorphism $\varphi\colon \pi^{-1}(U) \xrightarrow{\sim} U \times F$ above U, i.e. such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

The pair (φ, U) is said to be a *bundle chart* (or *local trivialization*) of the fibration, a family $(\varphi_i, U_i)_{i\in I}$ of bundle charts with $M = \bigcup_{i\in I} U_i$ is said to be a *bundle atlas for* E. In this situation, M is called the *basis*, E the *total space*, π the *projection*, F the *typical fiber* and $E_x \equiv \pi^{-1}(\{x\})$ the *fiber at* $x \in M$.

One commonly writes E/M or just E instead of $\pi\colon E \to M$ or $E \to M$. Furthermore, if $M_0 \subset M$, we use occasionally the notation $E/M_0 := \pi^{-1}(M_0)$.

It is an immediate consequence of the definition that each fiber E_x is a submanifold of E diffeomorphic to the typical fiber F. For any two bundle charts (φ_i, U_i), (φ_j, U_j) the composition $\varphi_j \circ \varphi_i^{-1}$ defines a diffeomorphism on $(U_i \cap U_j) \times F$; the corresponding maps $\phi_{ij}\colon U_i \cap U_j \to \mathrm{Diff}(F)$ into the group of diffeomorphisms of the typical fiber F are called *transition functions*.

A bundle atlas provides automatically a differentiable atlas for the manifold E and determines in that way the differentiable structure of E. Moreover in a canonical way, for $U \subset M$ open, $\pi|U\colon E/U \to U$ is a fiber bundle as well.

DEFINITION 1.0.21 (Trivial fiber bundle). A fiber bundle $\pi\colon E \to M$ is said to be *trivial*, if there exists a *trivialization*, i.e. a global bundle chart (φ, M).

DEFINITION 1.0.22 (Vector bundle). A locally trivial fibration $\pi\colon E \to M$ with an m-dimensional real vector space F as typical fiber is said to be an m*-dimensional vector bundle over* M, if there exists a bundle atlas $(\varphi_i, U_i)_{i\in I}$ for E such that the diffeomorphisms

$$\varphi_j \circ \varphi_i^{-1}\colon \{x\} \times F \to \{x\} \times F, \quad x \in U_i \cap U_j,$$

are linear isomorphisms of F.

Each fiber E_x then carries the structure of a real vector space such that the bundle charts above are fiberwise linear: for $x \in U$ the restriction $\varphi|E_x$ maps the fiber E_x linearly to $\{x\} \times F$. Without restriction of generality one can take $F = \mathbb{R}^n$ as the typical fiber.

DEFINITION 1.0.23 (Subbundle). Let $\pi\colon E \to M$ an m-dimensional vector bundle and $k \le m$. A subset $E_0 \subset E$ is said to be an k*-dimensional subbundle of* E, if about each point $x \in M$ there exists a fiberwise linear bundle chart $\varphi\colon E/U \to U \times \mathbb{R}^m$ for E such that $\varphi(E_0/U) = U \times \big(\mathbb{R}^k \times \{0\}\big)$. Then $\pi|E_0\colon E_0 \to M$ itself is a k-dimensional vector bundle over M.

DEFINITION 1.0.24 (Bundle homomorphism). Let E and E' be vector bundles over the same base manifold M. A differentiable map $\phi\colon E \to E'$ is called *bundle homomorphism* or *homomorphism of vector bundles*, if ϕ is a map over M and linear in each fiber, i.e., if ϕ maps each E_x to E'_x and each $\phi_x = \phi|E_x\colon E_x \to E'_x$ is given by a linear map. This constitutes the category $\mathscr{V}_M$ of vector bundles over M.

DEFINITION 1.0.25 (Section). Let $\pi\colon E \to M$ be a vector bundle. A *section of* E is a differentiable map $A\colon M \to E$ such that $\pi \circ A = \mathrm{id}_M$ (i.e. a right inverse to π). The set $\Gamma(E)$ of sections of E constitutes a $C^\infty(M)$-module in a natural way via $(\varphi A)(x) = \varphi(x)\, A(x)$, $\varphi \in C^\infty(M)$. The value of a section A at $x \in M$ is also denoted A_x instead of $A(x)$.

REMARK 1.0.26 (Local frame). Let $E \to M$ be an m-dimensional vector bundle and $x_0 \in M$. A *local frame for E at x_0* consists of an open neighbourhood U of x_0, together with sections $e_1, \dots, e_m \in \Gamma(E/U)$ such that for any $x \in U$ the family $\big(e_1(x), \dots, e_m(x)\big)$ provides an $\mathbb{R}$-basis of E_x. By means of appropriate bundle charts it is possible to construct local frames for E at any $x_0 \in M$. Then to each section $A \in \Gamma(E)$ there exist uniquely determined functions $a^i \in C^\infty(U)$ such that $A|U = \sum_{i=1}^m a^i e_i$.

When constructing fibrations one often starts with a basis M, a typical fiber F and a family $(E_x)_{x\in M}$ of manifolds E_x diffeomorphic to F. Then $E := \dot\bigcup_{x\in M} E_x$ ($\equiv \bigcup_x \{x\} \times E_x$) and $\pi\colon E \to M$, $E_x \ni e \mapsto x$ gives the total space E, at first just as a set with the corresponding projection. The still missing topology and differentiable structure on E, as well as appropriate bundle charts, are then typically provided by canonical pre-bundle charts: A *pre-bundle chart of* E is a pair (φ, U), consisting of an open subset $U \subset M$ and a fiberwise diffeomorphic bijection $\varphi\colon E/U = \dot\bigcup_{x\in U} E_x \to U \times F$ over U. A family $(\varphi_i, U_i)_{i\in I}$ of pre-bundle charts such that $\bigcup_{i\in I} U_i = M$ is called a *pre-bundle atlas for* E, if all transition maps

$$\begin{tikzcd}[column sep=small]
 & E/(U_i \cap U_j) \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\
(U_i \cap U_j) \times F \arrow[rr] & & (U_i \cap U_j) \times F
\end{tikzcd}$$

are differentiable, and thus diffeomorphisms.

LEMMA 1.0.27. *To each pre-bundle atlas for E there exists precisely one topology and differentiable structure on E which make $\pi\colon E \to M$ a locally trivial fibration with typical fiber F and the pre-bundle atlas to a bundle atlas.*

PROOF. Let $e \in E$ and $x := \pi(e) \in M$. Via a pre-bundle chart (φ, U) with $x \in U$ we have $\varphi\colon E/U \xrightarrow{\sim} U \times F$ where e is mapped to some point $(x, v) \in U \times F$. A basis of neighbourhoods at $e \in E$ for the wanted topology on E is found by pulling back via φ a basis of open sets at (x, v) in $U \times F$. The remaining claimed properties are then easily checked. $\square$

EXAMPLE 1.0.28 (Tangent bundle). Let M be an n-dimensional manifold. The tangent spaces T_pM, $p \in M$ are isomorphic to $\mathbb{R}^n$ as vector spaces (and hence as manifolds) and thus as described above they form a locally trivial fibration $TM := \dot\bigcup_{x\in M} T_xM \to M$: Each chart (h, U) for M induces a pre-bundle chart for TM via

$$\varphi_{(h,U)}\colon TM/U \to U \times \mathbb{R}^n, \quad v \mapsto \big(\pi(v), v(h^1), \dots, v(h^n)\big).$$

For any further chart (k, V) for M the transition between the pre-bundle charts is given by

$$(U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n, \quad (x, w) \mapsto \big(x, J_{h(x)}(k \circ h^{-1})w\big),$$

and hence is differentiable. Thus $TM \to M$ constitutes a fiber bundle. Moreover, since the bundle charts $\varphi_{(h,U)}$ are linear in each fiber, $TM \to M$ defines an n-dimensional vector bundle, the *tangent bundle of* M.

DEFINITION 1.0.29 (Induced fibration). Let $f\colon M \to N$ be a differentiable map between manifolds and $\pi\colon E \to N$ a locally trivial fibration with typical fiber F. Then also $f^*E := \dot{\bigcup}_{x\in M} E_{f(x)} \to M$ with the canonical projection is a locally trivial fibration with typical fiber F. To this end bundle charts (φ, U) for E provide fiberwise "induced" pre-bundle charts $\big(f^*\varphi, f^{-1}(U)\big)$ for $f^*E \equiv \big\{(x,e) \in M \times E : f(x) = \pi(e)\big\}$ via

$$f^*\varphi\colon f^*E/f^{-1}(U) \to f^{-1}(U) \times F, \quad f^*\varphi|(f^*E)_x \equiv \varphi|E_{f(x)} \text{ for } x \in f^{-1}(U).$$

These induced charts change in a differentiable way, and by Lemma 1.0.27, f^*E is a locally trivial fibration with base M, called the *fibration induced from E by f* or the *pullback fibration under f*.

EXAMPLE 1.0.30 (Induced vector bundle). Let $f\colon M \to N$ be a differentiable map between manifolds, and $E \to N$ be a vector bundle. Then $f^*E \to M$ is a vector bundle as well, the so-called *pullback of E under f*. For a bundle homomorphism $\phi\colon E \to E'$ over N there is again fiberwise a bundle homomorphism $f^*\phi\colon f^*E \to f^*E'$ over M, defined via $f^*\phi|(f^*E)_x \equiv \phi|E_{f(x)}$. This constitutes a covariant functor $f^*\colon \mathscr{V}_N \to \mathscr{V}_M$.

EXAMPLE 1.0.31. Let $f\colon M \to N$ be a differentiable map between manifolds. There is a canonical bundle homomorphism $df\colon TM \to f^*TN$ over M fiberwise explained by the differential $df_x\colon T_xM \to T_{f(x)}N$.

DEFINITION 1.0.32 (Section, vector field along a map). Let $f\colon M \to N$ be a differentiable map between manifolds, and E a vector bundle over N. The elements of the $C^\infty(M)$-module

$$\Gamma(f^*E) \equiv \{A\colon M \to E \mid A \text{ differentiable with } \pi \circ A = f\}$$

are called the *sections along f*, or in the special case of the $C^\infty(M)$-module $\Gamma(f^*TN)$, the *vector fields along f*. In particular, if $I \subset \mathbb{R}$ is an interval and $\alpha\colon I \to N$ a differentiable curve, then

$$\Gamma(\alpha^*E) \equiv \{\sigma\colon I \to E \mid \sigma \text{ differentiable with } \sigma(t) \in E_{\alpha(t)} \text{ for each } t \in I\},$$

and the vector field along α given by

$$\dot\alpha \in \Gamma(\alpha^*TN), \quad \dot\alpha_t := \dot\alpha(t),$$

is called the *tangential vector field along α*.

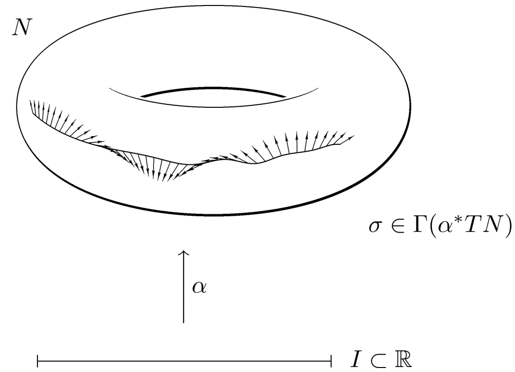


Figure 1.0.4. Vector field σ along the curve α

THEOREM 1.0.33 (Linear algebra for vector bundles). *Let $\mathscr{V}$ be the category of finite-dimensional real vector spaces and $\mathscr{V}_M$ the category of vector bundles over a manifold M. Further let*

$$\mathcal{F}\colon \mathscr{V}^{\times r}\times\mathscr{V}^{\times s}\to\mathscr{V}$$

be an r-times covariant and s-times contravariant functor which is differentiable in the sense that the maps induced by $\mathcal{F}$

$$\begin{aligned}\operatorname{Hom}(V_1,V_1')\times\cdots\times\operatorname{Hom}(V_r,V_r')\times\operatorname{Hom}(W_1',W_1)\times\cdots\times\operatorname{Hom}(W_s',W_s)\\ \to\operatorname{Hom}\big(\mathcal{F}(V_1,\ldots,V_r,W_1,\ldots,W_s),\mathcal{F}(V_1',\ldots,V_r',W_1',\ldots,W_s')\big)\end{aligned}$$

are differentiable. Then, by fiberwise application, $\mathcal{F}$ induces canonically an r-times covariant and s-times contravariant functor

$$\mathcal{F}_M\colon \mathscr{V}_M^{\times r}\times\mathscr{V}_M^{\times s}\to\mathscr{V}_M.$$

In a sloppy form Theorem 1.0.33 means the following: One decomposes vector bundle, bundle charts, resp. bundle homomorphisms, into its fiber parts, applies fiberwise the construction rule $\mathcal{F}$ in $\mathscr{V}$, and glues the result again together to new bundles, bundle charts and morphisms. The differentiability condition on $\mathcal{F}$ guarantees automatically the conditions of Lemma 1.0.27, necessary to give the still missing differentiable structures. Canonical examples for suitable functors are:

$\mathcal{F}(V_1,\ldots,V_r,W_1,\ldots,W_s)$	r	s
$V_1\oplus\cdots\oplus V_r$	r	0
$V_1\otimes\cdots\otimes V_r$	r	0
W^*	0	1
$\operatorname{Hom}(W,V)$	1	1
$\operatorname{Mult}(W_1,\ldots,W_s;V)$	1	s
$\operatorname{Bil}(W_1,W_2;\mathbb{R})$	0	2
$\operatorname{Alt}^k(W,V)$	1	1

In the case $W_1=\cdots=W_s=W$ one writes $\operatorname{Mult}(W^s;V)$ for $\operatorname{Mult}(W_1,\ldots,W_s;V)$. Usually one also writes furthermore $\mathcal{F}$ instead of $\mathcal{F}_M$, e.g. $E_1\oplus E_2$ instead of $E_1\oplus_M E_2$ for vector bundle E_1,E_2 over M.

Canonical isomorphisms in $\mathscr{V}$ carry over to canonical isomorphisms in $\mathscr{V}_M$. Typical examples are among others:

$\mathcal{F}(V_1,\ldots,V_r,W_1,\ldots,W_s)\cong$	$\mathcal{F}'(V_1,\ldots,V_r,W_1,\ldots,W_s)$
$\operatorname{Hom}(W,V)$	$W^*\otimes V$
$W_1^*\otimes W_2^*$	$(W_1\otimes W_2)^*$
$\operatorname{Bil}(W,W;\mathbb{R})$	$W^*\otimes W^*$
$\operatorname{Mult}(W_1,\ldots,W_s;V)$	$W_1^*\otimes\cdots\otimes W_s^*\otimes V$

DEFINITION 1.0.34 (Vector field). Let M be a manifold and $\pi\colon TM\to M$ the tangent bundle of M. The elements of the $C^\infty(M)$-module $\Gamma(TM)$ are called *vector fields on M*.

Vector fields can be read as derivations by means of the canonical $C^\infty(M)$-isomorphism

$$\Gamma(TM)\to\operatorname{Der}_{\mathbb{R}}C^\infty(M),\quad A\mapsto(f\mapsto Af);$$

here for $f\in C^\infty(M)$ the function $Af\colon M\to\mathbb{R}$ is explained by $Af(x):=A_x(f)$. This gives the product rule $A(fg)=f\,Ag+g\,Af$ for $f,g\in C^\infty(M)$. For an arbitrary map $A\colon M \to TM$ with $\pi \circ A = \mathrm{id}_M$ one verifies that $A \in \Gamma(TM)$ if and only if $Af \in C^\infty(M)$ for each function $f \in C^\infty(M)$.

EXAMPLE 1.0.35 (Vector fields in coordinates). Let $A \in \Gamma(TM)$ be a vector field on M and (h,U) be a chart for M. There exist uniquely determined functions $a_i \in C^\infty(U)$ such that $A|U = \sum a_i\,\partial_i$; here $\partial_i = \frac{\partial}{\partial h^i}$ denotes for $i = 1,\dots,n$ the derivation defined by

$$\big(\tfrac{\partial}{\partial h^i}\big)_x(f) = \tfrac{\partial}{\partial x^i}(f\circ h^{-1})\big(h(x)\big), \quad x \in U,$$

(see Example 1.0.16). In the special case $M = U \subset \mathbb{R}^n$, according to the canonical trivialization $TU \cong U \times \mathbb{R}^n$ (via the global chart id_U), each vector field $A \in \Gamma(TU)$ is of the form $A = (\mathrm{id}_U, a)$ where $a \in C^\infty(U;\mathbb{R}^n)$, and the map

$$C^\infty(U;\mathbb{R}^n) \xrightarrow{\sim} \Gamma(TU), \quad a \mapsto (\mathrm{id}_U, a)$$

is a $C^\infty(U)$-isomorphism. In this situation the canonical vector fields to the constant maps $(x \mapsto e_i) \in C^\infty(U;\mathbb{R}^n)$ are denoted by D_i (or D if $n=1$); as derivations the D_i operate via $D_i f = \frac{\partial}{\partial x^i} f$ for $f \in C^\infty(U)$ (and for $n=1$ again by $Df = \frac{d}{dx} f$).

DEFINITION 1.0.36 (Cotangent bundle, differential form). Let M be a manifold. The vector bundle $T^*M \equiv (TM)^*$ over M is called the *cotangent bundle of* M; the elements of $A^1(M) := \Gamma(T^*M)$ are denoted *differential forms on* M.

For $f \in C^\infty(M)$ let $df \in A^1(M)$ be the differential form defined by

$$(df)_x \equiv T_x f \in T_x^* M.$$

Given $\alpha \in A^1(M)$ and (h,U) a chart for M, there are unique functions $\alpha_i \in C^\infty(U)$ such that $\alpha|U = \sum \alpha_i\, dh^i$. Note that $dh^i\big(\frac{\partial}{\partial h^j}\big) = \delta_{ij}$ for $i,j = 1,\dots,n$.

REMARK 1.0.37 (Integral curve). Vector fields can be integrated to integral curves. Let $A \in \Gamma(TM)$ be a vector field on M and $x \in M$. A differentiable curve $\varphi\colon I \to M$ (where $I \subset \mathbb{R}$ is an open interval about 0) is said to be an *integral curve to the vector field A with starting point x*, if

$$\varphi(0) = x \quad \text{and} \quad \dot\varphi(t) = A\big(\varphi(t)\big) \quad \text{for } t \in I.$$

DEFINITION 1.0.38 (Local flow). A *local flow* on a manifold M is a differentiable map $\phi\colon D \to M$, where $D \subset \mathbb{R} \times M$ is an open neighbourhood of $\{0\} \times M$ and each $I_x := \{t \in \mathbb{R} : (t,x) \in D\}$ an interval, such that the following two conditions are satisfied:

(i) $\phi(0,x) = x$
(ii) $\phi\big(s, \phi(t,x)\big) = \phi(s+t, x)$ whenever the left-hand side is explained.

For any $x \in M$ the curve $\varphi_x\colon I_x \to M$, $t \mapsto \phi(t,x)$ is called *flow line with starting point* x. (As a consequence of condition (ii) along with the fact that D is open, flow lines are automatically maximal).

REMARK 1.0.39. Via reduction to the *Existence and Uniqueness Theorem* for solutions of first order ordinary differential equations, we conclude that to any vector field A on a manifold M there exists a local flow ϕ on M whose flow lines coincide with the maximal integral curves to A, i.e. such that for $\varphi_x(t) = \phi(t,x)$ the following flow equation holds:

$$\dot\varphi_x(t) = A\big(\varphi_x(t)\big), \quad \varphi_x(0) = x. \tag{1.0.1}$$

1.1. Stochastic Flows

In the same way as a vector field on a differentiable manifold induces a flow, second order differential operators induce stochastic flows with similar properties. In this sense, Brownian motion on a Riemannian manifold M appears as the stochastic flow associated to the canonical Laplacian on M, the so-called Laplace-Beltrami operator. The new feature of stochastic flows is that the flow curves depend on a random parameter and behave irregularly as functions of time [**29**]. This irregularity reveals an irreversibility of time which is inherent to stochastic phenomena.

Let M be a differentiable manifold of dimension n and denote by

$$TM \xrightarrow{\pi} M$$

its *tangent bundle*. In particular, from a set-theoretical point of view, we have

$$TM = \dot{\cup}_{x\in M} T_xM, \quad \pi|T_xM = x.$$

The space of smooth sections of TM is denoted by

$$\begin{aligned}\Gamma(TM) &= \{A\colon M \to TM \text{ smooth} \mid \pi\circ A = \mathrm{id}_M\}\\ &= \{A\colon M\to TM \text{ smooth} \mid A(x)\in T_xM \text{ for all } x\in M\}\end{aligned}$$

and constitutes the *vector fields* on M. As usual, we identify vector fields on M and $\mathbb{R}$-derivations on $C^\infty(M)$ as follows:

$$\Gamma(TM) \mathrel{\widehat{=}} \big\{A\colon C^\infty(M)\to C^\infty(M)\ \mathbb{R}\text{-linear} \mid A(fg) = fA(g)+gA(f)\ \forall f,g\in C^\infty(M)\big\}$$

where a vector field $A\in\Gamma(TM)$ is considered as $\mathbb{R}$-derivation via

$$A(f)(x) := df_x\,A(x)\in\mathbb{R}, \quad x\in M, \tag{1.1.1}$$

using the *differential* $df_x\colon T_xM\to\mathbb{R}$ of f at x.

There is a dynamical point of view to vector fields on manifolds: it associates to each vector field a dynamical system given by the flow of the vector field.

1.1.1. Flow of a vector field. Given a vector field $A\in\Gamma(TM)$. For each $x\in M$ we consider the smooth curve $t\mapsto x(t)$ in M with the properties

$$x(0) = x \ \text{ and } \ \dot{x}(t) = A(x(t)).$$

We write $\phi_t(x) := x(t)$. In this way, for $A\in\Gamma(TM)$, the *flow* to A is given by

$$\begin{cases}\frac{d}{dt}\phi_t = A(\phi_t),\\ \phi_0 = \mathrm{id}_M\,.\end{cases} \tag{1.1.2}$$

System (1.1.2) is understood in the sense that for any $f\in C_c^\infty(M)$ (space of compactly supported smooth functions on M) the following conditions hold:

$$\begin{cases}\frac{d}{dt}(f\circ\phi_t) = A(f)\circ\phi_t,\\ f\circ\phi_0 = f.\end{cases} \tag{1.1.3}$$

Indeed, by the chain rule along with definition (1.1.1), we have for each $f\in C_c^\infty(M)$,

$$\frac{d}{dt}(f\circ\phi_t) = (df)_{\phi_t}\,\frac{d}{dt}\phi_t = (df)_{\phi_t}\,A(\phi_t) = A(f)(\phi_t).$$

In integrated form, for each $f\in C_c^\infty(M)$, the conditions (1.1.3) write as:

$$f\circ\phi_t(x) - f(x) - \int_0^t A(f)(\phi_s(x))\,ds = 0, \quad t\geq 0,\ x\in M. \tag{1.1.4}$$

As usual, the curve

$$\phi_\bullet(x)\colon\ t \mapsto \phi_t(x)$$

is called *flow curve* (or *integral curve*) to A starting at x.

REMARK 1.1.1. Defining $P_t f := f \circ \phi_t$, we observe that $\frac{d}{dt} P_t f = P_t(A(f))$, in particular

$$\frac{d}{dt}\Big|_{t=0} P_t f = A(f). \tag{1.1.5}$$

In other words, from the knowledge of the flow ϕ_t, the underlying vector field A can be recovered by taking the derivative at zero as in Eq. (1.1.5).

1.1.2. Flow to a second order differential operator. Now let L be a second order partial differential operator (PDO) on M, e.g. of the form

$$L = A_0 + \sum_{i=1}^{r} A_i^2, \tag{1.1.6}$$

where $A_0, A_1, \dots, A_r \in \Gamma(TM)$ for some $r \in \mathbb{N}$. Note that $A_i^2 = A_i \circ A_i$ is understood as composition of derivations, i.e.

$$A_i^2(f) = A_i(A_i(f)), \quad f \in C^\infty(M).$$

EXAMPLE 1.1.2. Let $M = \mathbb{R}^n$ and consider

$$A_0 = 0 \text{ and } A_i = \frac{\partial}{\partial x_i} \text{ for } i = 1, \dots, n.$$

Then $L = \Delta$ is the classical Laplace operator on $\mathbb{R}^n$.

Alternatively, we may consider partial differentiable operators L on M which locally in a chart (h, U) can be written as

$$L|U = \sum_{i=1}^{n} b_i \partial_i + \sum_{i,j=1}^{n} a_{ij} \partial_i \partial_j, \tag{1.1.7}$$

where $b \in C^\infty(U, \mathbb{R}^n)$ and $a \in C^\infty(U, \mathbb{R}^n \otimes \mathbb{R}^n)$ such that $a_{ij} = a_{ji}$ for all i, j (a symmetric). Here we use the notation $\partial_i = \frac{\partial}{\partial h_i}$.

Motivated by the example of a flow to a vector field (vector fields can be seen as first order differential operators) we want to investigate the question whether an analogous concept of flow exists for second order PDOs.

QUESTION. Is there a notion of a flow to L if L is a second order PDO given by (1.1.6) or (1.1.7)?

DEFINITION 1.1.3. Let $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \ge 0})$ be a filtered probability space, i.e. a probability space equipped with increasing sequence of sub-σ-algebras $\mathscr{F}_t$ of $\mathscr{F}$. An adapted continuous process

$$X_\bullet(x) \mathrel{\widehat{=}} (X_t(x))_{t \ge 0}$$

on $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t \ge 0})$ taking values in M, is called *flow process* to L (or *L-diffusion*) with starting point x if $X_0(x) = x$ and if, for all test functions $f \in C_c^\infty(M)$, the process

$$N_t^f(x) := f(X_t(x)) - f(x) - \int_0^t (Lf)(X_s(x))\, ds, \quad t \ge 0, \tag{1.1.8}$$

is a martingale, i.e.

$$\mathbb{E}^{\mathscr{F}_s}\underbrace{\left[f(X_t(x)) - f(X_s(x)) - \int_s^t (Lf)(X_r(x))\,dr\right]}_{= N_t^f(x) - N_s^f(x)} = 0, \quad \text{for all } s \leq t.$$

Note that, by definition, flow processes to a second order PDO depend on an additional random parameter $\omega \in \Omega$. For each $t \geq 0$, $X_t(x) \equiv (X_t(x,\omega))_{\omega\in\Omega}$ is an $\mathscr{F}_t$-measurable random variable. The defining equation (1.1.4) for flow curves translates to the martingale property of (1.1.8), i.e. the flow curve condition (1.1.4) only holds under conditional expectations. The theory of martingales gives a rigorous meaning to the idea of a process without systematic drift [**46**].

Flow processes will be constructed as solutions to certain stochastic differential equations on M, which degenerate to the flow equation (1.0.1) in the particular case of vector fields. The second order part of the differential operator causes the "flow lines" to depend now on random in an intriguing way. The paths of flow processes are still continuous, but are in general nowhere differentiable anymore.

REMARK 1.1.4. Since $N_0^f(x) = 0$, we get from the martingale property of $N^f(x)$ that

$$\mathbb{E}\big[N_t^f(x)\big] = \mathbb{E}\big[N_0^f(x)\big] = 0.$$

Hence, defining $P_t f(x) := \mathbb{E}\left[f(X_t(x))\right]$, we observe that

$$P_t f(x) = f(x) + \int_0^t \mathbb{E}\left[(Lf)(X_s(x))\right]\,ds,$$

and thus

$$\frac{d}{dt}P_t f(x) = \mathbb{E}\left[(Lf)(X_t(x))\right] = P_t(Lf)(x),$$

in particular

$$\frac{d}{dt}\Big|_{t=0}\mathbb{E}\left[f(X_t(x))\right] \equiv \frac{d}{dt}\Big|_{t=0} P_t f(x) = Lf(x).$$

The last formula shows that as for deterministic flows we can recover the operator L from its stochastic flow process. To this end however, we have to average over all possible trajectories starting from x.

For background on stochastic flows we refer to the monograph of Kunita [**29**].

EXAMPLE 1.1.5 (Brownian motion). Let $M = \mathbb{R}^n$ and $L = \frac{1}{2}\Delta$ where Δ is the Laplacian on $\mathbb{R}^n$. Let $X \equiv (X_t)$ be a Brownian motion on $\mathbb{R}^n$ starting at the origin. By Itô's formula [**38**], for $f \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} d(f\circ X_t) &= \sum_{i=1}^n \partial_i f(X_t)\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^n \partial_i\partial_j f(X_t)\,dX_t^i dX_t^j \\ &= \langle(\nabla f)(X_t), dX_t\rangle + \frac{1}{2}(\Delta f)(X_t)\,dt. \end{aligned}$$

Thus, for each $f \in C_c^\infty(\mathbb{R}^n)$,

$$f(X_t) - f(X_0) - \int_0^t \frac{1}{2}(\Delta f)(X_s)\,ds, \quad t \geq 0,$$

is a martingale. This means that the process

$$X_t(x) := x + X_t$$

is an L-diffusion to $\frac{1}{2}\Delta$ in the sense of Definition 1.1.3.

REMARKS 1.1.6. As for deterministic flows, we have to deal with the problem that stochastic flows may explode in finite times.

1. We allow $X_\bullet(x)$ to be defined only up to some stopping time $\zeta(x)$, i.e.

$$X_\bullet(x)|[0, \zeta(x)[$$

where

$$\{\zeta(x) < \infty\} \subset \left\{ \lim_{t \uparrow \zeta(x)} X_t(\omega) = \infty \text{ in } \hat{M} := M \,\dot{\cup}\, \{\infty\} \right\} \quad \mathbb{P}\text{-a.s.} \tag{1.1.9}$$

Here $\hat{M}$ denotes the one-point compactification of M. A stopping time $\zeta(x)$ with property (1.1.9) is called (maximal) *lifetime* for the process $X_\bullet(x)$ starting at x. In equivalent terms, let $U_n \subset M$ be open, relatively compact subsets exhausting M in the sense that

$$U_n \subset \bar{U}_n \subset U_{n+1} \subset \ldots, \quad \bar{U}_n \text{ compact, and } \cup_n U_n = M.$$

Then we have $\zeta(x) = \sup_n \tau_n(x)$ for the maximal lifetime of $X_\bullet(x)$ where $\tau_n(x)$ is the family of stopping times (*first exit times* of U_n) defined by

$$\tau_n(x) := \inf\{t \geq 0\colon\ X_t(x) \notin U_n\}.$$

2. For $f \in C^\infty(M)$ (not necessarily compactly supported), the process $N^f(x)$ will in general only be a *local* martingale [**38**], i.e. there exist stopping times $\tau_n \uparrow \zeta(x)$ such that

$$\forall n \in \mathbb{N}, \quad \big(N^f_{t\wedge\tau_n}(x)\big)_{t\geq 0} \text{ is a (true) martingale.}$$

3. The following two statements are equivalent (the proof will be given later):
   (a) The process
   $$f(X_\bullet(x)) = (f(X_t(x)))_{t\geq 0}$$
   is of locally bounded variation for all $f \in C_c^\infty(M)$.
   (b) The operator L is of first order, i.e. L is a vector field (in which case the flow is deterministic).

In other words, flow processes have "nice paths" (for instance, paths of bounded variation) if and only if the corresponding operator is first order (i.e. a vector field).

1.1.3. What are L-diffusions good for? Before discussing the problem of how to construct L-diffusions, we want to study some implications to indicate the usefulness and power of this concept. In the following two examples we only assume existence of an L-diffusion to a given operator L.

A. (*Dirichlet problem*) Let $\varnothing \neq D \subsetneq M$ be an open, connected, relatively compact domain, $\varphi \in C(\partial D)$ and let L be a second order PDO on M. The *Dirichlet problem* (DP) is the problem to find a function $u \in C(\bar{D}) \cap C^2(D)$ such that

$$\begin{cases} Lu = 0 \text{ on } D \\ u|_{\partial D} = \varphi. \end{cases} \tag{DP}$$

Suppose that there is an L-diffusion $(X_t(x))_{t\geq 0}$. We choose a sequence of open domains $D_n \uparrow D$ such that $\bar{D}_n \subset D$, and for each n, we consider the *first exit time* of D_n,

$$\tau_n(x) = \inf\{t \geq 0,\ X_t(x) \notin D_n\}.$$

Then $\tau_n(x) \uparrow \tau(x)$ where

$$\tau(x) = \sup_n \tau_n(x) = \inf\{t \geq 0,\ X_t(x) \notin D\}.$$

Now assume that u is a solution to (DP). We may choose test functions $u_n \in C_c^\infty(M)$ such that $u_n|D_n = u|D_n$ and $\operatorname{supp} u_n \subset D$. Then, by the property of an L-diffusion,

$$N_t(x) := u_n(X_t(x)) - u_n(x) - \int_0^t (Lu_n)(X_r(x))\, dr$$

is a martingale. We suppose that $x \in D_n$. Then

$$\begin{aligned} N_{t\wedge\tau_n(x)}(x) &= u_n(X_{t\wedge\tau_n(x)}(x)) - u_n(x) - \int_0^{t\wedge\tau_n(x)} \underbrace{(Lu_n)(X_r(x))}_{=0}\, dr \\ &= u(X_{t\wedge\tau_n(x)}(x)) - u(x) \end{aligned} \tag{1.1.10}$$

is also a martingale (here we used that the integral in (1.1.10) is zero since $Lu_n = Lu = 0$ on D_n). Thus we get

$$\mathbb{E}\left[N_{t\wedge\tau_n(x)}(x)\right] = \mathbb{E}\left[N_0(x)\right] = 0$$

which shows that for each $n \in \mathbb{N}$,

$$u(x) = \mathbb{E}\left[u(X_{t\wedge\tau_n(x)}(x))\right]. \tag{1.1.11}$$

From Eq. (1.1.11) we may conclude by dominated convergence and since $\tau_n(x) \uparrow \tau$ that

$$u(x) = \lim_{n\to\infty} \mathbb{E}\left[u(X_{t\wedge\tau_n(x)}(x))\right] = \mathbb{E}\left[\lim_{n\to\infty} u(X_{t\wedge\tau_n(x)}(x))\right] = \mathbb{E}\left[u(X_{t\wedge\tau(x)}(x))\right].$$

We now make the *hypothesis* that $\tau(x) < \infty$ a.s. (the process exits the domain D in finite time). Then

$$\begin{aligned} u(x) &= \lim_{t\to\infty} \mathbb{E}\left[u(X_{t\wedge\tau(x)}(x))\right] = \mathbb{E}\left[\lim_{t\to\infty} u(X_{t\wedge\tau(x)}(x))\right] \\ &= \mathbb{E}\left[u(X_{\tau(x)}(x))\right] = \mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] \end{aligned}$$

where for the last equality we used the boundary condition $u|\partial D = \varphi$. Note that by passing to the image measure $\mu_x := \mathbb{P} \circ X_{\tau(x)}(x)^{-1}$ on the boundary we get

$$\mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] = \int_{\partial D} \varphi(z)\, \mu_x(dz).$$

NOTATION 1.1.7. The measure μ_x, defined on Borel sets $A \subset \partial D$,

$$\mu_x(A) = \mathbb{P}\left\{X_{\tau(x)}(x) \in A\right\},$$

is called *exit measure* from the domain D of the diffusion $X_t(x)$. It represents the probability that the process X_t, when started at x in D, exits the domain D through the boundary set A.

Conclusions. From the discussion of the Dirichlet problem above we can make the following two observations.

(a) (Uniqueness) Under the hypothesis

$$\boxed{\tau(x) < \infty \text{ a.s. for all } x \in D}$$

we have uniqueness of the solutions to the Dirichlet problem (DP). It will be shown later that this hypothesis concerns non-degeneracy of the operator L.

(b) (Existence) Under the hypothesis

$$\boxed{\tau(x) \to 0 \ \text{ if } D \ni x \to a \in \partial D}$$

we have

$$\mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right] \to \varphi(a), \quad \text{ if } D \ni x \to a \in \partial D.$$

Thus one may define $u(x) := \mathbb{E}\left[\varphi(X_{\tau(x)}(x))\right]$. It can be shown then that u is L-harmonic on D if it is twice differentiable; thus under the hypothesis in (b), u will then satisfy the boundary condition and hence solve (DP). The hypothesis in (b) is obviously a regularity condition on the boundary ∂D.

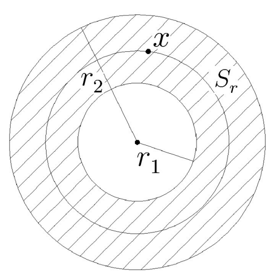
Note that in the arguments above we nowhere used the explicit form of the operator L nor of the domain D. We only used the general properties of a stochastic flow process associated to the given operator L. For a more complete discussion of the Dirichlet problem see [**43, 2**].

EXAMPLES 1.1.8.

(1) Let $M = \mathbb{R}^2 \backslash \{0\}$ and $D = \left\{x \in \mathbb{R}^2 : r_1 < |x| < r_2\right\}$ with $0 < r_1 < r_2$. Consider the operator

$$L = \frac{1}{2}\,\frac{\partial^2}{\partial \vartheta^2}$$

where ϑ denotes the angle when passing to polar coordinates on M. If u is a solution of (DP), then $u + v(r)$ is a solution of (DP) as well, for any radial function $v(r)$ satisfying $v(r_1) = v(r_2) = 0$. Hence, uniqueness of solutions fails.



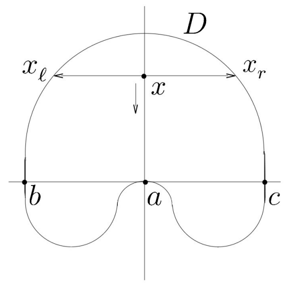
Note: For $x \in D$ with $|x| = r$, let $S_r = \left\{x \in \mathbb{R}^2 : |x| = r\right\}$. Then, the flow process $X_\bullet(x)$ to L is easily seen to be a (one-dimensional) Brownian motion on S_r. In particular,

$$\tau(x) = +\infty \text{ a.s.}$$

(2) Let $M = \mathbb{R}^2$ and consider the operator

$$L = \frac{1}{2}\,\frac{\partial^2}{\partial x_1^2}$$

on a domain D in $\mathbb{R}^2$ of the following shape:



Then, for $x = (x_1, x_2) \in D$, the flow process $X_\bullet(x)$ starting at x is a (one-dimensional) Brownian motion on $\mathbb{R} \times \{x_2\}$. In other words, flow processes move on horizontal lines. In particular, when started at $x \in D$, the process can only exit at two points (e.g. x_ℓ and x_r in the picture). Letting x vertically approach a, by symmetry of the one-dimensional Brownian motion, we see that there exists a solution of (DP) if and only if

$$\varphi(a) = \frac{\varphi(b) + \varphi(c)}{2}.$$

B. (*Heat equation*) Let L be a second order PDO on M and fix $f \in C(M)$. The *heat equation* on M with initial condition f concerns the problem of finding a real-valued function $u = u(t, x)$ defined on $\mathbb{R}_+ \times M$ such that

$$\text{(HE)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = Lu & \text{on }]0, \infty[\times M, \\ u|_{t=0} = f. \end{cases}$$

Suppose now that there is an L-diffusion $X_\bullet(x)$. It is straightforward to see that the "time-space process" $(t, X_t(x))$ will then be a $\hat{L}$-diffusion for the parabolic operator

$$\hat{L} = \frac{\partial}{\partial t} + L$$

with starting point $(0, x)$. By definition, this means that for all $\varphi \in C^2(\mathbb{R}_+ \times M)$,

$$d\varphi(t, X_t(x)) - \big(\hat{L}\varphi\big)(t, X_t(x))\, dt \overset{\mathrm{m}}{=} 0$$

where $\overset{\mathrm{m}}{=}$ denotes equality modulo differentials of local martingales.

From now on we assume non-explosion of the L-diffusion. In other words, we adopt the *hypothesis* that $\zeta(x) = +\infty$ a.s. for all $x \in M$, i.e.

$$\boxed{\mathbb{P}\big\{X_t(x) \in M,\ \forall t \geq 0\big\} = 1, \quad \forall x \in M.}$$

Suppose now that u is a *bounded* solution of (HE). We fix $t \geq 0$ and consider the restriction $u|[0, t] \times M$. Then

$$u(t-s, X_s(x)) - u(t, x) - \int_0^s \left[\left(\frac{\partial}{\partial r} + L\right) u(t-r, \cdot)\right](X_r(x))\, dr, \quad 0 \leq s < t,$$

is a local martingale. In other words, fixing $t > 0$, we have for $0 \leq s < t$,

$$\text{(1.1.12)} \qquad \begin{aligned} u(t-s, X_s(x)) = u(t, x) + \int_0^s \underbrace{\left(\frac{\partial}{\partial r} + L\right) u(t-r, \cdot)}_{=0,\ \text{since } u \text{ solves (HE)}}(X_r(x))dr \\ + (\text{local martingale})_s. \end{aligned}$$

Since the integral in (1.1.12) vanishes, we see that the local martingale term in (1.1.12) is actually a bounded local martingale (since $u(t-s, X_s(x)) - u(t,x)$ is bounded) and hence a true martingale (equal to zero at time 0). Using the martingale property we first take expectations and then pass to the limit as $s \uparrow t$ to obtain

$$u(t,x) = \mathbb{E}\left[u(t-s, X_s(x))\right] \to \mathbb{E}\left[u(0, X_t(x))\right] = \mathbb{E}\left[f(X_t(x))\right], \quad \text{as } s \uparrow t, \tag{1.1.13}$$

where for the limit in (1.1.13) we used dominated convergence (recall that u is bounded).

Conclusion. Under the hypothesis $\zeta(x) = +\infty$ for all $x \in M$, we have uniqueness of bounded solutions to the heat equation (HE). Solutions are necessarily of the form

$$\boxed{u(t,x) = \mathbb{E}\left[f(X_t(x))\right]}$$

Interpretation. The solution $u(t,x)$ at time t and at point x can be constructed as follows: run an L-diffusion process starting from x up time t, apply the initial condition f to the obtained random position $X_t(x)$ at time t and average over all possible paths.

REMARK 1.1.9. If we drop the hypothesis of infinite lifetime $\zeta(x) = +\infty$ for all $x \in M$, then uniqueness of bounded solutions to the heat equation can no longer be expected. There exists always a minimal solution u to the heat equation (HE) in the sense that $u(t,x) \to 0$ as $x \to \infty$ in the one-point compactification $\hat{M} = M \cup \{\infty\}$ of M. Let $\sigma_n \uparrow \zeta(x)$ be an increasing sequence of stopping times. Then the argument above shows

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[u(t - t\wedge\sigma_n, X_{t\wedge\sigma_n}(x))\right] \\ &= \mathbb{E}\left[\lim_{n\to\infty} u(t - t\wedge\sigma_n, X_{t\wedge\sigma_n}(x))\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{t<\zeta(x)\}} u(0, X_t(x))\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{t<\zeta(x)\}} f(X_t(x))\right]. \end{aligned}$$

This gives for the minimal solution the representation

$$\boxed{u(t,x) = \mathbb{E}\left[\mathbb{1}_{\{t<\zeta(x)\}} f(X_t(x))\right]}$$

1.1.4. Γ-operators and quadratic variation.

DEFINITION 1.1.10. Let $L\colon C^\infty(M) \to C^\infty(M)$ be a linear mapping (for instance a second order PDO). The Γ-*operator associated to* L (“*l’operateur carré du champ*”) is the bilinear map

$$\Gamma\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M) \text{ given as}$$

$$\boxed{\Gamma(f,g) := \frac{1}{2}\Big(L(fg) - fL(g) - gL(f)\Big)}$$

EXAMPLE 1.1.11. Let L be a second order PDO on M without constant term (i.e. $L1 = 0$). Suppose that in a local chart (h, U) for M the operator L writes as

$$L|C_U^\infty(M) = \sum_{i,j=1}^n a_{ij}\, \partial_i\partial_j + \sum_{i=1}^n b_i\, \partial_i$$

where $C_U^\infty(M) = \{f \in C^\infty(M) : \operatorname{supp} f \subset U\}$ and $\partial_i = \frac{\partial}{\partial h_i}$. Then

$$\Gamma(f,g) = \sum_{i,j=1}^n a_{ij}(\partial_i f)(\partial_j g), \quad \forall f, g \in C_U^\infty(M).$$

For instance, in the special case that $M = \mathbb{R}^n$ and $L = \Delta$, we find

$$\Gamma(f,f) = |\nabla f|^2.$$

REMARK 1.1.12. Let L be a second order PDO. Then the following equivalence holds:

$$\Gamma(f,g) = 0 \ \ \forall f,g \in C^\infty(M) \text{ if and only if } L \text{ is of first order, i.e. } L \in \Gamma(TM).$$

For instance, if $L = A_0 + \sum_{i=1}^r A_i^2$, then

$$\Gamma(f,g) = \sum_{i=1}^r A_i(f)A_i(g),$$

and in particular

$$\Gamma \equiv 0 \quad \text{if and only if} \quad A_1 = A_2 = \ldots = A_r = 0.$$

REMARK 1.1.13. A continuous real-valued stochastic process $(X_t)_{t\geq 0}$ is called a *semimartingale* if it can be decomposed as

$$X_t = X_0 + M_t + A_t \tag{1.1.14}$$

where M is a local martingale and A an adapted process of locally bounded variation (with $M_0 = A_0 = 0$). The representation of a semimartingale X as in (1.1.14) (Doob-Meyer decomposition) is unique: if $\mathscr{M}_0$ denotes the class of local martingales starting from 0 and $\mathscr{A}_0$ is the class of adapted process with paths of locally bounded variation starting from 0, then $\mathscr{M}_0 \cap \mathscr{A}_0 = 0$.

DEFINITION 1.1.14. Let X be a continuous adapted process taking values in a manifold M. Then X is called *semimartingale on M* if

$$f(X) \equiv (f(X_t))_{t\geq 0}$$

is a real-valued semimartingale for all $f \in C^\infty(M)$.

REMARK 1.1.15 (Semimartingale with lifetime). As already noted, semimartingales are often defined only up to some predictable stopping time $\xi > 0$. By a transformation of time, if required, infinite lifetime can always be achieved. For instance, let X be semimartingale defined on $[0,\xi[$ and let $(\tau_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite stopping times such that $\tau_0 = 0$, $\tau_n < \xi$ and $\tau_n \uparrow \xi$, then

$$\tau_{n+r} := \left(\tau_n + \tfrac{r}{1-r}\right) \wedge \tau_{n+1}, \quad 0 \leq r < 1,$$

defines a continuous time-change $(\tau_t)_{t\geq 0}$ with $\tau_0 = 0$ and $\tau_\infty = \xi$, and the time-changed process $\hat{X}$: $\hat{X}_t := X_{\tau_t}$ is a semimartingale (with respect to the time-changed filtration) of infinite lifetime.

Obviously the semimartingale property is a local property.

REMARK 1.1.16. Let ξ be a predictable stopping time and X be an M-valued process defined on $[0,\xi[$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of finite stopping times such that $\tau_0 = 0$, $\tau_n \leq \tau_{n+1}$ for $n \in \mathbb{N}$ and $\sup_n \tau_n = \xi$. The following conditions are equivalent:

(i) X is an M-valued semimartingale.

(ii) For any $n \in \mathbb{N}$ the stopped process X^{τ_n} is a semimartingale.

(iii) For any $n \in \mathbb{N}$ the restriction $X|[\tau_n, \tau_{n+1}[$ is a semimartingale, i.e., the process $(Y_t^n)_{t\in\mathbb{R}_+}$ with $Y_t^n := X_{(\tau_n+t)\wedge\tau_{n+1}}$ is a semimartingale with respect to the filtration $(\mathscr{F}_t^n)_{t\in\mathbb{R}_+}$ shifted by τ_n, i.e. $\mathscr{F}_t^n := \mathscr{F}_{\tau_n+t}$.

REMARK 1.1.17. If X has maximal lifetime ζ, i.e.,

$$\{\zeta < \infty\} \subset \left\{\lim_{t\uparrow\zeta} X_t = \infty \text{ in } \hat{M} = M \dot{\cup} \{\infty\}\right\} \text{ a.s.},$$

then $f(X)$ is well-defined as a process globally on $\mathbb{R}_+$ for all $f \in C_c^\infty(M)$ (with the convention $f(\infty) = 0$). For $f \in C^\infty(M)$, in general,

$$f(X) \equiv (f(X_t))_{t<\zeta}$$

is only a semimartingale with lifetime ζ.

PROPOSITION 1.1.18. *Let $L\colon C^\infty(M) \to C^\infty(M)$ be an $\mathbb{R}$-linear map and X be a semimartingale on M such that for all $f \in C^\infty(M)$,*

$$N_t^f := f(X_t) - f(X_0) - \int_0^t Lf(X_r)\, dr$$

is a continuous local martingale (of same lifetime as X) (i.e. $d(f(X)) - Lf(X)\,dt \overset{\mathrm{m}}{=} 0$ where $\overset{\mathrm{m}}{=}$ denotes equality modulo differentials of local martingales). Then, for all $f, g \in C^\infty(M)$, the quadratic variation $[f(X), g(X)]$ of $f(X)$ and $g(X)$ is given by

$$d\,[f(X), g(X)] \equiv d\big[N^f, N^g\big] = 2\,\Gamma(f,g)(X)\,dt.$$

In particular, $\Gamma(f,f)(X) \geq 0$ *a.s.*

PROOF. Let $f \in C^\infty(M, \mathbb{R}^r)$ and $\phi \in C^\infty(\mathbb{R}^r)$. Writing as above $\overset{\mathrm{m}}{=}$ for equality modulo differentials of local martingales, we have

$$d(\phi \circ f)(X) \overset{\mathrm{m}}{=} L(\phi \circ f)(X)\,dt. \tag{1.1.15}$$

Developing the left-hand side in Eq. (1.1.15) by Itô's formula, the function ϕ being applied to the semimartingale $f(X)$, we get

$$\begin{aligned}
&d\big(\phi(f(X))\big)\\
&\quad= \sum_{i=1}^r (D_i\phi)\big(f(X)\big)\, d\big(f^i(X)\big) + \frac{1}{2}\sum_{i,j=1}^r (D_iD_j\phi)\big(f(X)\big)\, d[f^i(X), f^j(X)]\\
&\quad\overset{\mathrm{m}}{=} \sum_{i=1}^r (D_i\phi)\big(f(X)\big)\, (Lf^i)(X)\,dt + \frac{1}{2}\sum_{i,j=1}^r (D_iD_j\phi)\big(f(X)\big)\, d[f^i(X), f^j(X)]
\end{aligned}$$

where $D_i = \partial/\partial x_i$. By equating the drift parts we find

$$\left(L(\phi\circ f) - \sum_{i=1}^r ((D_i\phi)\circ f)\,(Lf^i)\right)(X)\,dt = \tfrac{1}{2}\sum_{i,j=1}^r (D_iD_j\phi)\big(f(X)\big)\, d[f^i(X), f^j(X)].$$

Taking now $r = 2$ and considering the special case $\phi(x,y) = xy$, we get with $f = (f^1, f^2)$,

$$\big(L(f^1f^2) - f^1L(f^2) - f^2L(f^1)\big)\,(X)\,dt = d\left[f^1(X), f^2(X)\right].$$

This completes the proof since $\big(L(f^1f^2) - f^1L(f^2) - f^2L(f^1)\big)\,(X) = 2\Gamma(f^1,f^2)(X)$. ∎

LEMMA 1.1.19. *For an $\mathbb{R}$-linear map $L\colon C^\infty(M) \to C^\infty(M)$ the following statements are equivalent:*

(i) *L is a second order* PDO *(without constant term)*

(ii) *L satisfies the second order chain rule, i.e. for all $f \in C^\infty(M,\mathbb{R}^r)$ and $\phi \in C^\infty(\mathbb{R}^r)$,*

$$L(\phi\circ f)=\sum_{i=1}^r (D_i\phi\circ f)(Lf^i)+\sum_{i,j=1}^r (D_iD_j\phi\circ f)\,\Gamma(f^i,f^j).$$

PROOF. (i) $\Rightarrow$ (ii): Write L in local coordinates as

$$L|C_U^\infty(M)=\sum_{i,j=1}^n a_{ij}\,\partial_i\partial_j+\sum_{i=1}^n b_i\,\partial_i$$

and use that $\Gamma(f,g)=\sum_{i,j=1}^n a_{ij}\,\partial_i f\partial_j g$.

(ii) $\Rightarrow$ (i): Determine the action of L on functions φ written in local coordinates (h,U) via

$$L(\varphi)|U=L(\varphi\circ h^{-1}\circ h)\equiv L(\phi\circ f)$$

where $\phi=\varphi\circ h^{-1}$ and $f=h$. Details are left as an exercise to the reader. $\square$

COROLLARY 1.1.20. *Let $L\colon C^\infty(M)\to C^\infty(M)$ be an $\mathbb{R}$-linear mapping. Suppose that for each $x\in M$ there exists a semimartingale X on M such that $X_0=x$ and such that for each $f\in C^\infty(M)$,*

$$f(X_t)-f(x)-\int_0^t Lf(X)\,dr$$

is a local martingale. Then L is necessarily a PDO *of order at most* 2.

In addition, X has "nice" trajectories (e.g. in the sense that $[f(X),f(X)]=0$ for all $f\in C^\infty(M)$) if and only if L is first order.

PROOF. As in the proof of Proposition 1.1.18, for all $f\in C^\infty(M,\mathbb{R}^r)$ and $\phi\in C^\infty(\mathbb{R}^r)$, we have

$$\Big(L(\phi\circ f)-\sum_{i=1}^r (D_i\phi\circ f)(Lf^i)+\sum_{i,j=1}^r (D_iD_j\phi\circ f)\,\Gamma(f^i,f^j)\Big)(X)=0,$$

so that L is a second order PDO by Lemma 1.1.19. The second claim uses

$$d[f(X),g(X)]=2\,\Gamma(f,g)(X)\,dt,\quad f,g\in C^\infty(M).$$ $\square$

1.2. Construction of Stochastic Flows

Flows to vector fields are classically constructed as solutions of ordinary differential equations on manifolds. In the same way, stochastic flows can be constructed as solutions to stochastic differential equations (SDE) on manifolds. We start by recalling same basic facts about stochastic differential equations on $\mathbb{R}^n$.

1.2.1. Stochastic differential equations on Euclidean space.

EXAMPLE 1.2.1 (SDE on $\mathbb{R}^n$). Given $\beta\colon\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ and in addition a function

$$\sigma\colon\mathbb{R}_+\times\mathbb{R}^n\to\mathrm{Hom}(\mathbb{R}^r,\mathbb{R}^n)\equiv\mathrm{Matr}(n\times r;\mathbb{R}).$$

Let B be a Brownian motion on $\mathbb{R}^r$. Now one wants to find a continuous semimartingale Y on $\mathbb{R}^n$ such that

$$\boxed{dY_t=\beta(t,Y_t)\,dt+\sigma(t,Y_t)dB_t}$$

in the sense of Itô, i.e.

$$Y_t = Y_0 + \int_0^t \beta(s, Y_s)\, ds + \int_0^t \sigma(s, Y_s)\, dB_s. \tag{1.2.1}$$

In Eq. (1.2.1) the first term describes the "systematic part" (*drift term*) in the evolution of Y, whereas the second integral represents the "fluctuating part" (*diffusion term*).

DEFINITION 1.2.2. An $\mathbb{R}^n$-valued stochastic process $(Y_t)_{t\geq 0}$ is called *Itô process* if it has a representation as

$$Y_t = Y_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dB_s$$

where

- Y_0 is $\mathscr{F}_0$-measurable;
- K_s and H_s are adapted processes taking values in $\mathbb{R}^n$, resp. $\mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^n)$;
- $\mathbb{E}\big[\int_0^t |K_s|\, ds\big] < \infty$ and $\mathbb{E}\big[\int_0^t H_s^2\, ds\big] < \infty$ for each $t \geq 0$.

PROPOSITION 1.2.3. *Let* $\beta\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^n)$ *be continuous functions. For a continuous semimartingale Y on $\mathbb{R}^n$, defined up to some predictable stopping time τ (i.e. there exists a sequence of stopping times $\tau_n < \tau$ with $\tau_n \uparrow \tau$), the following conditions are equivalent:*

(a) *Y is a solution of the* SDE

$$dY_t = \beta(t, Y_t)\, dt + \sigma(t, Y_t)\, dB_t \quad \textit{on } [0, \tau[\tag{1.2.2}$$

i.e.,

$$Y_t = Y_0 + \int_0^t \beta(s, Y_s)\, ds + \int_0^t \sigma(s, Y_s)\, dB_s, \quad \forall 0 \leq t < \tau \textit{ a.s.}$$

(b) *For all $f \in C^\infty(\mathbb{R}^n)$,*

$$d\big(f(Y)\big) = (Lf)(t, Y)\, dt + \sum_{k=1}^n \sum_{i=1}^r \sigma_{ki}(t, Y)\, D_k f(Y)\, dB^i \quad \textit{on } [0, \tau[$$

where

$$L = \sum_{k=1}^n \beta_k\, D_k + \frac{1}{2} \sum_{k,\ell=1}^n (\sigma\sigma^*)_{k\ell} D_k D_\ell,$$

where σ^* is the transpose of σ, and $(\sigma\sigma^*)_{k\ell} = \sum_{i=1}^r \sigma_{ki}\sigma_{\ell i}$. In particular, every solution of (1.2.2) *is an L-diffusion on $[0, \tau[$ in the sense that*

$$d\big(f(Y)\big) - Lf(t, Y)\, dt = d(\textit{local martingale}) \textit{ on } [0, \tau[.$$

PROOF. (a) $\Rightarrow$ (b) Let Y be a solution of SDE (1.2.2). Then

$$dY^k dY^\ell \equiv d[Y^k, Y^\ell] = (\sigma\sigma^*)_{k\ell}(t, Y)\, dt$$

where $[Y^k, Y^\ell]$ represents the quadratic covariation of Y^k and Y^ℓ. By Itô's formula we get

$$\begin{aligned} d\big(f(Y)\big) &= \sum_{k=1}^n D_k f(Y) \Big(\beta_k(t, Y)\, dt + \sum_{i=1}^r \sigma_{ki}(t, Y)\, dB^i\Big) \\ &\quad + \frac{1}{2} \sum_{k,\ell=1}^n D_k D_\ell f(Y)\, \underbrace{(\sigma\sigma^*)_{k\ell}(t, Y)\, dt}_{= d[Y^k, Y^\ell]} \end{aligned}$$

$$= Lf(t,Y)\,dt + \sum_{k=1}^{n}\sum_{i=1}^{r}\sigma_{ki}(t,Y)\,D_k f(t,Y)\,dB^i$$
$$= Lf(t,Y)\,dt + d(\text{local martingale}).$$

(b) $\Rightarrow$ (a) Take $f(x) = x_\ell$. Then $D_k f = \delta_{k\ell}$ and $Lf = \beta_\ell$, thus

$$dY^\ell = \beta_\ell(t,Y)\,dt + \sum_{i=1}^{r}\sigma_{\ell i}(t,Y)dB^i \quad \text{for each } \ell = 1,\dots,n.$$

This shows that Y solves SDE (1.2.2) on $[0,\tau[$. $\square$

PROPOSITION 1.2.4 (Itô SDE on $\mathbb{R}^n$; case of global Lipschitz conditions). *Let Z be a continuous semimartingale on $\mathbb{R}^r$ and*

$$\alpha\colon \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^r,\mathbb{R}^n)\ (= \mathrm{Matr}(n\times r;\mathbb{R}))$$

such that

$$\exists L>0,\ \ |\alpha(y)-\alpha(z)| \le L|y-z|\ \ \forall y,z\in\mathbb{R}^n \quad (\textit{global Lipschitz conditions}).$$

Then, for each $\mathscr{F}_0$-measurable $\mathbb{R}^n$-valued random variable x_0, there exists a unique continuous semimartingale $(X_t)_{t\ge 0}$ on $\mathbb{R}^n$ such that

(1.2.3) $$dX = \alpha(X)\,dZ \ \ \textit{and}\ \ X_0 = x_0.$$

Uniqueness holds in the following sense: suppose that Y is another continuous semimartingale such that $dY = \alpha(Y)\,dZ$ and $Y_0 = x_0$, then $X_t = Y_t$ for all t a.s.

PROOF. The proof is standard in Stochastic Analysis, see for instance [**37**] or [**22**]. $\square$

PROPOSITION 1.2.5 (Itô SDEs on $\mathbb{R}^n$: case of the *local Lipschitz coefficients*). *Let Z be a continuous semimartingale on $\mathbb{R}^r$ and let*

$$\alpha : \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^r,\mathbb{R}^n),$$

be locally Lipschitz, i.e. for each compact $K\subset\mathbb{R}^n$ there exists a constant $L_K>0$ such that

$$\forall y,z\in K,\quad |\alpha(y)-\alpha(z)| \le L_K|y-z|.$$

Then, for any x_0 $\mathscr{F}_0$-measurable, there exists a unique maximal solution $X|[0,\zeta[$ of the SDE

$$dX = \alpha(X)\,dZ,\quad X_0 = x_0.$$

Uniqueness holds in the sense that if $Y|[0,\xi[$ is another solution and $y_0 = x_0$, then $\xi\le\zeta$ a.s. and $X|[0,\xi[= Y$.

PROOF. The proof is reduced to Proposition 1.2.4 by a standard truncation method. We briefly sketch the argument, since it will be used several times in the sequel. Let $B(0,R) = \{x\in\mathbb{R}^n\colon |x|\le R\}$ where $R = 1,2,\dots$ and choose test functions $\phi_R\in C_c^\infty(\mathbb{R}^n)$ such that $\phi_R|B(0,R)\equiv 1$. For $R>0$ consider the "truncated SDE"

(1.2.4) $$dX^R = \alpha^R(X^R)\,dZ,\quad X_0^R = x_0,$$

where $\alpha^R := \phi_R\,\alpha$ is now global Lipschitz. By Proposition 1.2.4 there is a unique solution X^R to (1.2.4). Then

$$X|[0,\tau_R[\ := X^R|[0,\tau_R[$$

is well-defined by uniqueness, where

$$\tau_R = \inf\{t\ge 0 : X_t^R \notin B(0,R)\}.$$

This finally defines X on the stochastic interval $[0, \zeta[$ where $\zeta = \sup_R \tau_R$. Uniqueness of X is deduced from the uniqueness of $X|[0, \tau_R[$. □

EXAMPLE 1.2.6. Consider the following Itô SDE on $\mathbb{R}^n$:

$$dX = \underbrace{\beta(X)}_{n\times 1}\, dt + \underbrace{\sigma(X)}_{n\times r}\, \underbrace{dB}_{r\times 1} \tag{1.2.5}$$

where B is Brownian motion on $\mathbb{R}^r$. Then the space-time process $Z_t = (t, B_t)$ is a semimartingale on $\mathbb{R}^{r+1}$ and SDE (1.2.5) can be written as

$$dX = \begin{pmatrix}\beta(X)\\ \sigma(X)\end{pmatrix}\begin{pmatrix}dt\\ dB\end{pmatrix} = \alpha(X)\, dZ$$

where $\alpha(X) := \binom{\beta(X)}{\sigma(X)}$. Thus, under a local Lipschitz condition on the coefficients β and σ, the SDE

$$dX = \beta(X)\, dt + \sigma(X)\, dB \tag{1.2.6}$$

has a unique strong solution for every given initial condition x_0. By Proposition 1.2.3, maximal solutions of Eq. (1.2.6) are L-diffusions to the operator

$$L = \sum_{i=1}^n \beta_i \partial_i + \frac{1}{2} \sum_{i,j=1}^n (\sigma\sigma^*)_{ij} \partial_i \partial_j,$$

where $\partial_i = \partial/\partial x_i$ is the derivative in direction i.

DEFINITION 1.2.7 (PDO in Hörmander form). For a vector field $A \in \Gamma(TM)$ on M (read as a derivation) let $A^2(f) := A(A(f))$, $f \in C^\infty(M)$. A map $L\colon C^\infty(M) \to C^\infty(M)$ is called a *partial differential operator* (PDO) *in Hörmander form*, if there exist vector fields $A_0, A_1, \ldots, A_r$ on M such that L can be written as

$$L = A_0 + \sum_{i=1}^r A_i^2.$$

In the special case $M = \mathbb{R}^n$ and $A_i := D_i = \frac{\partial}{\partial x^i}$ $(i = 1, \ldots, n)$ for instance, $\Delta = \sum_{i=1}^n A_i^2$ is the Euclidean Laplacian.

1.2.2. Stratonovich differentials.

DEFINITION 1.2.8. For continuous real-valued semimartingales X and Y let

$$X \circ dY := X dY + \frac{1}{2} d[X, Y]$$

be the *Stratonovich differential*. Here $X dY$ is the usual Itô differential and $d[X, Y] = dX dY$ the differential of the quadratic covariation of X and Y. The integral

$$\int_0^t X \circ dY = \int_0^t X\, dY + \frac{1}{2}[X, Y]_t \tag{1.2.7}$$

is called *Stratonovich integral* of X with respect to Y.

Formula (1.2.7) gives the relation between the Stratonovich integral and the usual Itô integral. Since Stratonovich integrals can always be converted back to Itô integrals, their use in our context will be only formal and for the sake of convenient notations.

REMARK 1.2.9. We have the following properties of Stratonovich differential, respectively Stratonovich integrals.

1. (*Associativity*) $X \circ (Y \circ dZ) = (XY) \circ dZ$, i.e.,

$$X \circ d\left(\int_0^\bullet Y \circ dZ\right) = (XY) \circ dZ.$$

Indeed, we have

$$\begin{aligned} X \circ (Y \circ dZ) &= X \circ d\left(\int_0^\bullet Y \circ dZ\right) \\ &= X\, d\left(\int_0^\bullet Y \circ dZ\right) + \frac{1}{2} dX\, d\left(\int_0^\bullet Y \circ dZ\right) \\ &= X(Y\,dZ) + \frac{1}{2}\, X\, dY\, dZ + \frac{1}{2} dX\left(Y\,dZ + \frac{1}{2}\, dY\,dZ\right) \\ &= (XY)dZ + \frac{1}{2}\,(X\,dY + Y\,dX + dX\,dY)dZ \\ &= (XY)dZ + \frac{1}{2}\, d(XY)dZ \\ &= (XY) \circ dZ. \end{aligned}$$

2. (*Product rule*) $d(XY) = X \circ dY + Y \circ dX$

PROOF. By Itô's formula we have

$$d(XY) = X\,dY + Y\,dX + dX\,dY = X \circ dY + Y \circ dX. \qquad \square$$

PROPOSITION 1.2.10 (Itô-Stratonovich formula). *Let X be a continuous $\mathbb{R}^n$-valued semimartingale and $f \in C^3(\mathbb{R}^n)$. Then*

$$d\big(f(X)\big) = \sum_{i=1}^n (D_i f)(X) \circ dX^i \equiv \langle \nabla f(X), \circ\, dX \rangle. \tag{1.2.8}$$

PROOF. By Itô's formula, we have

$$d(D_i f(X)) = \sum_{k=1}^n (D_i D_k f)(X)\, dX^k + \tfrac{1}{2} \sum_{k,\ell=1}^n (D_i D_k D_\ell f)(X)\, dX^k dX^\ell.$$

Hence we get

$$\begin{aligned} \sum_{i=1}^n (D_i f)(X) \circ dX^i &= \sum_{i=1}^n (D_i f)(X)\, dX^i + \tfrac{1}{2} \sum_{i=1}^n d(D_i f(X)) dX^i \\ &= \sum_{i=1}^n (D_i f)(X)\, dX^i + \tfrac{1}{2} \sum_{i,k=1}^n (D_i D_k f(X))\, dX^k dX^i \\ &= d\big(f(X)\big). \end{aligned} \qquad \square$$

Formula (1.2.8) shows the main advantage of the Stratonovich differential: it converts Itô's formula into the usual chain rule of classical analysis. Hence, at least formally, classical differential calculus can be applied in calculations involving Stratonovich differentials.

PROPOSITION 1.2.11. *Let $\beta\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $\sigma\colon \mathbb{R}_+ \times \mathbb{R}^n \to$ $\mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^n)$ be C^1. Furthermore, let B be a Brownian motion on $\mathbb{R}^r$. For a semimartingale Y on $\mathbb{R}^n$ (defined up to some predictable stopping time τ) the following conditions are equivalent:*

(i) *The semimartingale Y is a solution of the Stratonovich* SDE

$$dY = \beta(t,Y)\,dt + \sigma(t,Y)\circ dB, \tag{1.2.9}$$

i.e.

$$Y_t = Y_0 + \int_0^t \beta(s,Y_s)\,ds + \int_0^t \sigma(s,Y_s)\circ dB_s, \quad \textit{for } 0\le t<\tau \textit{ a.s.}$$

(ii) *For all $f\in C^\infty(\mathbb{R}^n)$,*

$$d\big(f(Y)\big) = (Lf)(t,Y)\,dt + \sum_{k=1}^r (A_kf)(t,Y)dB^k \quad \textit{on } [0,\tau[$$

where

$$L = A_0 + \frac{1}{2}\sum_{k=1}^r A_k^2,$$

with the vector fields $A_i\in\Gamma(T\mathbb{R}^n)$ defined as

$$A_0 = \sum_{i=1}^n \beta_i D_i, \quad A_k = \sum_{i=1}^n \sigma_{ik}D_i, \quad k=1,\dots,r. \tag{1.2.10}$$

PROOF. (i) ⇒ (ii) By the Itô-Stratonovich formula (Proposition 1.2.10) we have

$$\begin{aligned}
d\big(f(Y)\big) &= \sum_{i=1}^n (D_if)(Y)\circ dY^i\\
&= \sum_{i=1}^n (D_if)(Y)\,\beta_i(t,Y)\,dt + \sum_{i=1}^n (D_if)(Y)\left(\sum_{k=1}^r \sigma_{ik}(t,Y)\circ dB^k\right)\\
&= (A_0f)(t,Y)\,dt + \sum_{k=1}^r (A_kf)(t,Y)\circ dB^k\\
&= (A_0f)(t,Y)\,dt + \sum_{k=1}^r (A_kf)(t,Y)\,dB^k + \frac{1}{2}\sum_{k=1}^r d\big((A_kf)(t,Y)\big)dB^k.
\end{aligned}$$

Since

$$d(A_kf(t,Y)) = \partial_t(A_kf)(t,Y)\,dt + (A_0A_kf)(t,Y)\,dt + \sum_{\ell=1}^r (A_\ell A_kf)(t,Y)\circ dB^\ell,$$

we observe that

$$d(A_kf(t,Y))\,dB^k = (A_k^2f)(t,Y)\,dt.$$

and hence

$$d\big(f(Y)\big) = \underbrace{\left((A_0f)(t,Y) + \frac{1}{2}\sum_{k=1}^r (A_k^2f)(t,Y)\right)}_{=(Lf)(t,Y)}dt + \sum_{k=1}^r (A_kf)(t,Y)\,dB^k.$$

(ii) ⇒ (i) It is sufficient to take $f(x)=x_\ell$. □

COROLLARY 1.2.12. *Solutions to the Stratonovich* SDE

$$dY = \beta(t,Y)\,dt + \sigma(t,Y)\circ dB$$

define L-diffusions for the operator

$$L = A_0 + \tfrac{1}{2}\sum_{i=1}^{r} A_i^2 \quad \textit{with } A_0, A_1, \ldots, A_r \textit{ as in } \text{Eq. (1.2.10)},$$

in the sense that

$$d(f\circ Y) - (Lf)(t,Y)\,dt \overset{\mathrm{m}}{=} 0$$

for all $f \in C^\infty(\mathbb{R}^n)$.

1.2.3. Stochastic differential equations on manifolds. In this section we describe the construction of L-diffusions as solutions of stochastic differential equations on manifolds [**11, 17**].

DEFINITION 1.2.13 (Stochastic differential equation on M). Let M be a differentiable manifold, $\pi\colon TM \to M$ its tangent bundle and E a finite dimensional vector space (without restrictions $E = \mathbb{R}^r$). A *stochastic differential equation on* M is a pair (A, Z) where

1. Z is a semimartingale taking values in E;
2. $A\colon M\times E \to TM$ is a smooth homomorphism of vector bundles over M, i.e.

$$(x,e) \longmapsto A(x)e := A(x,e)$$

$$\begin{tikzcd} M\times E \arrow[r, "A"'] \arrow[d, "\mathrm{pr}_1"'] & TM \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}"'] & M \end{tikzcd}$$

REMARK 1.2.14. Formally the homomorphism A may be considered as section $A \in \Gamma(E^*\otimes TM)$. In particular, we have

$$\begin{cases} \forall x\in M, & A(x)\in \mathrm{Hom}(E, T_xM),\\ \forall e\in E, & A(\cdot)e \in \Gamma(TM). \end{cases}$$

NOTATION 1.2.15. For the SDE (A,Z) we also write

$$\boxed{dX = A(X)\circ dZ}$$

or

$$\boxed{dX = \sum_{i=1}^{r} A_i(X)\circ dZ^i}$$

where $A_i = A(\cdot)e_i \in \Gamma(TM)$ and $e_1,\ldots,e_r$ is a basis of E.

DEFINITION 1.2.16 (Solution of a stochastic differential equation). Let (A,Z) be an SDE on M and let $x_0\colon \Omega\to M$ be $\mathscr{F}_0$-measurable. An adapted continuous process $X|[0,\zeta[\, \equiv (X_t)_{t<\zeta}$ taking values in M, defined up to the stopping time ζ, is called solution to the SDE

$$dX = A(X)\circ dZ \tag{1.2.11}$$

with initial condition $X_0 = x_0$, if for all $f\in C_c^\infty(M)$ the following conditions are satisfied:

(i) $f(X)$ is a semimartingale;

(ii) for any stopping time τ such that $0 \le \tau < \zeta$, we have

$$f(X_\tau) = f(X_0) + \int_0^\tau (df)_{X_s} A(X_s) \circ dZ_s. \tag{1.2.12}$$

We call X *maximal solution* of the SDE (1.2.11) if

$$\{\zeta < \infty\} \subset \left\{ \lim_{t \uparrow \zeta} X_t = \infty \text{ in } \hat{M} = M \,\dot{\cup}\, \{\infty\} \right\} \text{ a.s.}$$

Note: The integral in (1.2.12) is defined using the linear functional

$$E \xrightarrow{A(x)} T_x M \xrightarrow{(df)_x} \mathbb{R}, \quad x \in M.$$

REMARK 1.2.17. We adopt the convention $X_t(\omega) := \infty$ for $\zeta(\omega) \le t < \infty$ and $f(\infty) = 0$ for $f \in C_c^\infty(M)$. Then we may write, for all $t \ge 0$,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t (df)_{X_s} A(X_s) \circ dZ_s \\ &= f(X_0) + \sum_{i=1}^r \int_0^t (df)_{X_s} A_i(X_s) \circ dZ_s^i \\ &= f(X_0) + \sum_{i=1}^r \int_0^t (A_i f)(X_s) \circ dZ_s^i \quad \text{with } A_i = A(\cdot) e_i. \end{aligned}$$

EXAMPLE 1.2.18. Let $E = \mathbb{R}^{r+1}$ and $Z = (t, Z^1, \ldots, Z^r)$ where $(Z^1, \ldots, Z^r)$ is a Brownian motion on $\mathbb{R}^r$. Denote the standard basis of $\mathbb{R}^{r+1}$ by $(e_0, e_1, \ldots, e_r)$. Letting

$$A\colon M \times E \to TM$$

be a homomorphism of vector bundles over M, we consider the vector fields

$$A_i := A(\cdot) e_i \in \Gamma(TM), \quad i = 0, 1, \ldots, r.$$

Then the SDE

$$\boxed{dX = A(X) \circ dZ} \tag{1.2.13}$$

writes as

$$\boxed{dX = A_0(X)\, dt + \sum_{i=1}^r A_i(X) \circ dZ^i}$$

and for each $f \in C_c^\infty(M)$ we have

$$\begin{aligned} d\big(f(X)\big) &= (df)_X A(X) \circ dZ \\ &= \sum_{i=0}^r (df)_X A(X) e_i \circ dZ^i \\ &= \sum_{i=0}^r (df)_X A_i(X) \circ dZ^i \\ &= \sum_{i=0}^r (A_i f)(X) \circ dZ^i \\ &= (A_0 f)(X)\, dt + \sum_{i=1}^r (A_i f)(X) \circ dZ^i \end{aligned}$$

$$= (A_0 f)(X)\,dt + \sum_{i=1}^{r} \left((A_i f)(X)\,dZ^i + \frac{1}{2} d\big((A_i f)(X)\big)\,dZ^i \right).$$

Taking into account that

$$d\big((A_i f)(X)\big) = \sum_{j=1}^{r} (A_j A_i f)(X)\,dZ^j + d(\text{terms of bounded variation}),$$

we see that

$$d\big((A_i f)(X)\big)\,dZ^i = (A_i^2 f)(X)\,dt,$$

where we used that $dZ^i dZ^j = \delta_{ij}\,dt$ for $1 \le i, j \le r$. Hence we get

$$\begin{aligned} d\big(f(X)\big) &= (A_0 f)(X)\,dt + \frac{1}{2}\sum_{j=1}^{r}(A_i^2 f)(X)\,dt + \sum_{i=1}^{r}(A_i f)(X)\,dZ^i \\ &= (Lf)(X)\,dt + \sum_{i=1}^{r}(A_i f)(X)\,dZ^i. \end{aligned}$$

COROLLARY 1.2.19. *Let $L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2$ and let X be a solution to Eq.* (1.2.13). *Then, for all $f \in C_c^\infty(M)$,*

$$d\big(f(X)\big) - (Lf)(X)\,dt \overset{\mathrm{m}}{=} 0$$

where $\overset{\mathrm{m}}{=}$ denotes equality modulo differentials of martingales. In other words, maximal solutions to the SDE

$$\boxed{dX = A(X) \circ dZ}$$

are L-diffusions to the operator $L = A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2$.

THEOREM 1.2.20 (SDE: Existence and uniqueness of solutions; $M = \mathbb{R}^n$). *Let (A, Z) be an* SDE *on $M = \mathbb{R}^n$ and x_0 an $\mathscr{F}_0$-measurable random variable taking values in $\mathbb{R}^n$. Then there exists a unique maximal solution X (with maximal lifetime $\zeta > 0$ a.s.) of the* SDE

$$dX = A(X) \circ dZ \tag{1.2.14}$$

with initial condition $X_0 = x_0$. Uniqueness holds in the following sense: if $Y|[0,\xi[$ is another solution of (1.2.14) *to the same initial condition, then $\xi \le \zeta$ a.s. and $X|[0,\xi[= Y$ a.s.*

PROOF. As in the proof of Proposition 1.2.5 let $B(0,R) = \{x \in \mathbb{R}^n : |x| \le R\}$ where $R = 1, 2, \ldots$ and choose test functions $\phi_R \in C_c^\infty(\mathbb{R}^n)$ such that $\phi_R | B(0,R) \equiv 1$. Since

$$A \in \Gamma(\mathrm{Hom}(\mathbb{R}^r, TM)),$$

we have for each $x \in \mathbb{R}^n$ the linear map

$$A(x)\colon \mathbb{R}^r \to T_x M.$$

In this way A gives rise to a smooth map $\mathbb{R}^n \to \mathrm{Matr}(n \times r; \mathbb{R})$.

Consider now the "truncated SDE"

$$dX^R = A^R(X^R) \circ dZ \tag{1.2.15}$$

where $A^R = \phi_R\, A$. By Proposition 1.2.4, the truncated SDE (1.2.15) has a unique global solution X^R with initial condition $X_0^R = x_0$, i.e., for each R there exists a continuous

$\mathbb{R}^n$-valued semimartingale $(X_t^R)_{t\geq 0}$ satisfying $X_0^R = x_0$ such that (1.2.15) holds in the Itô-Stratonovich sense. In terms of the stopping times

$$\tau_R := \inf\left\{t \geq 0 : X_t^R \notin B(0,R)\right\},$$

we have for $R < R'$,

$$X^{R'}|[0,\tau_R[= X^R|[0,\tau_R[\quad \text{a.s.}$$

Hence a stochastic process X (with lifetime $\zeta = \lim_{R\uparrow\infty} \tau_R$) is well-defined via

$$X|[0,\tau_R[= X^R|[0,\tau_R[.$$

For each $f \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(f) \subset B(0,R)$ (with R sufficiently large), we have

$$\begin{aligned}
d\big(f(X)\big) &= d\big(f(X^R)\big)\\
&= \sum_{k=1}^n (D_k f(X^R)) \circ d(X^R)^k \quad \text{(using Itô-Stratonovich formula)}\\
&= \langle \nabla f(X^R),\, \circ\, dX^R\rangle\\
&= \langle \nabla f(X^R), \phi_R(X^R)\, A(X^R) \circ dZ\rangle\\
&= \langle \nabla f(X), A(X) \circ dZ\rangle\\
&= \sum_{i=1}^r \langle \nabla f(X), A_i(X) \circ dZ^i\rangle\\
&= \sum_{i=1}^r (df)_X A_i(X) \circ dZ^i\\
&= (df)_X A(X) \circ dZ.
\end{aligned}$$

Hence, X is the unique solution to Eq. (1.2.14) with initial condition $X_0 = x_0$. Note that X is a solution of $dX = A(X) \circ dZ$ in the Itô-Stratonovich sense (in $\mathbb{R}^n$) if and only if $\forall f \in C_c^\infty(\mathbb{R}^n)$,

$$d\big(f(X)\big) = (df)_X A(X) \circ dZ. \qquad \square$$

THEOREM 1.2.21 (SDE: Existence and uniqueness of solutions; general case). *Let (A,Z) be an* SDE *on a differentiable manifold M and let $x_0 \colon \Omega \to M$ be $\mathscr{F}_0$-measurable. There exists a unique maximal solution $X|[0,\zeta[$ (where $\zeta > 0$ a.s.) of the* SDE

$$dX = A(X) \circ dZ$$

with initial condition $X_0 = x_0$. Uniqueness holds in the sense that if $Y|[0,\xi[$ is another solution with $Y_0 = x_0$, then $\xi \leq \zeta$ a.s. and $X|[0,\xi[= Y$ a.s.

We shall reduce Theorem 1.2.21 to Theorem 1.2.20 via embedding the manifold M into a high-dimensional Euclidean space.

WHITNEY'S EMBEDDING THEOREM. *Each manifold M of dimension n can be embedded into $\mathbb{R}^{n+k}$ as a closed submanifold (for k sufficiently large, e.g. $k = n+1$), i.e.,*

$$M \hookrightarrow \iota(M) \subset \mathbb{R}^{n+k}$$

where $\iota : M \to \iota(M)$ is a diffeomorphism and $\iota(M) \subset \mathbb{R}^{n+k}$ a closed submanifold.

PROOF (OF THEOREM 1.2.21). We choose a Whitney embedding (in general not intrinsic)

$$M \underset{\text{diffeom.}}{\overset{\iota}{\hookrightarrow}} \iota(M) \subset \mathbb{R}^{n+k}$$

and identify M and $\iota(M)$; in particular for each $x \in M$ the tangent space T_xM is then a linear subspace of $\mathbb{R}^{n+k}$ according to

$$T_xM \xhookrightarrow{d\iota_x} T_x\mathbb{R}^{n+k} \equiv \mathbb{R}^{n+k}.$$

Vector fields $A_1, \ldots, A_r \in \Gamma(TM)$ can be extended to vector fields

$$\bar{A}_1, \ldots, \bar{A}_r \in \Gamma(T\mathbb{R}^{n+k}) \equiv C^\infty(\mathbb{R}^{n+k}; \mathbb{R}^{n+k}) \quad \text{with } \bar{A}_i|M = A_i,$$

i.e. $\bar{A}_i \circ \iota = d\iota \circ A_i$. Hence a given bundle map

$$A: M \times \mathbb{R}^r \to TM, \quad (x, z) \mapsto A(x)z = \sum_{i=1}^r A_i(x)z^i$$

has a continuation

$$\bar{A}: \mathbb{R}^{n+k} \times \mathbb{R}^r \to \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}, \quad (x, z) \mapsto \bar{A}(x)z = \sum_{i=1}^r \bar{A}_i(x)z^i.$$

The idea is to consider in place of the original SDE

$$(*) \qquad dX = A(X) \circ dZ \ \text{ on } M$$

the SDE

$$(\bar{*}) \qquad dX = \bar{A}(X) \circ dZ \ \text{ on } \mathbb{R}^{n+k}.$$

First of all it is clear that any solution of $(*)$ in M provides a solution of $(\bar{*})$ in $\mathbb{R}^{n+k}$. More precisely: If X is a solution to $(*)$ with starting value $X_0 = x_0$, then $\bar{X} := \iota \circ X$ solves equation $(\bar{*})$ with starting value $\bar{X}_0 = \iota \circ x_0$. Indeed if $\bar{f} \in C_c^\infty(\mathbb{R}^{n+k})$, then $f := \bar{f}|M = \bar{f} \circ \iota \in C_c^\infty(M)$, and we have:

$$\begin{aligned}
d\big(\bar{f}(\bar{X})\big) = d\big(f(X)\big) &= \sum_{i=1}^r (df)_X\, A_i(X) \circ dZ^i \\
&= \sum_{i=1}^r (d\bar{f})_{\bar{X}}\, (d\iota)_X\, A_i(X) \circ dZ^i \\
&= \sum_{i=1}^r (d\bar{f})_{\bar{X}}\, \bar{A}_i(\iota \circ X) \circ dZ^i \\
&= \sum_{i=1}^r (d\bar{f})_{\bar{X}}\, \bar{A}_i(\bar{X}) \circ dZ^i.
\end{aligned}$$

This implies in particular uniqueness of solutions to $(*)$, since equation $(\bar{*})$ has a unique solution to a given initial condition.

To establish existence of solutions to $(*)$ we first remark that any test function $f \in C_c^\infty(M)$ has a continuation $\bar{f} \in C_c^\infty(\mathbb{R}^{n+k})$ such that $\bar{f}|M \equiv \bar{f} \circ \iota = f$. We make the following important observation.

Each solution $X|[0, \zeta[$ of $(\bar{*})$ in $\mathbb{R}^{n+k}$ with $X_0 = x_0$ which stays on M for $t < \zeta$ (where x_0 is an M-valued $\mathscr{F}_0$-measurable random variable) gives a solution of $(*)$.

Hence, to complete the proof it is sufficient to show the following lemma. □

LEMMA 1.2.22. *If $X|[0, \zeta[$ is the maximal solution of $(\bar{*})$ in $\mathbb{R}^{n+k}$ with $X_0 = x_0$, then*

$$\{t < \zeta\} \subset \{X_t \in M\}, \quad \textit{for all } t \ \textit{a.s.}$$

Observe that it is enough to verify Lemma 1.2.22 for one specific continuation $\bar{A}$ of A.

PROOF (OF LEMMA 1.2.22). Let

$$\perp M = \left\{(x,v) \in M \times \mathbb{R}^{n+k} \mid v \in (T_xM)^\perp\right\},$$

be the normal bundle of M and consider M embedded into $\perp M$ as zero section:

$$M \hookrightarrow \perp M, \quad x \mapsto (x,0)$$

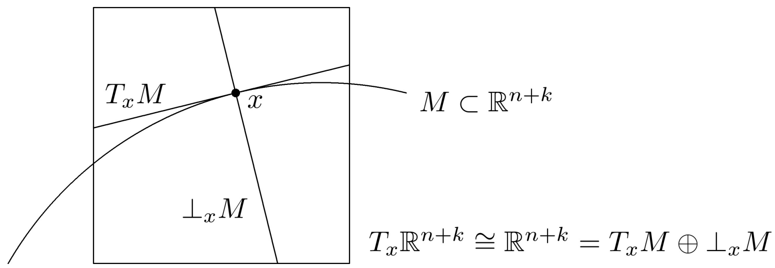


Figure 1.2.1. Normal bundle $\perp M$

Fact: There is a smooth function $\varepsilon\colon M \to \,]0,\infty[$ such that the map

$$\tau_\varepsilon(M) := \left\{(x,v) \in \perp M\colon |v| < \varepsilon(x)\right\} \xrightarrow{\cong} \bigcup_{x\in M} \{y \in \mathbb{R}^{n+k} : |y-x| < \varepsilon(x)\},$$
$$(x,v) \longmapsto x+v,$$

is a diffeomorphism from the tubular neighbourhood $\tau_\varepsilon(M)$ of M of radius ε onto the indicated part in $\mathbb{R}^{n+k}$. This follows from the local inversion theorem since the given map has full rank along the zero section of $\perp M$.

Note that both

$$\pi\colon \tau_\varepsilon(M) \to M, \quad (x,v) \mapsto x$$
$$\mathrm{dist}^2(\cdot, M)\colon \tau_\varepsilon(M) \to \mathbb{R}, \quad (x,v) \mapsto |v|^2,$$

are smooth maps.

Now letting $R > 0$ be sufficiently large such that

$$M \cap B(0, R+1) \neq \varnothing,$$

then

$$\varepsilon_R = \inf\{\varepsilon(x) \mid x \in M \cap B(0,R+1)\} > 0.$$

We choose a decreasing smooth function $\lambda : [0,\infty[\to [0,1]$ of the form

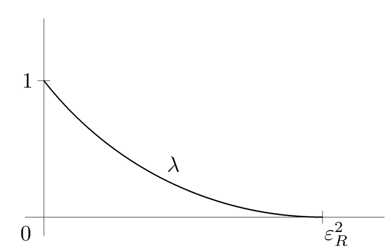


Figure 1.2.2. Cut-off function λ

and a test function $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+k})$ such that $\varphi|B(0,R) \equiv 1$ and $\operatorname{supp}(\varphi) \subset B(0,R+1)$. Consider the map

$$\begin{aligned} &\bar{A}^R\colon \mathbb{R}^{n+k} \times \mathbb{R}^r \to \mathbb{R}^{n+k} \times \mathbb{R}^{n+k}, \\ &\bar{A}^R(y,z) := \begin{cases} \varphi(y)\, \lambda(\operatorname{dist}^2(y,M))\, A(\pi(y))\, z & \text{if } y \in \tau_\varepsilon(M), \\ 0 & \text{if } y \notin \tau_\varepsilon(M). \end{cases} \end{aligned}$$

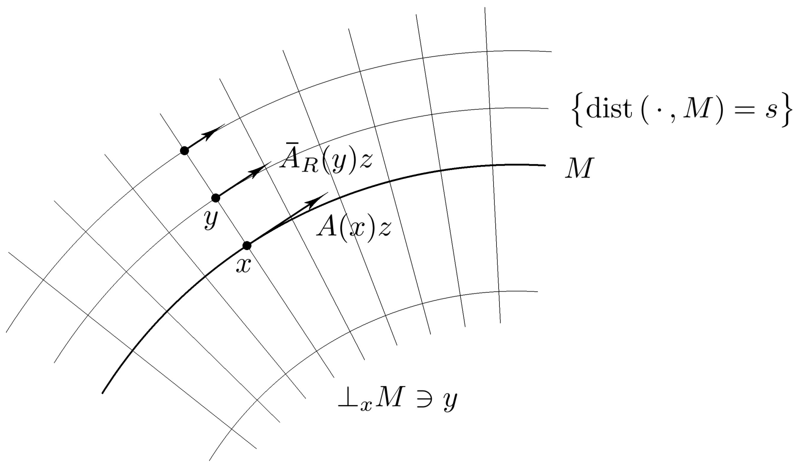


Figure 1.2.3. Extended coefficients of the SDE

Let X be the solution of

$$dX = \bar{A}^R(X) \circ dZ, \quad X_0 = x_0.$$

Consider the test function $f \in C_c^\infty(\mathbb{R}^{n+k})$ given as

$$f(y) = \varphi(y)\, \lambda(\operatorname{dist}^2(y,M)).$$

Then

$$\begin{aligned} d\big(f(X)\big) &= (df)_X \bar{A}^R(X) \circ dZ \\ &= \langle \nabla f(X),\, \bar{A}^R(X) \circ dZ \rangle \\ &= 0 \quad \text{on } [0, \tau_R[, \end{aligned}$$

where $\tau_R := \inf\{t \ge 0\colon X_t \notin B(0,R)\}$. Indeed, f is constant on each submanifold of the form

$$\{\operatorname{dist}(\cdot, M) = s\} \cap B(0,R), \quad s < \varepsilon_R,$$

whereas $\bar{A}^R(y,z)$ is tangent to such submanifolds. Thus, for all $y \in B(0,R)$ and $z \in \mathbb{R}^r$,

$$\nabla f(y) \perp \bar{A}^R(y)z.$$

Hence, for any solution X of $(\bar{*})$, we obtain that

$$f(X) \equiv \text{constant on } [0, \tau_R[\ \text{ a.s.}$$

Since R is arbitrary, this completes the proof of the Lemma. $\square$

Solutions to an SDE on M of the type (1.2.11) are by definition semimartingales on M as defined above: A continuous adapted process X with values in M is a *semimartingale* on M if, for each $f \in C_c^\infty(M)$, the composition $f \circ X$ provides a continuous real-valued semimartingale. It is easy to see that each M-valued semimartingale can be obtained as solution of an SDE on M.

THEOREM 1.2.23 (Manifold-valued semimartingales as solutions of an SDE). *Every semimartingale on a manifold M is given as solution of an* SDE *of the type* (1.2.11).

PROOF. Let X be an arbitrary semimartingale on M. Without loss of generality (after an eventual change of time), we may assume that X has infinite lifetime. Choosing a Whitney embedding $\iota\colon M \hookrightarrow \mathbb{R}^{n+k}$ we may consider the semimartingale $Z := \iota \circ X$ taking values in $E := \mathbb{R}^{n+k}$. Let $A\colon M \times E \to TM$ be the bundle homomorphism which is fiberwise the orthogonal projection $A(x)\colon \mathbb{R}^{n+k} \to T_xM$ of $\mathbb{R}^{n+k}$ onto $T_xM \subset T_x\mathbb{R}^{n+k} = \mathbb{R}^{n+k}$. We show that X solves the equation

$$dX = A(X) \circ dZ.$$

Let $f \in C_c^\infty(M)$ be given. We choose a continuation $\bar{f} \in C_c^\infty(\mathbb{R}^{n+k})$ where $\bar{f} \circ \iota = f$ such that $\bar{f}$ is constant locally about M on the normal subspaces $\perp_x M$ (this is $\bar{f}(y) = f(x)$ for $y \in \perp_x M$ sufficiently small). Now let $x \in M$ and $z \in \mathbb{R}^{n+k}$. By decomposing $z = z_0 + z^\perp$ where $z_0 \in T_xM$ and $z^\perp \in \perp_x M$, we obtain:

$$(df)_x A(x) z = (d\bar{f})_{\iota(x)} (d\iota)_x\, A(x) z = (d\bar{f})_{\iota(x)} z_0 = (d\bar{f})_{\iota(x)} z.$$

But then

$$\begin{aligned} d\big(f(X)\big) = d\big(\bar{f}(\iota(X))\big) &= \sum_{i=1}^{n+k} \big(D_i \bar{f}\big)(\iota(X)) \circ dZ^i \\ &= \sum_{i=1}^{n+k} (df)_X A(X) e_i \circ dZ^i = (df)_X A(X) \circ dZ \end{aligned}$$

which gives the claim. □

REMARK 1.2.24. Let M and N be differentiable manifolds. For semimartingales X on M, respectively X' on N, both adapted to the same filtration, consider the product semimartingale $\tilde{X} := (X, X')$ taking values in $M \times N$. Suppose that

$$dX = A(X) \circ dZ, \quad \text{resp.} \quad dX' = A'(X') \circ dZ' \tag{1.2.16}$$

with bundle maps $A\colon M \times \mathbb{R}^k \to TM$ over M, respectively $A'\colon N \times \mathbb{R}^{k'} \to TN$ over N. Then $\tilde{X}$ solves the "composed" SDE

$$d\tilde{X} = \tilde{A}(\tilde{X}) \circ d\tilde{Z} \tag{1.2.17}$$

driven by the $\mathbb{R}^k \times \mathbb{R}^{k'}$-valued semimartingale $\tilde{Z} := (Z, Z')$ where

$$\tilde{A}\big(x, x'\big)(z, z') := \big(A(x)z, A'(x')z'\big) \in T_xM \oplus T_{x'}N \equiv T_{(x,x')}(M \times N)$$

defines a bundle map $\tilde{A}\colon (M \times N) \times (\mathbb{R}^k \times \mathbb{R}^{k'}) \to T(M \times N)$ over $M \times N$.

PROOF. Let $\iota\colon M \hookrightarrow \mathbb{R}^\ell$ and $\iota'\colon N \hookrightarrow \mathbb{R}^{\ell'}$ be Whitney embeddings. Any function $f \in C^\infty(M \times N)$ factorizes as $f = \bar{f} \circ (\iota, \iota')$ for some $\bar{f} \in C^\infty(\mathbb{R}^\ell \times \mathbb{R}^{\ell'})$. Let $\bar{X} = \iota(X)$ and $\bar{X}' = \iota'(X')$. Then for $f \in C^\infty(M \times N)$, the semimartingale $f(\tilde{X}) = \bar{f}(\bar{X}, \bar{X}')$ satisfies

$$\begin{aligned} d\big(f(\tilde{X})\big) &= d\big(\bar{f}(\bar{X}, \bar{X}')\big) = (d\bar{f})(\bar{X}, \bar{X}') \circ d(\bar{X}, \bar{X}') \\ &= (d\bar{f})(\bar{X}, \bar{X}') \circ (d\bar{X}, d\bar{X}') = (d\bar{f})(\bar{X}, \bar{X}') \circ \big(d(\iota(X)), d(\iota(X'))\big) \\ &= \bar{f}_*\big(\iota_* A(X) \circ dZ, \iota'_* A'(X') \circ dZ'\big) = \big(\bar{f}_*(\iota, \iota')_*\big) \tilde{A}(\tilde{X}) \circ d\tilde{Z} \end{aligned}$$

$$= f_* \tilde{A}(\tilde{X}) \circ d\tilde{Z} \equiv \sum_i f_* \tilde{A}_i(\tilde{X}) \circ d\tilde{Z}^i$$

which proves the claim. $\square$

1.3. Quadratic Variation and Integration of one-forms

In this section we give canonical constructions related to continuous semimartingales on a manifold M, including the quadratic variation of continuous semimartingales with respect to bilinear forms on TM and the integral of one-forms on M along semimartingales, see [**12**] for more details. In the particular case $M = \mathbb{R}^n$ endowed with the Euclidean metric this notion of the quadratic variation reduces to the usual quadratic variation of a semimartingale.

Both notions (quadratic variation and integration of one-forms) can be deduced from a unified construction principle within the framework of second order differential geometry. We postpone this point of view and develop the theory first only as far as needed for martingale theory on manifolds.

We start with an elementary technical lemma on continuous processes, which is quite useful as it allows a spatial localization of continuous adapted processes, besides the usual localization in time through a localizing sequence of stopping times. The lemma basically reduces to properties of continuous paths.

LEMMA 1.3.1. *Let $(V_k)_{k\in\mathbb{N}}$ be a countable covering of M by open sets V_k and X be a continuous adapted M-valued process. Then there exists a non-decreasing sequence $(\tau_n)_{n\geq 0}$ of stopping times with $\tau_0 = 0$ and $\sup_n \tau_n = \infty$, such that on each of the intervals $[\tau_n, \tau_{n+1}] \cap \big(\mathbb{R}_+ \times \{\tau_n < \tau_{n+1}\}\big)$ the process X takes values only in one of the V_k.*

PROOF OF THE LEMMA. First of all, we choose a refinement $(W_k)_{k\in\mathbb{N}}$ to $(V_k)_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$ the closure $\bar{W}_k$ of W_k is still contained in one of the $V_{n(k)}$. We construct a sequence $(\tau_n^k)_{0\leq k\leq n,\, n\geq 0}$ of stopping times which after a suitable renumbering will satisfy the claimed assertions. Let $\tau_0^0 := 0$. Suppose that τ_n^k is already constructed up to a certain n, then let

$$\tau_{n+1}^0 := \tau_n^n, \text{ and } \tau_{n+1}^k := \inf\{t \geq \tau_{n+1}^{k-1} \colon X_t \notin W_k\} \quad \text{for } k = 1, \dots, n+1.$$

It remains to verify that $\sup_{n\geq 0} \sup_{k\leq n} \tau_n^k = \infty$. Let's suppose that there exists $\omega \in \Omega$ such that $t_0 := \sup_{n\geq 0} \sup_{k\leq n} \tau_n^k(\omega) < \infty$. Then we know $X_{t_0}(\omega) \in W_\ell$ for some ℓ, and by continuity even $X_t(\omega) \in W_\ell$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ with some sufficiently small $\varepsilon > 0$. By definition of t_0 there exists $n_0 \in \mathbb{N}$, $n_0 \geq \ell$ such that $\tau_{n_0}^0(\omega) > t_0 - \varepsilon$, with the consequence that then $\tau_{n_0}^\ell(\omega) \geq t_0 + \epsilon$ which gives a contradiction. $\square$

Given a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\in\mathbb{R}_+})$ satisfying the usual conditions, we denote by $\mathscr{S}$ be the vector space of real-valued continuous semimartingales:

$$\mathscr{S} = \mathscr{M} \oplus \mathscr{A}$$

where $\mathscr{M}$ denotes the space of continuous local martingales and $\mathscr{A}$ the space of continuous adapted processes, starting at 0 almost surely, which are pathwise locally of bounded variation.

We start by stating an elementary but useful representation lemma.

LEMMA 1.3.2. *Let M be an arbitrary differentiable manifold. There exist finitely many functions $h^1, \dots, h^\ell \in C^\infty(M)$ such that the following properties hold:*

(i) *Each function $f \in C^\infty(M)$ factorizes through $(h^1, \dots, h^\ell)$ as $f = \bar{f} \circ (h^1, \dots, h^\ell)$ for some $\bar{f} \in C^\infty(\mathbb{R}^\ell)$.*

(ii) *Each section $b \in \Gamma(T^*M \otimes T^*M)$ can be written as $b = \sum_{i,j=1}^{\ell} b_{ij}\, dh^i \otimes dh^j$ with functions $b_{ij} \in C^\infty(M)$.*

(iii) *Each differential form $\alpha \in \Gamma(T^*M)$ can be written as $\alpha = \sum_{i=1}^{\ell} \alpha_i\, dh^i$ with functions $\alpha_i \in C^\infty(M)$.*

(iv) *If X is a semimartingale on M, then every continuous adapted $T^*M \otimes T^*M$-valued process B above X (i.e., $B_t \in T^*_{X_t}M \otimes T^*_{X_t}M$ for $t \in \mathbb{R}_+$) which is a semimartingale in the sense that $B_t(V,U)$ is a real semimartingale for any vector fields $V, U \in \Gamma(TM)$, has a representation of the form $B = \sum_{i,j=1}^{\ell} B_{ij}\left(dh^i \otimes dh^j\right) \circ X$ with continuous adapted real-valued processes B_{ij}.*

(v) *If X is a semimartingale on M, then every continuous adapted T^*M-valued process J above X (i.e., $J_t \in T^*_{X_t}M$ for $t \in \mathbb{R}_+$) which is a semimartingale in the sense that $J_t(V)$ is a real semimartingale for any vector fields $V \in \Gamma(TM)$, has a representation of the form $J = \sum_{i=1}^{\ell} J_i\,(dh^i \circ X)$ with continuous adapted real-valued processes J_i.*

PROOF. We represent M via a Whitney embedding $h\colon M \hookrightarrow \mathbb{R}^\ell$ as a closed submanifold of some $\mathbb{R}^\ell$. Then there exists a differentiable partition $(\varphi_\lambda)_{\lambda\in\Lambda}$ of the unity on M and a family $(I_\lambda)_{\lambda\in\Lambda}$ of subsets $I_\lambda \subset \{1,\dots,\ell\}$ with the following property: for each $\lambda \in \Lambda$ the family $(h^i)_{i\in I_\lambda}$ define a chart for M on some open neighbourhood of $\operatorname{supp}(\varphi_\lambda)$.

Part (i) is evident: One defines $\bar f | h(M)$ through $f = \bar f \circ h$ and extends $\bar f$ constantly along the normal subspaces $\perp_x M$ to an open neighbourhood of $M \cong h(M)$, and finally smoothens $\bar f$ by multiplication with a function identical to 1 locally about $h(M)$ and vanishing outside a suitable larger tubular neighbourhood.

To part (ii): Note that $\varphi_\lambda\, b = \sum_{i,j=1}^{\ell} b_{ij}^\lambda\, dh^i \otimes dh^j$ with $b_{ij}^\lambda \in C^\infty(M)$ such that $\operatorname{supp}(b_{ij}^\lambda) \subset \operatorname{supp}(\varphi_\lambda)$ and $b_{ij}^\lambda := 0$ for $\{i,j\} \not\subset I_\lambda$, but then

$$b = \sum_{i,j=1}^{\ell} b_{ij}\, dh^i \otimes dh^j \quad \text{where } b_{ij} := \sum_\lambda b_{ij}^\lambda.$$

The proof of part (iii) is analogous to (ii).

To (iv): Analogously to (ii) we first write $\varphi_\lambda(X)B = \sum_{i,j=1}^{\ell} B_{ij}^\lambda\,(dh^i \otimes dh^j) \circ X$ with appropriate continuous $\mathbb{R}$-valued processes B_{ij}^λ, namely $B_{ij}^\lambda := \varphi_\lambda(X) B\big(\frac{\partial}{\partial h^i}, \frac{\partial}{\partial h^j}\big)$ for $\{i,j\} \subset I_\lambda$ and $B_{ij}^\lambda := 0$ for $\{i,j\} \not\subset I_\lambda$. Summation over λ then gives the claim.

The proof of (v) is again carried out analogously. ☐

THEOREM 1.3.3. *Let X be an M-valued semimartingale. There exists a unique linear mapping $\Gamma(T^*M \otimes T^*M) \to \mathscr{A}$, $b \mapsto \int b(dX, dX)$, such that for all $f, g \in C^\infty(M)$,*

$$df \otimes dg \;\mapsto\; [f(X), g(X)] \tag{1.3.1}$$

$$f\, b \;\mapsto\; \int \big(f(X)\big)\, b(dX, dX). \tag{1.3.2}$$

Here, by definition, $b(dX,dX) := d\int b(dX,dX)$ and $[f(X), g(X)]$ in item (1.3.1) *is the quadratic covariation process of $f(X)$ and $g(X)$.*

DEFINITION 1.3.4 (b-quadratic variation). The process $\int b(dX,dX)$ is called *integral of b along X* or *b-quadratic variation of X*. The random variable $\big(\int b(dX,dX)\big)_t$ giving its value at time t is written as $\int_0^t b(dX,dX)$.

PROOF (OF THEOREM 1.3.3). By Lemma 1.3.2 (ii) each section $b \in \Gamma(T^*M \otimes T^*M)$ can be represented as $b = \sum b_{ij}\, dh^i \otimes dh^j$. We define

$$\int b(dX, dX) := \sum \int \big(b_{ij}(X)\big)\, d[h^i(X), h^j(X)]. \tag{1.3.3}$$

Then uniqueness is obvious; to prove existence it remains to show that (1.3.3) is well-defined. To this end assume that

$$b = \sum_{\text{finite}} u_\nu\, df^\nu \otimes dg^\nu = 0.$$

We need to check that

$$\sum_\nu u_\nu(X)\, d[f^\nu(X), g^\nu(X)] = 0$$

as well. Without loss of generality, by means of Lemma 1.3.1, we may assume that h is already a global chart for M. According to Lemma 1.3.2 (i) we write $u_\nu = \bar{u}_\nu \circ h$, $f^\nu = \bar{f}^\nu \circ h$ and $g^\nu = \bar{g}^\nu \circ h$ in terms of appropriate extensions $\bar{u}_\nu, \bar{f}^\nu, \bar{g}^\nu \in C^\infty(\mathbb{R}^\ell)$. Defining $\bar{X} = h \circ X$, the claim then follows from the following calculation:

$$\begin{aligned}
\sum_\nu u_\nu(X)\, d[f^\nu(X), g^\nu(X)] &= \sum_\nu \bar{u}_\nu(\bar{X})\, d[\bar{f}^\nu(\bar{X}), \bar{g}^\nu(\bar{X})] \\
&= \sum_{i,j} \sum_\nu \bar{u}_\nu(\bar{X})\ (D_i \bar{f}^\nu)(\bar{X})\, (D_j \bar{g}^\nu)(\bar{X})\, d[\bar{X}^i, \bar{X}^j] \\
&= \sum_{i,j} \Big(\sum_\nu u_\nu\, df^\nu \otimes dg^\nu\Big) \Big(\big(\tfrac{\partial}{\partial h^i}\big)_X, \big(\tfrac{\partial}{\partial h^j}\big)_X\Big)\, d[\bar{X}^i, \bar{X}^j] = 0.
\end{aligned}$$ □

COROLLARY 1.3.5. *The b-quadratic variation $\int b(dX, dX)$ depends only on the symmetric part of b. In particular, $\int b(dX, dX) = 0$ if b is antisymmetric.*

PROOF. Defining $\bar{b}(v, w) := b(w, v)$, the assignment $b \mapsto \int \bar{b}(dX, dX)$ has the defining properties (1.3.1) and (1.3.2) as well. □

The next remark is again an immediate consequence of the defining properties (1.3.1) and (1.3.2) of the b-quadratic variation.

REMARK 1.3.6. The b-quadratic variation of a semimartingale commutes with time-change. More precisely, the following holds: Let X be an M-valued semimartingale, let $(\tau_t)_{t \geq 0}$ be a continuous finite time-change, and consider the time-changed semimartingale $\hat{X}$ defined by $\hat{X}_t = X_{\tau_t}$ (w.r.t. the time-changed filtration $(\hat{\mathscr{F}}_t)_{t\geq 0} := (\mathscr{F}_{\tau_t})_{t \geq 0}$). Then

$$\int_0^t b(d\hat{X}, d\hat{X}) = \int_{\tau_0}^{\tau_t} b(dX, dX).$$

In particular, for an arbitrary stopping time τ, if we denote by $X_t^\tau = X_{t\wedge\tau}$ the semimartingale stopped at the random time τ, then the formula $\int b(dX^\tau, dX^\tau) = \big(\int b(dX, dX)\big)^\tau$ where on the right-hand side the process $\int b(dX^\tau, dX^\tau)$ is stopped at time τ.

REMARK 1.3.7. (i) (Induced form) Let $\phi\colon M \to N$ be a differentiable map between manifolds, E be a vector bundle over N and $s \in \mathbb{N} \cup \{0\}$. Each multilinear form $L \in \Gamma(T^*N^{\otimes s} \otimes E)$ taking values in E induces via

$$(\phi^* L)_p(w_1, \dots, w_s) := L_{\phi(p)}(d\phi_p w_1, \dots, d\phi_p w_s), \quad w_i \in T_pM, p \in M,$$

a multilinear form $\phi^*L \in \Gamma(T^*M^{\otimes s} \otimes \phi^*E)$ with values in ϕ^*E, called *pullback* of L via ϕ. In particular, to each $X \in \Gamma(E)$ there is the induced section $\phi^*X \in \Gamma(\phi^*E)$ with $(\phi^*X)_p = X_{\phi(p)}$, $p \in M$.

(ii) (Induced frame) Let $e_1, \dots, e_m \in \Gamma(E/U)$ be a local frame for E. Then

$$\phi^*e_1, \dots, \phi^*e_m \in \Gamma\big(\phi^*E/\phi^{-1}(U)\big)$$

is a local frame for ϕ^*E. Hence, to each section $Y \in \Gamma(\phi^*E)$, there exist uniquely determined functions $b^1, \dots, b^m \in C^\infty\big(\phi^{-1}(U)\big)$ such that $Y|\phi^{-1}(U) = \sum b^i\, \phi^*e_i$.

THEOREM 1.3.8 (Pullback formula for the b-quadratic variation). *Let $\phi\colon M \to N$ be a differentiable map and $b \in \Gamma(T^*N \otimes T^*N)$. Then, for any semimartingale X on M,*

$$\int(\phi^*b)\,(dX, dX) = \int b\big(d(\phi \circ X), d(\phi \circ X)\big). \tag{1.3.4}$$

PROOF. The left-hand side of (1.3.4) satisfies the defining properties for the b-quadratic variation of $\phi(X)$. $\square$

We now turn to the problem of integrating one-forms on M along M-valued semimartingales, see [**21**].

THEOREM 1.3.9. *Let X be a semimartingale taking values in M. There is a unique linear mapping*

$$\Gamma(T^*M) \equiv A^1(M) \to \mathscr{S}, \quad \alpha \mapsto \int \alpha(\circ\, dX) \equiv \int_X \alpha,$$

such that for all $f \in C^\infty(M)$,

$$df \;\mapsto\; f(X) - f(X_0) \tag{1.3.5}$$

$$f\alpha \;\mapsto\; \int f(X) \circ \alpha(\circ\, dX). \tag{1.3.6}$$

On the right-hand side of (1.3.6) we have the Stratonovich integral of the process $f(X)$ with respect to the semimartingale $\int \alpha(\circ\, dX)$, thus

$$f(X) \,\circ\, \alpha(\circ\, dX) \equiv f(X) \,\circ\, d\Big(\int \alpha(\circ\, dX)\Big).$$

DEFINITION 1.3.10 (Stratonovich integral of one-forms along semimartingales). The process $\int \alpha(\circ\, dX)$ is called the *Stratonovich integral of α along X*. We also use the notation $\int_X \alpha$ for $\int \alpha(\circ\, dX)$.

PROOF (OF THEOREM 1.3.9). By Lemma 1.3.2 (iii) differential forms $\alpha \in \Gamma(T^*M)$ can be represented as $\alpha = \sum_i \alpha_i\, dh^i$ with functions $\alpha_i \in C^\infty(M)$. We define

$$\int_X \alpha := \sum_i \int \alpha_i(X) \circ d(h^i(X)). \tag{1.3.7}$$

Uniqueness is again obvious; it is thus sufficient to show that formula (1.3.7) is well-defined. To this end, we have to verify that if $\alpha = \sum_{\text{finite}} u_\nu\, df^\nu = 0$ then

$$\sum_\nu u_\nu(X) \circ d(f^\nu(X)) = 0$$

holds as well. Proceeding as in the proof of Theorem 1.3.3, without loss of generality, we assume again that h is already a global chart for M. But then we have

$$\sum_\nu u_\nu(X) \circ d(f^\nu(X)) = \sum_\nu \bar{u}_\nu(\bar{X}) \circ d(\bar{f}^\nu(\bar{X}))$$

$$= \sum_i \sum_\nu \bar{u}_\nu(\bar{X}) \circ \big(D_i \bar{f}^\nu(\bar{X}) \circ d\bar{X}^i\big)$$
$$= \sum_i \Big(\Big(\sum_\nu u_\nu \, df^\nu\Big) \big(\tfrac{\partial}{\partial h^i}\big)_X\Big) \circ d\bar{X}^i = 0,$$

which gives the claim. □

EXAMPLE 1.3.11. In the special case of a deterministic C^1 curve X in M, say $X_t = x(t)$, which is trivially a semimartingale, we obtain

$$\int_X \alpha = \int \alpha\big(\dot{x}(t)\big)\, dt, \quad \alpha \in \Gamma(T^*M). \tag{1.3.8}$$

Indeed, the right-hand side of (1.3.8) obviously has the defining properties of $\int_X \alpha$.

REMARK 1.3.12. Stratonovich integration of differential forms α along semimartingales commutes with time-change. More precisely, the following holds: Let X be a semimartingale taking values in M, $(\tau_t)_{t\geq 0}$ a continuous finite time-change, and consider the time-changed semimartingale $\hat{X}$ defined by $\hat{X}_t := X_{\tau_t}$ (with respect to the time-changed filtration $(\hat{\mathscr{F}}_t)_{t\geq 0} := (\mathscr{F}_{\tau_t})_{t\geq 0}$). Then

$$\int_0^t \alpha(\circ\, d\hat{X}) = \int_{\tau_0}^{\tau_t} \alpha(\circ\, dX).$$

In particular, for an arbitrary stopping time τ, if we denote by $X_t^\tau = X_{t\wedge\tau}$ the semimartingale stopped at the random time τ, then the formula

$$\int_{X^\tau} \alpha = \Big(\int_X \alpha\Big)^\tau$$

holds where on the right-hand side the semimartingale $\int_X \alpha$ is stopped at time τ.

THEOREM 1.3.13 (Pullback formula for the Stratonovich integral of a one-form). *Let $\phi\colon M \to N$ be a differentiable map and $\alpha \in A^1(N) \equiv \Gamma(T^*N)$. Then, for any semimartingale X on M,*

$$\int_X \phi^*\alpha = \int_{\phi\circ X} \alpha. \tag{1.3.9}$$

PROOF. The left-hand side of Eq. (1.3.9) satisfies the defining properties for the Stratonovich integral of α along $\phi \circ X$. □

REMARK 1.3.14. Let $\alpha, \beta \in \Gamma(T^*M)$. Then $\alpha \otimes \beta \in \Gamma(T^*M \otimes T^*M)$ and for the quadratic covariation process of $\int_X \alpha$ and $\int_X \beta$ we have the formula:

$$\Big[\int_X \alpha, \int_X \beta\Big] = \int (\alpha \otimes \beta)\,(dX, dX). \tag{1.3.10}$$

We continue with the observation that Theorems 1.3.3 and 1.3.9 can be slightly extended in an obvious way. In Eqs. (1.3.2) and (1.3.6), instead of $f(X)$ where $f \in C^\infty(M)$, more generally, continuous adapted $\mathbb{R}$-valued processes K may serve as multipliers.

THEOREM 1.3.15. *Let X be an M-valued semimartingale and let $\mathbb{B}$ be the real vector space of continuous adapted $T^*M \otimes T^*M$-valued processes B over X such that $B_t(V,U)$ are real semimartingales for any vector fields $V, U \in \Gamma(TM)$. There exists exactly one linear mapping*

$$\mathbb{B} \to \mathscr{A}, \quad B \mapsto \int B(dX, dX),$$

with the following properties:

$$b \circ X \mapsto \int b(dX, dX) \quad \textit{for any } b \in \Gamma(T^*M \otimes T^*M),$$
$$KB \mapsto \int K\,B(dX, dX) \quad \textit{for any continuous adapted real-valued processes } K.$$

Here $\int KB(dX,dX) := \int K\,d\big(\int B(dX,dX)\big)$.

PROOF. According to Lemma 1.3.2 (iv) each continuous adapted $T^*M \otimes T^*M$-valued process B over X has a representation as a finite sum of the form

$$B = \sum_\nu B_\nu \big(df^\nu \otimes dg^\nu\big) \circ X.$$

We set

$$\int B(dX, dX) := \sum_\nu B_\nu\, d[f^\nu(X), g^\nu(X)].$$

Well-definedness is verified as in the proof of Theorem 1.3.3. □

THEOREM 1.3.16. *Let X be an M-valued semimartingale and let $\mathbb{D}$ be the real vector space of continuous adapted T^*M-valued processes J over X such that $J_t(V)$ are real semimartingales for any vector field $V \in \Gamma(TM)$. There exists exactly one linear mapping $\mathbb{D} \to \mathscr{S}, \quad J \mapsto \int J(\circ\, dX) \equiv \int_X J$, with the following properties:*

$$\alpha \circ X \mapsto \int \alpha(\circ\, dX) = \int_X \alpha \quad \textit{for any } \alpha \in \Gamma(T^*M),$$
$$K\,J \mapsto \int K \circ J(\circ\, dX) \quad \textit{for any continuous adapted } \mathbb{R}\textit{-valued process } K.$$

Here $\int K \circ J(\circ\, dX) := \int K \circ d\big(\int J(\circ\, dX)\big)$.

PROOF. According to Lemma 1.3.2 (v) each continuous adapted T^*M-valued process J over X has a representation as a finite sum of the form

$$J = \sum_\nu J_\nu\,(df^\nu \circ X).$$

We set

$$\int J(\circ\, dX) := \sum_\nu J_\nu \circ d\big(f^\nu(X)\big).$$

Well-definedness is verified with the same calculation as in the proof of Theorem 1.3.9. □

The pullback formulas (1.3.4) and (1.3.9) carry over in an obvious way.

REMARK 1.3.17 (Pullback formulas). Let $\phi\colon M \to N$ be a differentiable map and X be a semimartingale on M.

(i) For a continuous adapted $T^*N \otimes T^*N$-valued process B over $\phi \circ X$ we have:

$$\int (\phi^* B)\,(dX, dX) = \int B\big(d(\phi(X)), d(\phi(X))\big).$$

(ii) For a continuous adapted T^*N-valued process J over $\phi \circ X$ we have:

$$\int_X \phi^* J = \int_{\phi(X)} J.$$

REMARK 1.3.18. Under a *complex differential form* α on a differentiable manifold M we understand a section $\alpha \in \Gamma(T^*M \otimes \mathbb{C})$. Decomposing α into its real and imaginary part, i.e., $\alpha = \alpha_1 + i\alpha_2$ where $\alpha_i \in \Gamma(T^*M)$ are real differential forms on M, we extend the Stratonovich integral of differential forms along M-valued semimartingales via

$$\int_X \alpha := \int_X \alpha_1 + i \int_X \alpha_2$$

to complex differential forms.

As an example for Stratonovich integration of one-forms we consider the winding of semimartingales in the plane. This notion generalizes the classical winding number of a (closed) differentiable curve in $\mathbb{C} \setminus \{0\}$, as defined in elementary function theory, to semimartingales in the plane. We identify the complex plane $\mathbb{C}$ with the Euclidean space $\mathbb{R}^2$.

REMARK 1.3.19 (Winding of a semimartingale in the plane). Let Z be a continuous $\mathbb{C}$-valued semimartingale such that $Z_0 \neq 0$ and Z does not hit the origin almost surely. Integration of the complex differential form $\alpha = dz/z$ on $\mathbb{C} \setminus \{0\}$ along Z,

$$\int_Z \alpha = \int \frac{1}{Z} \circ dZ \in \mathscr{S} + i\mathscr{S},$$

gives a continuous version of a logarithm along the paths of Z via

$$\log_\omega\big(Z_t(\omega)\big) - \log_\omega\big(Z_0(\omega)\big) := \Big(\int_Z \frac{dz}{z}\Big)_t(\omega), \quad t \geq 0, \quad \mathbb{P}\text{-almost all } \omega \in \Omega.$$

In other words, writing

$$Z_t \equiv |Z_t| e^{i\Theta_t}, \quad t \geq 0,$$

with a (pathwise) continuous version Θ_t of the argument of Z_t, then

$$\Theta_t = \Theta_0 + \operatorname{Im}\Big(\int_Z \frac{dz}{z}\Big)_t.$$

The process $\operatorname{Im}\int_Z \frac{dz}{z}$ is called *winding* of the semimartingale Z about the origin.

PROOF. It is sufficient to verify that, modulo indistinguishability,

$$\exp\Big(\int_0 \frac{1}{Z} \circ dZ\Big) = \frac{Z}{Z_0}.$$

But using the abbreviation $L := \int_Z dz/z \equiv \int Z^{-1} \circ dZ$, then

$$de^L = e^L \circ dL = (e^L/Z) \circ dZ,$$

and hence

$$d\Big(\frac{e^L}{Z}\Big) = e^L\Big(-\frac{1}{Z^2}\Big) \circ dZ + \frac{1}{Z}\Big(\frac{e^L}{Z}\Big) \circ dZ = 0. \qquad \square$$

In the sequel let $\mathscr{M}(\mathbb{C})$ denote the class of $\mathbb{C}$-valued local martingales. A local martingale $Z = X + iY \in \mathscr{M}(\mathbb{C})$ is said to be *conformal* if $[X, X] = [Y, Y]$ and $[X, Y] = 0$, or equivalently, if $dZ\,dZ = 0$.

REMARK 1.3.20. Stratonovich integrals of holomorphic differential forms along conformal martingales give local martingales. More precisely: Let Z be a conformal local martingale and $D \subset \mathbb{C}$ be a domain not left by Z a.s. For any complex differential form $\alpha = f(z)\,dz$ on D (where $f\colon D \to \mathbb{C}$ is a holomorphic function) the process

$$\int_Z \alpha \equiv \int f(Z) \circ dZ = \int f(Z)\,dZ \in \mathscr{M}(\mathbb{C})$$

is a conformal local martingale. On the other hand, a local martingale Z in $\mathbb{C}$ is already a conformal local martingale if $\int_Z \alpha \in \mathscr{M}(\mathbb{C})$ for $\alpha = z\,dz$.

PROOF. Indeed we have

$$f(Z)\circ dZ = f(Z)\,dZ + \tfrac{1}{2}\big(f'(Z)\big)\,dZdZ = f(Z)\,dZ \tag{1.3.11}$$

where the first equality in (1.3.11) results from the Itô formula for complex semimartingales (e.g. [**16**] Corollary to Theorem $4.46'$), whereas the second equality is a consequence of the conformity of Z. In addition local martingales of the type $N = \int f(Z)\,dZ$ are automatically conformal, since $dNdN = f(Z)^2\,dZdZ = 0$. The last statement follows with $f = \mathrm{id}$. □

In particular, if in the situation of Remark 1.3.20 the conformal local martingale Z is a Brownian motion on $\mathbb{C}$, then for each holomorphic function f the process $\int_Z f(z)\,dz$ is a conformal local martingale, and thus there exist independent one-dimensional Brownian motions B and β such that

$$\mathrm{Re}\int_Z f(z)\,dz = B_{T_t}, \quad \mathrm{Im}\int_Z f(z)\,dz = \beta_{T_t},$$

where the time-change is given by $T_t := \int_0^t \big(|f|^2\circ Z_s\big)\,ds$, $t\geq 0$. If $f\not\equiv 0$ then $T_\infty \equiv \infty$ $\mathbb{P}$-a.s., as is easily verified by using recurrence and the strong Markov property of the 2-dimensional Brownian motion:

LEMMA 1.3.21. *Let Z be a Brownian motion on $\mathbb{R}^2$ and $f\colon \mathbb{R}^2\to\mathbb{R}$ be a continuous, not identically vanishing function. Then $\int_0^\infty |f|^2(Z_s)\,ds \equiv \infty$, $\mathbb{P}$-a.s.*

We are going to summarize the results above in the case $f(z)\,dz = dz/z$.

COROLLARY 1.3.22. *Let $Z = X+iY$ be a BM in $\mathbb{C}$ starting from some point $z_0\neq 0$. Then $Z_t = |Z_t|\,e^{i\Theta_t}$ where*

$$\log|Z_t| - \log|Z_0| = \mathrm{Re}\int_0^t \frac{dZ}{Z} = \int_0^t \frac{X\,dX + Y\,dY}{|Z|^2}$$

$$\Theta_t - \Theta_0 = \mathrm{Im}\int_0^t \frac{dZ}{Z} = \int_0^t \frac{X\,dY - Y\,dX}{|Z|^2}.$$

In addition there exist independent one-dimensional Brownian motions B and β such that

$$\int_0^t \frac{dZ}{Z} = B_{T_t} + i\beta_{T_t}$$

with the time-change T_t given by $T_t := \int_0^t |Z_s|^{-2}\,ds$.

Since $T_\infty = \infty$ $\mathbb{P}$-a.s., one concludes from $\Theta_t - \Theta_0 = \beta_{T_t}$ that BM($\mathbb{C}$) winds with probability 1 arbitrary often clockwise and anti-clockwise about any given point, but unwinds again almost surely. On the other hand, $|Z|$ and B generate the same σ-algebra, hence

$$\mathscr{B}_\infty := \sigma\{|Z_s|\colon s\in\mathbb{R}_+\} = \sigma\{B_s\colon s\in\mathbb{R}_+\} \quad \text{modulo } \mathbb{P}\text{-nullsets};$$

indeed first of all $\sigma\{|Z_s|\colon s\leq t\} = \sigma\{\log|Z_s|\colon s\leq t\} = \sigma\{B_{T_s}\colon s\leq t\}$; on the other hand, the time-change $(T_t)_{t\geq 0}$ may be described in terms of B, as is seen from the formula

$$T_t = \inf\Big\{s\geq 0\colon |z_0|^2\int_0^s \exp(2B_r)\,dr > t\Big\}, \tag{1.3.12}$$

which is easily verified with the substitution $r = T_u$. As a consequence, the BM β describing the angular process is independent of the whole radial process, and hence independent of $\mathscr{B}_\infty \equiv \sigma\{|Z_s|\colon s \in \mathbb{R}_+\}$ and in particular of the time-change $(T_t)_{t\geq 0}$. Thus for any $\xi \in \mathbb{R}$:

$$\mathbb{E}^{\mathscr{B}_\infty}\big[\exp\big(i\xi(\Theta_t - \Theta_0)\big)\big] = \exp(-\xi^2/2\,T_t) \quad \mathbb{P}\text{-a.s.}$$

This formula allows to calculate the distribution of Θ_t for fixed t, and is moreover a useful tool for many explicit calculations related to the stochastic behaviour of BM in the plane (e.g. [**48**], [**49**]).

1.4. Linear Connections and Martingales on Manifolds

The aim of this section is to introduce martingales on manifolds. This task requires on the manifold a linear connection as additional geometric structure. We start the discussion by recalling basic notions from differential geometry; for more background on these topics the reader may consult [**13, 25, 26, 27**].

From a geometrical point of view we want to deal with the following situation. Let $\pi\colon E \to M$ be a vector bundle over a manifold M, for instance the tangent bundle TM of M, and let $\alpha\colon [0,1] \to M$ be a differentiable curve such that $\alpha(0) = p$ and $\alpha(1) = q$. We look for a canonical procedure to translate vectors $v \in E_p$ to E_q along the curve α.

If in addition E is endowed with a metric, in the sense that each fiber E_x carries a scalar product depending smoothly on x, then it is natural to demand in addition that angles are preserved by the translation along curves.

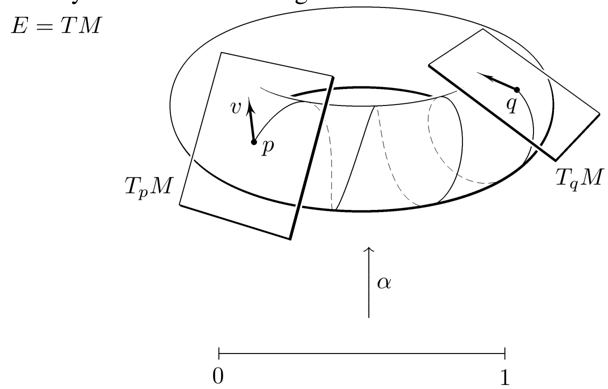


Figure 1.4.1. Parallel transport

The fibers of a vector bundle are all isomorphic to a fixed finite-dimensional vector space which however does not mean that there is a canonical way to identify them. The additional structure needed to relate fibers among each other in an intrinsic way is a "linear connection" in E. Such a structure encodes the information necessary to transport elements of one fiber of E along some curve to another fiber.

There are different (but equivalent) ways to introduce linear connections in a vector bundle E, for instance, as parallel transport, as covariant derivative, or horizontal splitting of TE. The most intuitive way is the concept of a parallel transport.

DEFINITION 1.4.1 (Parallel transport). Let $\pi\colon E \to M$ be a vector bundle over a differentiable manifold M. A *parallel transport* L in E is an assignment of a linear isomorphism $L_\alpha\colon E_p \to E_q$ to each differentiable path α from p to q in M such that the following properties hold:

(i) (Invariance under reparametrization) If $\alpha\colon [a,b] \to M$ is a differentiable curve then $L_{\alpha\circ\varphi} = L_\alpha$ for any differentiable reparametrization $\varphi\colon [a',b'] \to [a,b]$ such that $\varphi(a') = a$ and $\varphi(b') = b$.

(ii) (Transitivity) If $\alpha\colon [a,b] \to M$ and $a \le c \le b$ then $L_\alpha = L_{\alpha|[c,b]} \circ L_{\alpha|[a,c]}$.

(iii) (Behaviour under back-transport) $L_{\alpha^-} = L_\alpha^{-1}$ for $\alpha^-\colon [a,b] \to M$, $t \mapsto \alpha(a+b-t)$.

(iv) (Dependence on parameters) If α depends differentiably on parameters (e.g. if α is a differentiable family of curves), then L_α depends differentiably on these parameters as well.

(v) (First-Order-Axiom) For any $X \in \Gamma(E)$ and $v \in T_pM$ the *covariant derivative* $\nabla_v X$ *of* X *in direction* v,
$$\nabla_v X := \nabla_D(X\circ\alpha)(0) \in E_p \quad \text{for } \alpha\colon [-\varepsilon,\varepsilon] \to M\ C^\infty\text{-curve with } \alpha(0) = p \text{ and } \dot\alpha(0) = v,$$
is well-defined and independent of the choice of the curve α.

In (v) we use the following notion: for a differentiable curve $\alpha\colon [a,b] \to M$ and a C^∞ section $\sigma \in \Gamma(\alpha^*E)$, the *covariant derivative* $\nabla_D\sigma \in \Gamma(\alpha^*E)$ *of* σ *along* α with respect to L is defined as
$$(\nabla_D\sigma)(t) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L^{-1}_{\alpha|[t,t+\varepsilon]}\sigma(t+\varepsilon) \in E_{\alpha(t)}$$
if well-defined.

Before introducing the abstract notion of a covariant derivative on a vector bundle, we state the following Lemma.

LEMMA 1.4.2. *Let* E, F *be vector bundles over* M*, further* $K\colon \Gamma(E) \to \Gamma(F)$ *a* $C^\infty(M)$*-linear map and* $p \in M$*. Then* $K(A)_p = K(B)_p$ *for all sections* $A, B \in \Gamma(E)$ *with* $A_p = B_p$*. Thus* K *provides a section of the bundle* $\mathrm{Hom}(E,F) \cong E^* \otimes F$.

PROOF. It is sufficient to show that $A_p = 0$ already implies $K(A)_p = 0$. Fixing a local frame $e_1,\dots,e_m \in \Gamma(E/U)$ at p there exist uniquely determined functions $a^1,\dots,a^m \in C^\infty(U)$ such that $A|U = \sum_i a^i e_i$. In particular, we have $a^1(p) = \dots = a^m(p) = 0$. Now let $\psi \in C^\infty(M)$ such that $\psi(p) = 1$ and $\operatorname{supp}\psi \subset U$. In particular, $\bar e_i := \psi e_i \in \Gamma(E/U)$ and $\bar a^i := \psi a^i \in C^\infty(U)$ extend smoothly to global sections, resp. functions on M (being equal to 0 outside of U). Then $\psi^2 A = \sum_i \bar a^i \bar e_i$, and thus $K(A)_p = \psi(p)^2 K(A)_p = K(\psi^2 A)_p = \sum_i \bar a^i(p)\, K(\bar e_i)_p = 0$. □

DEFINITION 1.4.3 (Covariant derivative). Let E be a vector bundle over a manifold M. A *covariant derivative on* E is an $\mathbb{R}$-linear mapping
$$\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$$
satisfying the product rule
$$\nabla(fX) = df \otimes X + f\,\nabla X, \quad X \in \Gamma(E),\ f \in C^\infty(M).$$
Sections $X \in \Gamma(E)$ with the property that $\nabla X = 0$ are called *parallel*.

REMARK 1.4.4. Since according to Lemma 1.4.2,
$$\Gamma(T^*M \otimes E) \cong \mathrm{Hom}_{C^\infty(M)}\big(\Gamma(TM), \Gamma(E)\big),$$
a covariant derivative ∇ on E can equally be seen as $\mathbb{R}$-bilinear mapping
$$\Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (A,X) \mapsto \nabla_A X := (\nabla X)A.$$

In this notation a covariant derivative is $C^\infty(M)$-linear in the first argument and (as a consequence of the product rule) derivative in the second argument, i.e.,

(i) $\nabla_{fA}X = f\,\nabla_A X$;
(ii) $\nabla_A(fX) = (Af)X + f\,\nabla_A X$,

for all $A \in \Gamma(TM)$, $X \in \Gamma(E)$ and $f \in C^\infty(M)$.

REMARK 1.4.5. (i) Given $A \in \Gamma(TM)$, $X \in \Gamma(E)$ and $p \in M$, by Lemma 1.4.2, $(\nabla_A X)_p$ depends only on $A_p \in T_pM$. Hence, for $v \in T_pM$ choosing $A \in \Gamma(TM)$ such that $A_p = v$, then $\nabla_v X := (\nabla_A X)_p \in E_p$ is well-defined and is called *covariant derivative of X in direction v*.

(ii) For $v \in T_pM$, $X \in \Gamma(E)$ the covariant derivative $\nabla_v X$ depends only on the germ of X at p.

PROOF. For $p \in U \subset M$ open and $X|U \equiv 0$ we have to show that $\nabla_v X = 0$. To this end let $\psi \in C^\infty(M)$ such that $\operatorname{supp}\psi \subset U$ and $\psi(p) = 1$. But then $\psi X \equiv 0$, and thus $0 = \nabla_v(\psi X) = v(\psi)X_p + \psi(p)\nabla_v X = \nabla_v X$. $\square$

NOTATION 1.4.6 (Christoffel symbols). Let E be a vector bundle of rank m over an n-dimensional manifold M. Let ∇ be a covariant derivative on E and $e_1, \dots, e_m \in \Gamma(E/U)$ a local frame for E. If in addition (h, U) is a local chart for M and $\partial_i = \frac{\partial}{\partial h^i} \in \Gamma(TM/U)$ the corresponding local coordinate vector fields, then $(\partial_1, \dots, \partial_n)$ defines a local frame for TM, and the sections $\nabla_{\partial_i} e_j \in \Gamma(E/U)$ are well-defined by Remark 1.4.5. The uniquely determined functions $\Gamma^k_{ij} \in C^\infty(U)$ such that

$$\nabla_{\partial_i} e_j = \sum_{k=1}^m \Gamma^k_{ij} e_k$$

are called the *Christoffel symbols* of ∇ with respect to (h, U) and $e_1, \dots, e_m \in \Gamma(E/U)$. They determine the covariant derivative ∇ on E/U.

A covariant derivative on a vector bundle E in the sense of Definition 1.4.3 induces canonically a notion of covariant derivative of sections along maps. For a precise statement we come back to the notion of induced forms and frames as introduced in Remark 1.3.7.

DEFINITION 1.4.7 (Induced covariant derivative). Let $f\colon M \to N$ be a differentiable map between manifolds and ∇ a covariant derivative on a vector bundle E over N. There exists exactly one covariant derivative on the induced bundle f^*E over M (called *the covariant derivative on f^*E induced by f* and denoted again by ∇) such that

$$\nabla_w(f^*X) = \nabla_{df_p w} X \in E_{f(p)}, \quad X \in \Gamma(E),\ w \in T_pM,\ p \in M. \tag{1.4.1}$$

Indeed, let $e_1, \dots, e_m \in \Gamma(E/U)$ be a local frame for E and $Y \in \Gamma(f^*E)$ a global section. By Remark 1.3.7 (ii), Y has a unique representation on $f^{-1}(U)$ of the form $Y|f^{-1}(U) = \sum_i b^i f^* e_i$ where $b^i \in C^\infty(f^{-1}(U))$. For $w \in T_pM$, $p \in f^{-1}(U)$ we deduce from the product rule and property (1.4.1) of ∇ that

$$\nabla_w Y = \sum_{i=1}^m \Big(w(b^i)\,(e_i)_{f(p)} + b^i(p) \nabla_{df_p w} e_i \Big) \in E_{f(p)}. \tag{1.4.2}$$

This shows uniqueness of the induced covariant derivative. On the other hand, Eq. (1.4.2) defines a covariant derivative on f^*E which establishes existence.

If X is a section of E along a curve α on M, then Definition 1.4.7 gives in particular a notion of a covariant derivative X along α.

DEFINITION 1.4.8. Let ∇ be a covariant derivative on a vector bundle E over M and $\alpha\colon I \to M$ a differentiable curve defined on some real interval.

(i) (Covariant derivative for sections along curves) For sections $X \in \Gamma(\alpha^*E)$ along α the vector field $\nabla_D X \in \Gamma(\alpha^*E)$ is called the *covariant derivative of X along α*; here D denotes the canonical vector field on I.

(ii) (Parallel sections along curves) A section $X \in \Gamma(\alpha^*E)$ along α is said to be *parallel along* α (with respect to ∇) if $\nabla_D X = 0$. The linear subspace of $\Gamma(\alpha^*E)$ of parallel sections along α is denoted $\Gamma_{\mathrm{par}}(\alpha^*E)$.

DEFINITION 1.4.9 (Geodesics). Let M be a manifold and ∇ a covariant derivative on TM. A differentiable curve $\gamma\colon I \to M$ is said to be a *geodesic* if $\dot\gamma \in \Gamma(\gamma^*TM)$ is parallel along γ with respect to ∇, in other words, if $\nabla_D\dot\gamma = 0$.

REMARK 1.4.10 (Covariant derivative in coordinates). Let ∇ be a covariant derivative on a vector bundle E over M and $e_1, \ldots, e_m \in \Gamma(E/U)$ be a local frame for E. Let (h, U) be a local chart for M and $\partial_i = \frac{\partial}{\partial h^i}$ for $i = 1, \ldots, n$. Then

$$\nabla_{\partial_i} e_j = \sum_k \Gamma_{ij}^k e_k \quad \text{locally on } U.$$

We consider a section $X \in \Gamma(\alpha^*E)$ along a differentiable curve $\alpha\colon I \to M$. Fixing $t_0 \in I$ such that $\alpha(t_0) \in U$, then $X = \sum_{j=1}^m X^j\,\alpha^*e_j$ locally about t_0. By Definition 1.4.7 we get for t locally about t_0 (since $\dot\alpha(t) = \sum_{i=1}^n \dot\alpha^i(t)\,(\partial_i)_{\alpha(t)}$):

$$\begin{aligned}\big(\nabla_D X\big)(t) &= \sum_{j=1}^m \Big(\dot X^j(t)\,(e_j)_{\alpha(t)} + X^j(t)\nabla_{\dot\alpha(t)} e_j\Big)\\ &= \sum_{j=1}^m \Big(\dot X^j(t)\,(e_j)_{\alpha(t)} + \sum_{i=1}^n X^j(t)\,\dot\alpha^i(t)\,\nabla_{(\partial_i)_{\alpha(t)}} e_j\Big).\end{aligned}$$

Thus locally about t_0:

$$\nabla_D X = \sum_k \Big(D(X^k) + \sum_{i,j} X^j\,D(\alpha^i)\,(\Gamma_{ij}^k\circ\alpha)\Big)\;\alpha^*e_k. \tag{1.4.3}$$

THEOREM 1.4.11. *Let ∇ be a covariant derivative on a vector bundle E over M, further let $\alpha\colon I \to M$ be a differentiable curve, $t_0 \in I$ and $e \in E_{\alpha(t_0)}$. There exists exactly one section $X \in \Gamma_{\mathrm{par}}(\alpha^*E)$ along α such that $X(t_0) = e$.*

PROOF. The claim is reduced to the existence and uniqueness theorem for linear differential equations. Since it is sufficient to consider the local situation, we may assume the existence of a global chart (h, M) for M and a global frame $e_1, \ldots, e_m \in \Gamma(E)$ for E. Then there are uniquely determined coefficients $b^i \in \mathbb{R}$ such that $e = \sum_{i=1}^m b^i e_i$. Defining $c_{kj} := -\sum_{i=1}^n \dot\alpha^i\,(\Gamma_{ij}^k\circ\alpha) \in C^\infty(I)$, by Eq. (1.4.3) the requirement $\nabla_D X = 0$ together with $X_{t_0} = e$ is seen to be equivalent to the system of linear differential equations

$$\dot X^k = -\sum\nolimits_j c_{kj}X^j, \quad X^k(t_0) = b^k, \quad k = 1, \ldots, m. \tag{1.4.4}$$

It remains to recall that the unique solution to Eq. (1.4.4) is defined on all of I. □

DEFINITION 1.4.12 (Parallel transport; induced by ∇). Let ∇ be a covariant derivative on a vector bundle E over a manifold M and $\alpha\colon I \to M$ a differentiable curve. For $s, t \in I$ there is an isomorphism

$$/\!/_{s,t}\colon E_{\alpha(s)} \to E_{\alpha(t)}$$

explained by $/\!/_{s,t}e := X(t)$ where $X \in \Gamma_{\mathrm{par}}(\alpha^* E)$ is the unique parallel section along α such that $X(s) = e$. The isomorphism $/\!/_{s,t}$ is called the *parallel transport of* $E_{\alpha(s)}$ *to* $E_{\alpha(t)}$ *along* α.

REMARK 1.4.13. We have $/\!/_{s,t}^{-1} = /\!/_{t,s}$ and $/\!/_{t,t} = \mathrm{id}_{E_{\alpha(t)}}$. Each basis $e_1, \dots, e_m$ of $E_{\alpha(s)}$ can be extended to a global frame $\bar e_1, \dots, \bar e_m \in \Gamma(\alpha^* E)$ for $\alpha^* E$ via $\bar e_{i,t} := /\!/_{s,t} e_i$.

REMARK 1.4.14. The parallel transport associated to a covariant derivative ∇ according to Definition 1.4.12 defines a parallel transport in E in the sense of Definition 1.4.1. On the other hand the parallel transport determines again the underlying covariant derivative: If $X \in \Gamma(E)$, $v \in T_pM$ and $\alpha\colon I \to M$ a differentiable curve such that $\dot\alpha(0) = v$, then

$$\nabla_v X = \tfrac{d}{dt}\big|_{t=0} \,(/\!/_{0,t}^{-1} X_{\alpha(t)}) \in E_p.$$

PROOF. Let $e_1, \dots, e_m$ be a basis of E_p and $\bar e_1, \dots, \bar e_m \in \Gamma(\alpha^* E)$, $\bar e_{i,t} := /\!/_{0,t} e_i$, an extension to a global frame for $\alpha^* E$. Furthermore let $a^i \in C^\infty(I)$ be such that $\alpha^* X = \sum a^i \bar e_i$. Then $/\!/_{0,t}^{-1}(\alpha^* X)_t = \sum a^i(t) e_i$, and hence $\nabla_v X = \nabla_D(\alpha^* X)_0 = \sum \big(\dot a^i(0) e_i + a^i(0)(\nabla_D \bar e_i)_0\big) = \tfrac{d}{dt}\big|_{t=0} \,(/\!/_{0,t}^{-1} X_{\alpha(t)})$. □

Thus a parallel transport in E and a covariant derivation on E provide identical structures on E. We continue with a third equivalent point of view.

DEFINITION 1.4.15 (Horizontal splitting of TE). Let $\pi\colon E \to M$ be a vector bundle over a manifold M. A subbundle $H \subset TE$ is said to be a *horizontal splitting of* TE if the following two conditions hold:

(i) $TE = H \oplus \pi^* E$ (this is $T_eE = H_e \oplus E_{\pi(e)}$ for $e \in E$)

(ii) For $s \in \mathbb{R}^* \equiv \mathbb{R} \setminus \{0\}$ the subbundle H is compatible with the operation $\rho_s\colon E \to E$, $e \mapsto se$, in the sense that $(\rho_s)_* H_e = H_{se}$ for $e \in E$ and $s \in \mathbb{R}^*$.

REMARK 1.4.16. Let $\pi\colon E \to M$ be a vector bundle over M and $e \in E$ such that $p = \pi(e)$. The projection $\pi\colon E \to M$ is *submersive* at e, i.e. $(d\pi)_e\colon T_eE \to T_pM$ is surjective, with $\ker (d\pi)_e = T_e(\pi^{-1}p) = T_e(E_p) \cong E_p \subset T_eE$. Thus there is an exact sequence of vector bundles over E

$$\begin{tikzcd} 0 \arrow[r] & \pi^* E \arrow[r] & TE \arrow[r, "d\pi"] & \pi^* TM \arrow[r] \arrow[l, bend left, dashed, "h"] & 0\,, \end{tikzcd} \tag{1.4.5}$$

and the decomposition $TE = H \oplus \pi^* E$ induces a splitting of (1.4.5): $d\pi \circ h = \mathrm{id}$ where $h = (d\pi | H)^{-1}$. The differentiable splitting $h\colon \pi^* TM \xrightarrow{\sim} H \subset TE$ of the sequence (1.4.5) of vector bundles over M is called *horizontal lift*. Fiberwise, we have linear isomorphisms $(d\pi)_e | H_e\colon H_e \xrightarrow{\sim} T_pM$ and $h_e\colon T_pM \xrightarrow{\sim} H_e$.

NOTATION 1.4.17. Let $TE = H \oplus V$ with $V := \pi^* E$ be a horizontal splitting of TE; further let $w \in T_eE$ where $e \in E$ and $p = \pi(e) \in M$. We call w *horizontal* if $w \in H_e$, and *vertical* (in the sense "tangential to the submanifold E_p of E") if $w \in V_e \equiv E_p$.

In each fiber T_eE of TE the vertical space V_e is canonically given, however in general there is no canonical choice of a horizontal space H_e: a horizontal splitting provides exactly a selection of a horizontal complement H_e to V_e at each $e \in E$ in an $\mathbb{R}^*$-invariant way (differentiable depending on e).

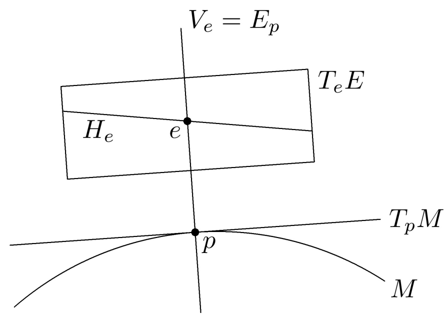


Figure 1.4.2. Horizontal splitting

THEOREM 1.4.18. *Horizontal splittings of TE and covariant derivatives on E are equivalent structures. For any vector bundle $\pi\colon E \to M$ over M, the following holds:*

(i) *A covariant derivative ∇ on E defines canonically a horizontal splitting H of TE, namely for each $e \in E$ with $p := \pi(e)$ via*

$$H_e := \left\{X_* v : v \in T_pM,\ X \in \Gamma(E) \text{ with } X(p) = e \text{ and } \nabla_v X = 0\right\} \subset T_eE.$$

(ii) *Inversely to* (i) *a horizontal splitting H of TE gives rise to a covariant derivation ∇ on E as follows:*

  (a) *For $X \in \Gamma(E)$ the covariant derivative $\nabla X \in \Gamma(T^*M \otimes E)$ is explained through the following homomorphism of vector bundles over M*

$$TM \xrightarrow{X_*} X^*TE \equiv X^*H \oplus X^*V \xrightarrow{\mathrm{pr}_V} X^*V = X^*\pi^*E = E,$$

  where pr_V denotes the projection onto the vertical subspace.

  (b) *For $\sigma \in \Gamma(\alpha^*E)$ where α is a differentiable curve in M the covariant derivative $\nabla_D\sigma \in \Gamma(\alpha^*E)$ is given as follows: $(\nabla_D\sigma)(t_0)$ is the image of $\big(\frac{\partial}{\partial t}\big)_{t_0}$ under*

$$\mathbb{R} = T\mathbb{R} \xrightarrow{\sigma_*} \sigma^*TE \equiv \sigma^*H \oplus \sigma^*V \xrightarrow{\mathrm{pr}_V} \sigma^*V = \sigma^*\pi^*E = E;$$

  here $\big(\frac{\partial}{\partial t}\big)_{t_0}$ is first mapped to $\dot\sigma(t_0)$ and then projected on the vertical component.

The constructions in (i) *and* (ii) *are inverse to each other.*

PROOF. (i) First of all, H_e as defined in (i) is a vector space. Indeed, to each $v \in T_pM$ there exists exactly one $w \in H_e$ such that $\pi_* w = v$, in other words, if $v \in T_pM$ and $X, \tilde X \in \Gamma(E)$ such that $X(p) = \tilde X(p) = e$ and $\nabla_v X = \nabla_v \tilde X = 0$, then $X_* v = \tilde X_* v$. Since this is a local statement at p, it is sufficient to consider the situation $E = U \times \mathbb{R}^m$ with $p \in U \subset \mathbb{R}^n$, where then $X\colon U \to \mathbb{R}^m$, $\tilde X\colon U \to \mathbb{R}^m$ and $\tilde X = A\,X$ for some differentiable map $A\colon U \to \mathrm{GL}(m;\mathbb{R})$, $A(p) =$ identity matrix. For $v \in T_pM$, one obtains from $\nabla_v \tilde X = 0$ together with $\nabla_v X = 0$ the equation $v(A)X_p = 0$ where v in $v(A)$ is applied as derivation componentwise to the matrix function A. This shows, as claimed, $d(AX)_p v = v(A)\,X_p + A(p)\,v(X) = A(p)\,v(X) = v(X) = (dX)_p v$. In particular, this shows

$$H_e = X_* T_pM = (dX)_p T_pM \tag{1.4.6}$$

in terms of a fixed section $X \in \Gamma(E)$ such that

$$X_p = e \quad \text{and} \quad (\nabla X)_p = 0 \tag{1.4.7}$$

from where the vector space structure of H_e is obvious. Existence of a section X with property (1.4.7) is immediate: it is sufficient to construct X locally about p and to extend it then to a smooth global section but locally in coordinates about p condition (1.4.7) reduces to find a function with prescribed 1-jet at the single point p.

Injectivity of $(dX)_p$ follows from $\pi_*(dX)_p = \mathrm{id}\,|T_pM$ and implies in particular $\dim H_e = \dim M$. Also $H_e \cap V_e = \{0\}$ is obvious since $w \in H_e$, say $w = X_*v$, implies $\pi_* w = (\pi \circ X)_* v = v$ whereas $w \in V_e$ just means that $\pi_* w = 0$. This proves $T_eE = H_e \oplus V_e$.

It remains to check that H defines a subbundle of TE, i.e., that H_e depends differentiably on $e \in E$. To this end, we fix a local chart (h, U) for M and assume without restriction of generality that $E \cong U \times \mathbb{R}^m$. If $\partial_i = \frac{\partial}{\partial h^i}$ is one of the basis vector fields over U and $X \in \Gamma(E)$ a non-vanishing section on U, i.e., $X : U \to \mathbb{R}^m$ differentiable and $X(p) \neq 0$ for all $p \in U$, then there exists a C^∞ function $A\colon U \to \mathrm{GL}(m;\mathbb{R})$ such that

$$\nabla_{\partial_i}(AX) = 0 \ \text{ on } U. \tag{1.4.8}$$

Note that condition (1.4.8) is equivalent to

$$\partial_i(A)X + A\,\nabla_{\partial_i} X = 0 \tag{1.4.9}$$

which gives a differential equation for A. For fixed $p \in U$ and $g \in \mathrm{GL}(m;\mathbb{R})$ let now $A = A_{i,p,g}\colon U \to \mathrm{GL}(m;\mathbb{R})$ denote the solution to (1.4.9) satisfying $A(p) = gX(p)$. Furthermore, choose for each $e \in E$ a matrix $g(e) \in \mathrm{GL}(m;\mathbb{R})$ depending differentiably on e such that $e = g(e)X_{\pi(e)}$. This construction gives to each $e \in E$ vector fields

$$X_i^{(e)} := A_{i,\pi(e),g(e)}\,X \in \Gamma(TU), \quad 1 \leq i \leq n,$$

and induced vector fields on E, namely

$$\bar\partial_i \in \Gamma(E), \quad (\bar\partial_i)_e := d(X_i^{(e)})_{\pi(e)}\,(\partial_i)_{\pi(e)}, \quad 1 \leq i \leq n,$$

such that $\big((\bar\partial_1)_e, \dots, (\bar\partial_n)_e\big)$ gives a basis for H_e for each $e \in E$.

Finally it is easy to see that H is compatible with the operation $\mathbb{R}^*$ which completes the proof of part (i) of Theorem 1.4.18.

(ii) The second part can be checked in an elementary way; verification of the product rule requires the $\mathbb{R}^*$-invariance of H (condition (ii) in Definition 1.4.15). □

According to Theorem 1.4.18 (ii) a section $X \in \Gamma(E)$ is parallel (i.e., $\nabla_v X$=0 for all $v \in TM$) if and only if X_*v is horizontal for any $v \in TM$. In the same way, a section $\sigma \in \Gamma(\alpha^*E)$ along $\alpha\colon I \to M$ is parallel (i.e., $\nabla_D \sigma = 0$) if and only if $\dot\sigma(t) \in H_{\sigma(t)}$ for all $t \in I$. Hence, as consequence of Theorem 1.4.11, we have the following result.

THEOREM 1.4.19. *Let $\pi\colon E \to M$ be a vector bundle over a differentiable manifold M and H a horizontal splitting of TE. Furthermore let $\alpha\colon I \to M$ be a differentiable curve and $e \in E_{\alpha(t_0)}$ for some $t_0 \in I$. Then there exists exactly one lift of α to a "horizontal curve" $u\colon I \to E$ above α with $u(t_0) = e$, i.e. such that $\pi \circ u = \alpha$, $u(t_0) = e$ and $\dot u(t) \in H_{u(t)}$ for $t \in I$.*

DEFINITION 1.4.20 (Linear connection). Let $\pi\colon E \to M$ be a vector bundle over a differentiable manifold M. A *linear connection in E* is a covariant derivative on E (or equivalently, a parallel transport in E or a horizontal splitting of TE). Linear connections in TM are simply called *linear connections on M*.

Let M be an n-dimensional manifold equipped with a linear connection ∇ in TM. By Definition 1.4.9, geodesics are curves with the property that their velocity field along the curve is parallel. According to Remark 1.4.10, in local coordinates (h, U), for a differentiable curve $\gamma\colon I \to M$ the condition $\nabla_D \dot\gamma = 0$ means that

$$\ddot\gamma^k + \sum_{i,j} (\Gamma^k_{ij} \circ \gamma)\, \dot\gamma^i\, \dot\gamma^j = 0, \quad k = 1, \dots, n, \tag{1.4.10}$$

with $\gamma^k = h^k \circ \gamma$ and Γ^k_{ij} the Christoffel symbols determined by $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij}\, \partial_k$. According to Theorem 1.4.18, condition $\nabla_D \dot\gamma = 0$ is equivalent to $\ddot\gamma(t) \in H_{\dot\gamma(t)}$ for each $t \in I$. The horizontal lift $h\colon \pi^* TM \xrightarrow{\sim} H \subset TTM$ induced by ∇ according to diagram (1.4.5), defines a canonical (horizontal) vector field ξ on TM, namely for $v \in TM$ via

$$T_{\pi(v)} M \xrightarrow{h_v} H_v \hookrightarrow T_v TM, \quad v \longmapsto \xi(v).$$

Obviously, ξ is not only a *second order differential equation* (i.e., a vector field on TM such that $(d\pi)_v\, \xi(v) = v$ for $v \in TM$), it is even a *spray* which means that in addition $\xi(sv) = (d\varrho_s)\big(s\xi(v)\big)$ holds for all $s \in \mathbb{R}^*$ and the multiplication $\varrho_s\colon TM \to TM$, $v \mapsto sv$. The vector field ξ on TM is called the *geodesic spray* to the linear connection ∇ on TM. In general, for a second order differential equation ξ, curves $\gamma\colon I \to M$ such that $\ddot\gamma(t) = \xi\big(\dot\gamma(t)\big)$ are called *integral curves of* ξ. Since $\pi \circ \dot\gamma = \gamma$, the relation $\pi_* \ddot\gamma = \dot\gamma$ holds trivially, so that in the case of the geodesic spray ξ:

$$\nabla_D \dot\gamma = 0 \iff \ddot\gamma(t) = \xi\big(\dot\gamma(t)\big) \quad \text{for any } t \in I.$$

Thus, a curve γ in M is a geodesic if and only if γ is an integral curve of the corresponding geodesic spray.

COROLLARY 1.4.21. *Let M be a manifold and ∇ a linear connection in TM. Given $p \in M$ and $v \in T_pM$, there exists a unique geodesic $\gamma = \gamma_v$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. In addition, $\gamma_{sv}(t) = \gamma_v(st)$ for $s, t \in \mathbb{R}$, if one of the two sides is defined.*

PROOF. In general, given a second order differential equation ξ, there exists to each $v \in T_pM$ a unique maximal integral curve γ such that $\dot\gamma(0) = v$ (in particular then $\gamma(0) = p$); this is an immediate consequence of the existence and uniqueness result for integral curves to vector fields. The condition that ξ is even a spray guarantees the addition: the integral curve to s-times the initial velocity corresponds to the original integral curve run through s-times as fast. $\square$

DEFINITION 1.4.22. Let M be a smooth manifold. A linear connection ∇ in TM is called *metrically complete* if every maximal geodesic is defined on all of $\mathbb{R}$.

DEFINITION 1.4.23 (Tensor field). Let TM and T^*M be the tangent bundle, resp., cotangent bundle of a differentiable manifold M. For $r, s \in \mathbb{N} \cup \{0\}$ the elements of $\Gamma(T^*M^{\otimes s} \otimes TM^{\otimes r})$ are called *tensor fields of type* (r, s) or (r, s)*-tensors* in short. For $s \in \mathbb{N}$, in terms of the canonical $C^\infty(M)$- linear isomorphism

$$\Gamma(T^*M^{\otimes s} \otimes TM) \cong \Gamma\big(\mathrm{Mult}_{\mathbb{R}}(TM^s; TM)\big) \cong \mathrm{Mult}_{C^\infty(M)}\big(\Gamma(TM)^s; \Gamma(TM)\big),$$

tensor fields of type $(1, s)$ correspond to $C^\infty(M)$-multilinear maps $\Gamma(TM)^s \to \Gamma(TM)$.

DEFINITION 1.4.24 (Torsion, curvature). Let M be a differentiable manifold and ∇ a linear connection on M.

(i) The map

$$T\colon \Gamma(TM)^2 \to \Gamma(TM), \quad (X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$$

is $C^\infty(M)$-bilinear and represents a (1,2)-tensor $T \in \Gamma(T^*M^{\otimes 2} \otimes TM)$, the so-called *torsion tensor* of ∇. Recall that for two vector fields $X, Y \in \Gamma(TM)$ the Lie product $[X,Y] \in \Gamma(TM)$ is defined as derivation via

$$[X,Y]f = X(Yf) - Y(Xf), \quad f \in C^\infty(M).$$

The connection ∇ is said to be *torsion-free* or *symmetric* if $T \equiv 0$.

(ii) The map

$$R\colon \Gamma(TM)^3 \to \Gamma(TM), \quad (X,Y,Z) \mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is $C^\infty(M)$-trilinear and represents a (1,3)-tensor $R \in \Gamma(T^*M^{\otimes 3} \otimes TM)$, the *curvature tensor* of the connection ∇. The tensor R may be written as $C^\infty(M)$-bilinear map

$$R\colon \Gamma(TM)^2 \to \mathrm{End}_{C^\infty(M)}\Gamma(TM) \cong \Gamma(\mathrm{End}TM)$$

and gives then a section $R \in \Gamma(T^*M^{\otimes 2} \otimes \mathrm{End}TM)$. This leads to the common notation $R(X,Y)Z \equiv R(X,Y,Z)$.

REMARK 1.4.25. A linear connection ∇ in TM is torsion-free if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all Christoffel symbols. In particular, a linear connection ∇ in TM can be "symmetrized" to a torsion-free connection by passing from ∇ to $\bar\nabla$ with the new Christoffel symbols $\bar\Gamma^k_{ij} := \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji})$.

PROOF. If (h,U) is a local chart for M and $\partial_1, \ldots, \partial_n$ the corresponding coordinate vector fields, then on U the Christoffel symbols are determined by $\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$. By the C^∞-bilinearity of T, the condition $T(\partial_i, \partial_j) = 0$ for all coordinate vector fields ∂_i, ∂_j implies already $T(X,Y)|U = 0$ for any $X, Y \in \Gamma(TM)$. Since $[\partial_i, \partial_j] \equiv 0$ we have $T(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i$ which proves the claim. ☐

REMARK 1.4.26. Let ∇ be a linear connection in TM. The symmetrized connection $\bar\nabla$ defined in Remark 1.4.25 is given by

$$\bar\nabla_X Y = \nabla_X Y - \frac{1}{2}T(X,Y) = \frac{1}{2}\big(\nabla_X Y + \nabla_Y X + [X,Y]\big), \quad X, Y \in \Gamma(TM).$$

Let $f\colon M \to N$ be a differentiable map between manifolds. Reading the curvature tensor R of a linear connection in TN as an element of $\Gamma(T^*N^{\otimes 2} \otimes \mathrm{End}TN)$ and taking into account that $f^*\mathrm{End}TN \cong \mathrm{End}(f^*TN)$, we obtain

$$f^*R \in \Gamma(T^*M^{\otimes 2} \otimes f^*\mathrm{End}TN) \cong \mathrm{Bil}_{C^\infty(M)}\big(\Gamma(TM), \Gamma(TM); \mathrm{End}_{C^\infty(M)} f^*TN\big).$$

In explicit terms, for $A, B \in \Gamma(TM)$, $Y \in \Gamma(f^*TN)$ and $p \in M$,

$$\big((f^*R)(A,B)Y\big)_p = R_{f(p)}(df_p A_p, df_p B_p, Y_p) \in T_{f(p)}N.$$

THEOREM 1.4.27 (Cartan's structural equations). *Let $f\colon M \to N$ be a differentiable map between manifolds and ∇ a linear connection in TN. Then, for $A, B \in \Gamma(TM)$, $Y \in \Gamma(f^*TN)$,*

$$(f^*T)(A,B) = \nabla_A(df B) - \nabla_B(df A) - df[A,B] \in \Gamma(f^*TN)$$
$$(f^*R)(A,B)Y = \nabla_A\nabla_B Y - \nabla_B\nabla_A Y - \nabla_{[A,B]}Y \in \Gamma(f^*TN).$$

(On the right-hand sides ∇ corresponds to the induced covariant derivative on f^*TM; see Definition 1.4.7).

PROOF. It is sufficient to verify the two equations locally. To this end, let (h, U) be a chart for N and $\partial_1, \ldots, \partial_d \in \Gamma(TN/U)$ the corresponding local frame for TN. Then also $f^*\partial_1, \ldots, f^*\partial_d \in \Gamma\big(f^*TN/f^{-1}(U)\big)$ is a local frame for f^*TN over M and on $f^{-1}(U)$ we have

$$df A = \sum A(h^i \circ f)\, f^*\partial_i, \quad df B = \sum B(h^i \circ f)\, f^*\partial_i$$
$$df[A,B] = \sum \big(A(B(h^i \circ f)) - B(A(h^i \circ f))\big)\, f^*\partial_i.$$

From this one obtains furthermore (always on $f^{-1}(U)$)

$$\nabla_A(df B) = \sum \big(A(B(h^i \circ f))\, f^*\partial_i + B(h^i \circ f)\nabla_A f^*\partial_i\big)$$
$$\nabla_B(df A) = \sum \big(B(A(h^i \circ f))\, f^*\partial_i + A(h^i \circ f)\nabla_B f^*\partial_i\big)\,.$$

On the other hand, we have $T(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i$ and hence

$$\begin{aligned}(f^*T)(A,B) &= \textstyle\sum_{i,j} A(h^i \circ f)\, B(h^j \circ f)\, f^*\big(T(\partial_i, \partial_j)\big)\\ &= \textstyle\sum_i \big(B(h^i \circ f)\nabla_A f^*\partial_i - A(h^i \circ f)\nabla_B f^*\partial_i\big)\,.\end{aligned}$$

This shows the first structural equation. The verification of the second equation is similar. □

DEFINITION 1.4.28 (Covariant derivative of a differential form). Let M be a manifold and ∇ a linear connection in TM. For a differential form $\alpha \in A^1(M) \equiv \Gamma(T^*M)$ and a vector field $A \in \Gamma(TM)$ let $\nabla_A \alpha \in A^1(M)$ be defined as

$$(\nabla_A \alpha)(B) := A(\alpha B) - \alpha(\nabla_A B), \quad B \in \Gamma(TM). \tag{1.4.11}$$

Note that $\nabla_A\alpha$ is well-defined by Lemma 1.4.2, as the right-hand side of (1.4.11) is $C^\infty(M)$-linear in B. We may write

$$\nabla\alpha \in \Gamma(T^*M \otimes T^*M) \equiv \Gamma(\mathrm{Bil}(TM, TM); \mathbb{R}), \quad \nabla\alpha(A,B) := (\nabla_A\alpha)(B).$$

DEFINITION 1.4.29 (Hessian). For $f \in C^\infty(M)$ the covariant derivative of $\alpha = df$,

$$\mathrm{Hess}(f) := \nabla df \in \Gamma(T^*M \otimes T^*M), \quad (\nabla df)(A,B) = AB\, f - (\nabla_A B)\, f,$$

is called *second fundamental form* (*Hessian*) *of* f.

REMARK 1.4.30. Let M be a manifold and ∇ a linear connection in TM. Then

$$\nabla df \in \mathrm{Bil}_{C^\infty(M)}\big(\Gamma(TM), \Gamma(TM); C^\infty(M)\big), \quad (A,B) \mapsto (\nabla df)(A,B),$$

is symmetric for each $f \in C^\infty(M)$ if and only if ∇ is *torsion-free*, i.e.,

$$T(A,B) \equiv \nabla_A B - \nabla_B A - [A,B] = 0$$

for all $A, B \in \Gamma(TM)$.

EXAMPLE 1.4.31. If $M = \mathbb{R}^n$ and ∇ the canonical connection on $\mathbb{R}^n$ defined by $\nabla_{D_i} D_j = 0$, then $(\nabla df)(D_i, D_j) = D_i D_j f$.

We now turn to a central concept of the stochastic calculus on manifolds, the notion of manifold-valued martingales.

DEFINITION 1.4.32 (∇-martingale). Let M be a manifold and ∇ be a linear connection in TM. Further let X be an M-valued semimartingale defined on some filtered probability space $\big(\Omega; \mathscr{F}; \mathbb{P}; (\mathscr{F}_t)_{t\ge 0}\big)$. Then X is called ∇*-martingale* (or simply *martingale*) if for any $f \in C^\infty(M)$:

$$d\big(f(X)\big) \overset{\mathrm{m}}{=} \tfrac{1}{2}\,(\nabla df)(dX, dX) \tag{1.4.12}$$

where $\overset{\mathrm{m}}{=}$ means equality modulo differentials of local martingales.

REMARK 1.4.33. Since $(\nabla df)(dX,dX)$ in Eq. (1.4.12) only depends on the symmetric part of ∇df, we may always assume that the linear connection ∇ is torsion-free. Symmetrization of the connection does not change the class of ∇-martingales.

A priori, martingales on M may be defined only up to some predictable stopping time. Since the concept of a martingale is invariant under time transformation (see Remark 1.3.6) and since by an appropriate time transformation infinite (or deterministic finite) lifetime can be achieved, we neglect this point in the notation.

EXAMPLE 1.4.34. In the special case of $M=\mathbb{R}^n$ equipped with the canonical linear connection ∇, we have $(\nabla df)(D_i,D_j)=D_iD_jf$, and hence ∇-martingales in the sense of Definition 1.4.32 coincide with the usual class of continuous local martingales on $\mathbb{R}^n$. Indeed, according to Itô's formula, a continuous $\mathbb{R}^n$-valued semimartingale X is a local martingale if and only if

$$d\big(f(X)\big)-\tfrac12\sum_{i,j}(D_iD_jf)(X)\,d[X^i,X^j]\in d\mathscr{M}$$

for all $f\in C^\infty(\mathbb{R}^n)$ which is exactly condition (1.4.12) of Definition 1.4.32.

REMARK 1.4.35 (∇-martingales as solutions of SDEs). Let ∇ be a linear connection on TM which without loss of generality is torsion-free. Let $A_0,A_1,\dots,A_r\in\Gamma(TM)$ and suppose that X solves the SDE

$$dX=A_0(X)\,dt+\sum_{i=1}^r A_i(X)\circ dZ^i. \tag{1.4.13}$$

Here Z may be an arbitrary continuous $\mathbb{R}^r$-valued semimartingale. Then for $f\in C^\infty(M)$ we have

$$d\big(f(X)\big)=\big(A_0f\big)(X)\,dt+\sum_{i=1}^r\big(A_if\big)(X)\,dZ^i+\frac12\sum_{i,j=1}^r\big(A_iA_jf\big)(X)\,d[Z^i,Z^j].$$

Since $(\nabla df)(A_i,A_j)=A_iA_jf-(\nabla_{A_i}A_j)f$ and since on the other hand

$$(\nabla df)(dX,dX)=\sum_{i,j=1}^r(\nabla df)(A_i,A_j)(X)\,d[Z^i,Z^j],$$

we obtain

$$\begin{aligned}&d\big(f(X)\big)-\frac12(\nabla df)(dX,dX)\\&\quad=(A_0f)(X)\,dt+\sum_{i=1}^r\big(A_if\big)(X)\,dZ^i+\frac12\sum_{i,j=1}^r\big(\nabla_{A_i}A_jf\big)(X)\,d[Z^i,Z^j].\end{aligned}$$

Denoting the drift of the semimartingale Z by Z^{drift}, we obtain that X is a ∇-martingale if and only if for any $f\in C^\infty(M)$,

$$\big(A_0f\big)(X)\,dt+\sum_{i=1}^r\big(A_if\big)(X)\,d(Z^{\text{drift}})^i+\frac12\sum_{i,j=1}^r\big(\nabla_{A_i}A_jf\big)(X)\,d[Z^i,Z^j]=0.$$

In the special case when Z is a Brownian motion on $\mathbb{R}^r$ we find that solutions X to the SDE (1.4.13) are ∇-martingales if

$$A_0=-\frac12\sum_{i=1}^r\nabla_{A_i}A_i.$$

1.5. Riemannian Metrics and Brownian Motions

The measurement of the distance between points and the length of curves on a manifold requires as additional structure a metric on the tangent bundle. Manifolds equipped with a metric are called Riemannian manifolds. Such a structure is also needed for the notion of Brownian motions on manifolds.

DEFINITION 1.5.1 (Riemannian metric). Let E be a vector bundle over M. A *Riemannian metric on E* is a section

$$g \in \Gamma(E^* \otimes E^*) \cong \Gamma\big(\mathrm{Bil}(E, E; \mathbb{R})\big) \cong \mathrm{Bil}_{C^\infty(M)}\big(\Gamma(E), \Gamma(E); C^\infty(M)\big)$$

such that $g_x \in \mathrm{Bil}(E_x, E_x; \mathbb{R})$ is symmetric and positive definite for any $x \in M$.

We often write $\langle\cdot,\cdot\rangle$ instead of g and then $\langle\cdot,\cdot\rangle_x$ for the scalar product g_x on the fiber E_x (depending differentiably on x in bundle charts). For a section $A \in \Gamma(E)$ we use the notation $|A|$ for $\sqrt{g(A,A)}$ (and write $|e|_x$ instead of $\sqrt{g_x(e,e)}$ for $e \in E_x$).

DEFINITION 1.5.2 (Riemannian manifold). A *Riemannian manifold* is a pair (M, g) consisting of a differentiable manifold M and a Riemannian metric g on the tangent bundle TM.

DEFINITION 1.5.3 (Length of curves). Let $\alpha\colon [a,b] \to M$ be a piecewise differentiable curve on a Riemannian manifold (M, g) such that $\alpha|[t_{i-1}, t_i]$ is differentiable for some partition $a = t_0 < t_1 < \cdots < t_r = b$ of the interval $[a, b]$. Then

$$L(\alpha) := \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} |\dot\alpha(t)|_{\alpha(t)}\, dt$$

is well-defined and called the *length of* α.

DEFINITION 1.5.4 (Isometry; local isometry). Let (M, g) and (N, h) be Riemannian manifolds. A differentiable map $f\colon M \to N$ is called *local isometry* if $g = f^*h$, i.e., if $g_p(u, v) = (f^*h)_p(u, v) \equiv h_{f(p)}\big((df)_p u, (df)_p v\big)$ for all $u, v \in T_pM$. If in addition f is a diffeomorphisms, then f is called an *isometry*.

The condition $g = f^*h$ means that, for any $p \in M$, the map $(df)_p\colon (T_pM, g_p) \to (T_{f(p)}N, h_{f(p)})$ is an isometry of Euclidean vector spaces. In particular, local isometries let the length of curves invariant.

DEFINITION 1.5.5 (Riemannian connection). Let (M, g) be a Riemannian manifold and ∇ a linear connection in TM. Then ∇ is called a *Riemannian connection* if all parallel transports

$$//_{s,t}\colon (T_{\alpha(s)}M, g_{\alpha(s)}) \to (T_{\alpha(t)}M, g_{\alpha(t)})$$

along differentiable curves α are isometries.

THEOREM 1.5.6 (Characterization of Riemannian connections). *Let (M, g) be a Riemannian manifold and ∇ a linear connection in TM. The following items equivalent:*

(i) *∇ is a Riemannian connection.*

(ii) (Ricci identity) *For all $X, Y, Z \in \Gamma(TM)$,*

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

(iii) *If $f\colon N \to M$ is a differentiable map, then for $X, Y \in \Gamma(f^*TM)$, $A \in \Gamma(TN)$,*

$$A\,\langle X, Y\rangle = \langle \nabla_A X, Y\rangle + \langle X, \nabla_A Y\rangle.$$

PROOF. (i) ⇒ (ii) Let $p \in M$ and $\alpha\colon I \to M$ a differentiable curve such that $\dot\alpha(0) = Z_p$. In terms of the parallel transport $/\!/_{s,t}$ along α we calculate

$$\begin{aligned} Z_p\langle X, Y\rangle &= \tfrac{d}{dt}\big|_{t=0} \langle X_{\alpha(t)}, Y_{\alpha(t)}\rangle_{\alpha(t)} \\ &= \tfrac{d}{dt}\big|_{t=0} \langle \sigma_{0,t}^{-1} X_{\alpha(t)}, \sigma_{0,t}^{-1} Y_{\alpha(t)}\rangle_p \\ &= \big\langle \tfrac{d}{dt}\big|_{t=0} (\sigma_{0,t}^{-1} X_{\alpha(t)}), Y_p\big\rangle_p + \big\langle X_p,\, \tfrac{d}{dt}\big|_{t=0} (\sigma_{0,t}^{-1} Y_{\alpha(t)})\big\rangle_p \\ &= \big\langle (\nabla_Z X)_p, Y_p\big\rangle_p + \big\langle X_p, (\nabla_Z Y)_p\big\rangle_p. \end{aligned}$$

(ii) ⇒ (iii) Let $X, Y \in \Gamma(f^*TM)$ and $A \in \Gamma(TN)$. Since for $\phi \in C^\infty(N)$,

$$A\,\langle \phi X, Y\rangle = (A\phi)\,\langle X, Y\rangle + \phi\, A\,\langle X, Y\rangle \quad \text{and}$$
$$\big\langle \nabla_A(\phi X), Y\big\rangle = (A\phi)\,\langle X, Y\rangle + \phi\,\langle \nabla_A X, Y\rangle,$$

it is sufficient to verify the statement for $X = f^*U$, $Y = f^*V$ where $U, V \in \Gamma(TM)$. But with $q \in N$ and $w := A_q$, we obtain

$$\begin{aligned} A_q\langle X, Y\rangle &= w\big(\langle U, V\rangle \circ f\big) = (df_q w)\,\langle U, V\rangle \\ &= \langle \nabla_{df_q w} U, V_{f(q)}\rangle + \langle U_{f(q)}, \nabla_{df_q w} V\rangle = \langle \nabla_w X, Y_q\rangle + \langle X_q, \nabla_w Y\rangle. \end{aligned}$$

(iii) ⇒ (i) Let X, Y be parallel vector fields along a differentiable curve α in M. Then $D\langle X, Y\rangle = \langle \nabla_D X, Y\rangle + \langle X, \nabla_D Y\rangle = 0$ which shows that $\langle X, Y\rangle$ is constant. □

Geodesics with respect to Riemannian connections are parameterized proportionally to arc length. Indeed, we have $D\langle\dot\gamma, \dot\gamma\rangle = 2\langle \nabla_D \dot\gamma, \dot\gamma\rangle = 0$, and hence $|\dot\gamma|$ is constant.

THEOREM 1.5.7 (of Levi-Civita). *On a Riemannian manifold (M, g) there exists a unique torsion-free Riemannian connection ∇ in TM.*

PROOF. For uniqueness it is sufficient to show that $\langle \nabla_X Y, Z\rangle$ is uniquely determined for $X, Y, Z \in \Gamma(TM)$. Indeed, from the Ricci identity we obtain

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle X, Z\rangle &= \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle. \end{aligned}$$

Adding the first two equations and subtracting the last one, along with the torsion-freeness of ∇, gives

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\quad + \langle [X, Y], Z\rangle - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle. \end{aligned} \tag{1.5.1}$$

The right-hand side of this equation is $C^\infty(M)$-linear in Z and determines the vector field $\nabla_X Y \in \Gamma(TM)$. It is straightforward to check that it defines a torsion-free Riemannian connection in TM. □

DEFINITION 1.5.8. The unique torsion-free Riemannian connection in TM for a Riemannian manifold (M, g) according to Theorem 1.5.7 is called *Levi-Civita connection* in TM and the associated parallel transport the *Levi-Civita parallelism*.

REMARK 1.5.9. Eq. (1.5.1) can be used to express the Levi-Civita connection of a Riemannian manifold (M, g) directly via the metric g. To this end, let (h, U) be a local chart for M and $\partial_i = \frac{\partial}{\partial h^i} \in \Gamma(TM/U)$ for $i = 1, \ldots, n$. Consider

$$g_{ij} := \big\langle \tfrac{\partial}{\partial h^i}, \tfrac{\partial}{\partial h^j}\big\rangle \in C^\infty(U),$$

and $g^{ij} \in C^\infty(U)$ with $\sum_j g^{ij} g_{jk} = \delta_{ik}$. Then, by means of Eq. (1.5.1)

$$2\langle \nabla_{\partial_i} \partial_j, \partial_k \rangle = \partial_i \langle \partial_j, \partial_k \rangle + \partial_j \langle \partial_k, \partial_i \rangle - \partial_k \langle \partial_i, \partial_j \rangle,$$

i.e., $2 \sum_m \Gamma^m_{ij} g_{mk} = \partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}$ from where the wanted relation follows:

$$\Gamma^\ell_{ij} = \tfrac{1}{2} \sum_k g^{k\ell} \left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij} \right). \tag{1.5.2}$$

EXAMPLE 1.5.10 (Levi-Civita connection on $\mathbb{R}^n$). Let $(M, g) = (\mathbb{R}^n, \text{eucl})$ with the canonical Riemannian metric

$$\text{eucl}(A, B) \equiv \langle A, B \rangle = \sum_{i=1}^n A^i B^i$$

for vector fields A, B on $\mathbb{R}^n$. Vector fields on $\mathbb{R}^n$ are interpreted equally as functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and as derivations on $C^\infty(\mathbb{R}^n)$: the constant maps $(x \mapsto e_i) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $i = 1, \ldots, n$, correspond to the derivations $D_1, \ldots, D_n$ where $D_i = \frac{\partial}{\partial x^i}$.

(i) According to Eq. (1.5.2), the Levi-Civita connection ∇ on $(\mathbb{R}^n, \text{eucl})$ is determined by $\nabla_{D_i} D_j = 0$ which for vector fields A, B on $\mathbb{R}^n$ means that

$$\nabla_A B = \sum_i A(B^i)\, D_i \equiv \sum_i A(B^i)\, e_i = \big(A(B^1), \ldots, A(B^n)\big). \tag{1.5.3}$$

This connection is also denoted by D, i.e. $\nabla_A B = D_A B$, since according to (1.5.3), $\nabla_v B = D_v B$ coincides with the directional derivative of $B \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ in direction $v \in \mathbb{R}^n$.

(ii) Linear connections induced by the Levi-Civita connection on $(\mathbb{R}^n, \text{eucl})$ are easy to determine. For instance, let M be a manifold and $f \colon M \to \mathbb{R}^n$ a differentiable map. To describe the induced covariant derivative on $f^* T\mathbb{R}^n$, we note that each section $A \in \Gamma(f^* T\mathbb{R}^n)$ writes as $\sum_{i=1}^n A^i f^* D_i$ with $A^i \in C^\infty(M)$. Then, for $X \in \Gamma(TM)$, the covariant derivative $\nabla_X A = \sum_{i=1}^n (\nabla_X A)^i f^* D_i \in \Gamma(f^* T\mathbb{R}^n)$ with respect to the induced linear connection in $f^* T\mathbb{R}^n$ is given by

$$(\nabla_X A)^i = X A^i, \quad i = 1, \ldots, n.$$

Indeed, by the product rule, we have

$$\nabla_X A = \sum (X A^i) f^* D_i + A^i \nabla_X f^* D_i$$

where $\nabla_X f^* D_i = D_{df X} D_i = 0$.

PROPOSITION 1.5.11 (Levi-Civita connections on Riemannian submanifolds). *Let (M, g) be a Riemannian submanifold of a Riemannian manifold $(\tilde M, \tilde g)$ in the sense that there is an embedding $M \overset{\iota}{\hookrightarrow} \tilde M$ such that $g = \iota^* \tilde g$. The homomorphism $d\iota_x \colon T_x M \to T_x \tilde M$ then is an isometry for each $x \in M$. For $x \in M$ let $\mathrm{pr}^\top_x \colon T_x \tilde M \to T_x M$ denote the orthogonal projection onto the linear subspace $T_x M \equiv d\iota_x T_x M \subset T_x \tilde M$. This gives a homomorphism of vector bundles $\mathrm{pr}^\top \colon \iota^* T\tilde M \to TM$ over M with $\mathrm{pr}^\top \circ d\iota = \mathrm{id}_{TM}$:*

$$\begin{tikzcd}
TM \arrow[r, "d\iota"] \arrow[dr] & \iota^* T\tilde M \arrow[l, bend right, "\mathrm{pr}^\top"'] \arrow[d] & T\tilde M \arrow[d] \\
& M \arrow[r, hook, "\iota"] & \tilde M
\end{tikzcd}$$

Let now ∇, respectively $\tilde{\nabla}$, denote the Levi-Civita connection in TM, respectively $T\tilde{M}$. Then, for $A, B \in \Gamma(TM)$,

$$\nabla_A B = \mathrm{pr}^\top \tilde{\nabla}_A (d\iota\, B). \tag{1.5.4}$$

In particular, ∇ is uniquely determined by $\tilde{\nabla}$.

PROOF. First of all, we remark that the right-hand side of (1.5.4) defines a linear connection ∇' in TM. By the uniqueness part of Theorem 1.5.7 it is hence enough to show that ∇' is a torsion-free Riemannian connection. Denoting by T', respectively $\tilde{T}$, the torsion tensor of ∇', respectively $\tilde{\nabla}$, then we have by Theorem 1.4.27 for $A, B \in \Gamma(TM)$:

$$\begin{aligned} T'(A,B) &= \nabla'_A B - \nabla'_B A - [A,B] \\ &= \mathrm{pr}^\top \big(\tilde{\nabla}_A(d\iota\, B) - \tilde{\nabla}_B(d\iota\, A) - d\iota\,[A,B]\big) \\ &= \mathrm{pr}^\top (\iota^*\tilde{T})(A,B) = 0. \end{aligned}$$

Here we used that $\tilde{T} \equiv 0$ since $\tilde{\nabla}$ is torsion-free, and hence ∇' is also torsion-free. Furthermore, by Theorem 1.5.6, we obtain for $A, B, C \in \Gamma(TM)$,

$$\begin{aligned} \langle \nabla'_C A, B\rangle + \langle A, \nabla'_C B\rangle &= \langle \tilde{\nabla}_C(d\iota\, A), d\iota\, B\rangle + \langle d\iota\, A, \tilde{\nabla}_C(d\iota B)\rangle \\ &= C\langle d\iota\, A, d\iota\, B\rangle = C\langle A, B\rangle. \end{aligned}$$

Thus ∇' satisfies the Ricci identity and is therefore a Riemannian connection according to Theorem 1.5.6. $\square$

EXAMPLE 1.5.12 (Riemannian submanifolds of $(\mathbb{R}^n, \mathrm{eucl})$). We specialize Proposition 1.5.11 to the case of a Riemannian submanifold of $\mathbb{R}^n$. Then $(M,g) \xhookrightarrow{\iota} (\mathbb{R}^n, \mathrm{eucl})$ with $g = \iota^*\,\mathrm{eucl}$. By means of the fiberwise isometric bundle embedding $d\iota\colon TM \hookrightarrow \iota^*T\mathbb{R}^n$ we have $TM \subset \iota^*T\mathbb{R}^n \cong M \times \mathbb{R}^n$ as a vector subbundle and then $\Gamma(TM) \subset \Gamma(\iota^*T\mathbb{R}^n) = C^\infty(M;\mathbb{R}^n)$: vector fields on M are hereby $\mathbb{R}^n$-valued C^∞-maps on M. In terms of the orthogonal projection $\mathrm{pr}^\top\colon M \times \mathbb{R}^n \to TM$, $(x,v) \mapsto \mathrm{pr}^\top_x v$, we then have according to (1.5.3) and (1.5.4),

$$\nabla_A B = \mathrm{pr}^\top (AB^1, \ldots, AB^n), \quad A, B \in \Gamma(TM). \tag{1.5.5}$$

REMARK 1.5.13. Let (M,g) be a Riemannian manifold and ∇ a Riemannian connection in TM. For $f \in C^\infty(M)$ consider $\mathrm{grad}\, f \in \Gamma(TM)$, the *gradient of* f, defined by

$$\langle \mathrm{grad}\, f, A\rangle = Af, \quad A \in \Gamma(TM).$$

Note that

$$(\nabla df)(A,B) = \langle \nabla_A \mathrm{grad}\, f, B\rangle.$$

Indeed, according to Theorem 1.5.6 (ii) (Ricci identity) we have

$$A\,\langle \mathrm{grad}\, f, B\rangle = \langle \nabla_A \mathrm{grad}\, f, B\rangle + \langle \mathrm{grad}\, f, \nabla_A B\rangle,$$

and hence $\langle \nabla_A \mathrm{grad}\, f, B\rangle = AB\, f - (\nabla_A B)\, f$.

THEOREM 1.5.14 (Martingales on submanifolds of $(\mathbb{R}^n, \mathrm{eucl})$). *Let $M \xhookrightarrow{\iota} \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ endowed with the induced metric $g = \iota^*$* eucl *and let ∇ be the Levi-Civita connection on M. Suppose that X is an M-valued semimartingale and $\bar{X} = \iota(X)$ its embedding into $\mathbb{R}^n$. Let*

$$\bar{X} = \bar{X}_0 + N + C$$

be the Doob-Meyer decomposition of $\bar{X}$ in $\mathbb{R}^n$, where $N \in \mathscr{M}_0(\mathbb{R}^n)$ and $C \in \mathscr{A}_0(\mathbb{R}^n)$. We consider $T_{X_t}M$ as a linear subspace of $\mathbb{R}^n$. Then X is a ∇-martingale if and only if

$$dC_t \perp T_{X_t}M \quad \text{for all } t \in \mathbb{R}_+, \text{ a.s.}, \tag{1.5.6}$$

where the last condition is understood in the sense that $\langle H_t, dC_t\rangle = 0$ for each piecewise continuous process H such that $H_t \in T_{X_t}M$ a.s. In particular, each continuous local martingale on $\mathbb{R}^n$ taking values in M is a ∇-martingale on M.

PROOF. For $f \in C^\infty(M)$ we denote by $\bar{f} \in C^\infty(\mathbb{R}^n)$ a continuation of f to $\mathbb{R}^n$, i.e. $f = \bar{f} \circ \iota$. For a vector field $A \in \Gamma(TM)$ let $\bar{A} = \iota_* A \in \Gamma(\iota^* T\mathbb{R}^n) \equiv C^\infty(M; \mathbb{R}^n)$. Calculating the Hessian with respect to the Levi-Civita connection ∇ on $(M, \iota^* \text{eucl})$, we then get for $A, B \in \Gamma(TM)$,

$$(\nabla df)(A, B) = (\operatorname{Hess}_{\mathbb{R}^n} \bar{f})(\bar{A}, \bar{B}). \tag{1.5.7}$$

Indeed, for $f \in C^\infty(M)$, we note that

$$\iota_* \operatorname{grad} f = (D_1\bar{f}, \dots, D_n\bar{f})|M \in C^\infty(M; \mathbb{R}^n).$$

Identifying $A \in \Gamma(TM)$ and $\iota_* A \in \Gamma(\iota^* T\mathbb{R}^n) = C^\infty(M; \mathbb{R}^n)$ then gives according to Eq. (1.5.5),

$$\begin{aligned}(\nabla df)(A, B) &= \langle \nabla_A \operatorname{grad} f, B\rangle_{TM} = \big\langle \operatorname{pr}^\top\big(A(D_1\bar{f}), \dots, A(D_n\bar{f})\big), B\big\rangle_{TM} \\ &= \big\langle \big(A(D_1\bar{f}), \dots, A(D_n\bar{f})\big), B\big\rangle_{\mathbb{R}^n} \\ &= \sum_{i,j} A^i B^j\, D_i D_j \bar{f} = (\operatorname{Hess}_{\mathbb{R}^n} \bar{f})(\bar{A}, \bar{B}),\end{aligned}$$

For $h \in C^\infty(M)$ with $\bar{h} \in C^\infty(\mathbb{R}^n)$, we thus obtain by Itô's formula along with pullback formula (1.3.4),

$$\begin{aligned} d\big(h(X)\big) = d\big(\bar{h}(\bar{X})\big) &= \sum_{i=1}^n D_i\bar{h}(\bar{X})\, d\bar{X}^i + \tfrac{1}{2} \sum_{i,j=1}^n D_i D_j \bar{h}(\bar{X})\, d\bar{X}^i d\bar{X}^j \\ &= \langle (\operatorname{grad}_{\mathbb{R}^n} \bar{h})(\bar{X}), d\bar{X}\rangle + \tfrac{1}{2}(\operatorname{Hess}_{\mathbb{R}^n} \bar{h})(d\bar{X}, d\bar{X}) \\ &\overset{\mathrm{m}}{=} \langle (\operatorname{grad}_{\mathbb{R}^n} \bar{h})(\bar{X}), dC\rangle + \tfrac{1}{2}(\nabla dh)(dX, dX).\end{aligned}$$

Hence X is a ∇-martingale if and only if $\langle (\operatorname{grad}_{\mathbb{R}^n} \bar{h})(\bar{X}), dC\rangle = 0$ for all continuations $\bar{h} \in C^\infty(\mathbb{R}^n)$ of functions $h \in C^\infty(M)$. Applied to the coordinate functions $h^1, \dots, h^n$ of the embedding $M \xhookrightarrow{\iota} \mathbb{R}^n$ this gives the claim. $\square$

For the rest of this section let (M, g) be a Riemannian manifold equipped with the Levi-Civita connection ∇.

DEFINITION 1.5.15 (Riemannian quadratic variation). Let X be a semimartingale taking values in a Riemannian manifold $(M, g) = (M, \langle\cdot,\cdot\rangle)$. The process

$$[X, X] := \int g(dX, dX) = \int \langle dX, dX\rangle$$

is called *Riemannian quadratic variation* of X.

DEFINITION 1.5.16 (Laplace-Beltrami operator). Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection on M. For $f \in C^\infty(M)$ let

$$\Delta f := \operatorname{trace} \nabla df \in C^\infty(M)$$

where $\nabla df \in \Gamma(T^*M \otimes T^*M)$ denotes the Hessian of f. In other words, $(\Delta f)(x) = \sum_i (\nabla df)(e_i, e_i)$ where $e_1, \ldots, e_n$ is some orthonormal basis for T_xM. The operator Δ is called *Laplace-Beltrami operator on* M.

In local coordinates (h, U) for M,

$$\nabla df|U = \sum_{i,j} \Big(\partial_i \partial_j\, f - \sum_k \Gamma^k_{ij}\, \partial_k f \Big)\, dh^i \otimes dh^j, \tag{1.5.8}$$

and thus

$$\Delta f|U = \sum_{i,j} g^{ij} \Big(\partial_i \partial_j\, f - \sum_k \Gamma^k_{ij}\, \partial_k f \Big) \tag{1.5.9}$$

where $g_{ij} = g(\partial_i, \partial_j)$ and $(g^{ij}) \equiv (g_{ij})^{-1}$; here we used that $g(\partial_i, \cdot\,) = \sum_j g_{ij}\, dh^j$ or equivalently $\sum_i g^{ji} g(\partial_i, \cdot\,) = dh^j$. In particular, we see that Δ is a second order differential operator on M (without constant term).

DEFINITION 1.5.17 (Brownian motion on a Riemannian manifold). Let (M, g) be a Riemannian manifold and X an adapted M-valued process with maximal lifetime ζ, defined on a filtered probability space $\big(\Omega; \mathscr{F}; \mathbb{P}; (\mathscr{F}_t)_{t \in \mathbb{R}_+}\big)$ satisfying the usual conditions. The process X is called a *Brownian motion on* (M, g) if, for any $f \in C^\infty(M)$, the real-valued process

$$f(X) - \frac{1}{2} \int (\Delta f)(X)\, dt$$

is a local martingale (with lifetime ζ). The class of Brownian motions on (M, g) is denoted by $\mathrm{BM}(M, g)$.

THEOREM 1.5.18 (Lévy's characterization of M-valued Brownian motions). *Let X be a semimartingale with maximal lifetime taking values in a Riemannian manifold (M, g). The following conditions are equivalent:*

(i) *X is* $\mathrm{BM}(M, g)$.
(ii) *X is a ∇-martingale with the property that $[f(X), f(X)] = \int |\operatorname{grad} f|^2(X)\, dt$ for every $f \in C^\infty(M)$.*
(iii) *X is a ∇-martingale with the property that $\int b(dX, dX) = \int (\operatorname{trace} b)(X)\, dt$ for every $b \in \Gamma(T^*M \otimes T^*M)$.*

In particular, for the Riemannian quadratic variation $[X, X] = \int g(dX, dX) \equiv \int \langle dX, dX \rangle$ of a Brownian motion X on M, we get

$$\int_0^t g(dX, dX) = n\, t$$

where $n = \dim M$.

PROOF. A. We verify that for an arbitrary M-valued semimartingale X the following two conditions are equivalent:

(a) $[f(X), f(X)] = \int |\operatorname{grad} f|^2(X)\, dt$
(b) $\int b(dX, dX) = \int (\operatorname{trace} b)(X)\, dt$ for every $b \in \Gamma(T^*M \otimes T^*M)$.

In particular, this shows (ii) $\Longleftrightarrow$ (iii). Indeed, for $f, h \in C^\infty(M)$ we have

$$\begin{aligned} \operatorname{trace}(df \otimes dh) &= \sum_i (df \otimes dh)(e_i, e_i) = \sum_i (df)(e_i)\, (dh)(e_i) \\ &= \sum_i \langle \operatorname{grad} f, e_i \rangle\, \langle \operatorname{grad} h, e_i \rangle = \langle \operatorname{grad} f, \operatorname{grad} h \rangle. \end{aligned}$$

Thus (b) ⇒ (a) is just the special case for $b = df \otimes df$. To show the converse direction (a) ⇒ (b), first note that (a) implies by polarization

$$[f(X), h(X)] = \textstyle\int \langle \operatorname{grad} f, \operatorname{grad} h\rangle(X)\,dt, \quad f, h \in C^\infty(M).$$

Thus $[f(X), h(X)] = \int (df \otimes dh)\,(dX, dX) = \int \operatorname{trace}(df \otimes dh)(X)\,dt$. By means of the uniqueness part of Theorem 1.3.3, we get

$$\textstyle\int b(dX, dX) = \int (\operatorname{trace} b)(X)\,dt$$

for any bilinear form $b \in \Gamma(T^*M \otimes T^*M)$.

B. (iii) ⇒ (i): Part A applied to the given ∇-martingale X shows $b(dX, dX) = (\operatorname{trace} b)(X)\,dt$ for bilinear forms $b \in \Gamma(T^*M \otimes T^*M)$, in particular for $b = \nabla df$,

$$d\big(f(X)\big) \overset{\text{m}}{=} \frac{1}{2}\nabla df(dX, dX) = \frac{1}{2}(\Delta f)(X)\,dt,$$

thus X is BM(M, g).

C. (i) ⇒ (ii): Now let X be BM(M, g) and $f \in C^\infty(M)$. According to $\nabla df^2 = 2\,(f\nabla df + df \otimes df)$ we first note $\Delta(f^2) = 2f\Delta f + 2\,|\operatorname{grad} f|^2$, thus

$$d\big(f^2(X)\big) \overset{\text{m}}{=} \frac{1}{2}(\Delta\, f^2)(X)\,dt = (f\Delta f)(X)\,dt + |\operatorname{grad} f|^2(X)\,dt.$$

On the other hand, by means of Itô's formula,

$$\begin{aligned} d\big(f^2(X)\big) &= 2\,f(X)\,d\big(f(X)\big) + d[f(X), f(X)] \\ &\overset{\text{m}}{=} f(X)\,(\Delta f)(X)\,dt + d[f(X), f(X)]. \end{aligned}$$

Uniqueness in the Doob-Meyer decomposition implies

$$[f(X), f(X)] = \int |\operatorname{grad} f|^2(X)\,dt.$$

Finally, once again by means of part A, the last formula gives

$$\nabla df(dX, dX) = (\operatorname{trace} \nabla df)(X)\,dt = (\Delta f)(X)\,dt$$

from where we conclude that X is a ∇-martingale. □

EXAMPLE 1.5.19. According to Lévy's characterization of flat Brownian motions, Brownian motions on $(\mathbb{R}^n, \text{eucl})$ in the sense of Definition 1.5.17 coincide with the usual class of $\mathbb{R}^n$-valued Brownian motions.

THEOREM 1.5.20 (M-valued Brownian motions as solutions of an SDE). *Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection on M. We consider the* SDE

$$dX = A_0(X)\,dt + A(X) \circ dZ \tag{1.5.10}$$

with $A_0 \in \Gamma(TM)$, $A \in \Gamma\big(\mathrm{Hom}(M \times \mathbb{R}^r, TM)\big)$, and Z a BM *on $\mathbb{R}^r$. Then maximal solutions to* (1.5.10) *are Brownian motions on (M, g) if the two subsequent conditions are satisfied:*

(i) $A_0 = -\frac{1}{2}\sum_i \nabla_{A_i} A_i$ *with $A_i \equiv A(\cdot)e_i$ for $i = 1, \ldots, r$.*
(ii) *The map $A(x)^*\colon T_xM \to \mathbb{R}^r$ is an isometric embedding for every $x \in M$, i.e., $A(x)A(x)^* = \mathrm{id}_{T_xM}$ where $A(x)^*$ is the adjoint to $A(x) \in \mathrm{Hom}(\mathbb{R}^r, T_xM)$.*

PROOF. Let X be a solution to Eq. (1.5.10) and assume that conditions (i) and (ii) are satisfied. According to Remark 1.4.35 condition (i) guarantees that X is a ∇-martingale. In addition, we have for $f \in C^\infty(M)$,

$$d\big(f(X)\big) \overset{\mathrm{m}}{=} \frac{1}{2} \sum_{i=1}^{r} (\nabla df)(A_i, A_i)(X)\, dt.$$

It is thus sufficient to verify that

$$\sum_{i=1}^{r} (\nabla df)(A_i, A_i) = \Delta f.$$

Letting $x \in M$ and $(a_1, \ldots, a_n)$ an orthonormal basis of $T_x M$, we obtain

$$\begin{aligned}(\Delta f)(x) = \operatorname{trace}(\nabla df)_x &= \sum_{i=1}^{n} (\nabla df)_x(a_i, a_i) \\ &= \sum_{i=1}^{n} (\nabla df)_x\big(A(x)\, A(x)^* a_i, A(x)\, A(x)^* a_i\big).\end{aligned}$$

Completing $\big(A(x)^* a_1, \ldots, A(x)^* a_n\big)$ to an orthonormal basis $(\tilde e_1, \ldots, \tilde e_r)$ of $\mathbb{R}^r$, taking into account that $(\operatorname{im} A(x)^*)^\perp = \ker A(x)$ and denoting by $(e_1, \ldots, e_r)$ the standard basis of $\mathbb{R}^r$, we obtain

$$\begin{aligned}(\Delta f)(x) &= \sum_{i=1}^{r} (\nabla df)_x\big(A(x)\tilde e_i, A(x)\tilde e_i\big) \\ &= \sum_{i=1}^{r} (\nabla df)_x\big(A(x)e_i, A(x)e_i\big) = \sum_{i=1}^{r} (\nabla df)_x\big(A_i(x), A_i(x)\big)\end{aligned}$$

which completes the proof. □

REMARK 1.5.21. Conditions (i) and (ii) of Theorem 1.5.20 can always be satisfied for r sufficiently large. For instance, let $M \hookrightarrow \mathbb{R}^r$ be a Whitney embedding. Then $T_x M$ can be seen as a subspace $\mathbb{R}^r$ for each $x \in M$. Defining $A \in \Gamma(\mathrm{Hom}(M \times \mathbb{R}^r, TM))$ fiberwise as orthogonal projection $A(x)\colon \mathbb{R}^r \to T_x M$ onto $T_x M$ and setting $A_0 = -\frac{1}{2} \sum_i \nabla_{A_i} A_i$, then every solution to the SDE (1.5.10) (with a given initial condition) is a Brownian motion on (M, g). The drawback of this construction is that to a given Riemannian manifold (M, g) there is no *canonical* choice of the coefficients A_0 and A; there is however a canonical SDE on the orthonormal frame bundle $\mathrm{O}(TM)$ over M such that its solutions project to Brownian motions on (M, g). We develop this construction in the next Section.

We conclude this Section with a specification of Theorem 1.5.20 in the case of submanifolds of $\mathbb{R}^n$.

THEOREM 1.5.22 (Brownian motions on submanifolds of $\mathbb{R}^n$). *Let $M \overset{\iota}{\hookrightarrow} \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ endowed with the induced Riemannian metric $g = \iota^*\,\mathrm{eucl}$. Consider the* SDE

$$dX = A(X) \circ dZ \tag{1.5.11}$$

where Z is a Brownian motion on $\mathbb{R}^n$ and

$$A \in \Gamma(\mathrm{Hom}(M \times \mathbb{R}^n, TM)), \quad (x, v) \mapsto A(x)v,$$

such that $A(x)\colon \mathbb{R}^n \to T_x M$ is the orthogonal projection onto $T_x M$. Then every solution to (1.5.11) *gives a Brownian motion on (M, g).*

PROOF. For each $x \in M$, the map $d\iota_x\colon T_xM \to \mathbb{R}^n$ is an isometric embedding and we consider T_xM as a linear subspace of $\mathbb{R}^n$. Note that $A(x)^* = d\iota_x$ and $A(x)A(x)^* = \mathrm{id}_{T_xM}$. In terms of the vector fields $A_i \equiv A(\cdot)e_i \in \Gamma(TM)$, $i = 1, \dots, n$, by Theorem 1.5.20, it is sufficient to verify that $\sum_{i=1}^n \nabla_{A_i} A_i = 0$ where ∇ denotes the Levi-Civita connection on M. For $e \in \mathbb{R}^n$ let $A^e := A(\cdot)e \in \Gamma(TM)$. We show that

$$\nabla_v A^e = 0 \quad \text{for all } e \in \operatorname{im} A(x)^* = (\ker A(x))^\perp,\ v \in T_xM.$$

To this end, let $Q(x) := A(x)^*A(x)\colon \mathbb{R}^n \to \mathbb{R}^n$. We note that $Q(x)^2 = Q(x)$ and $Q(x)e = e$ for $e \in \operatorname{im} A(x)^*$. Thus, using $QQ = Q$, we have

$$(dQ)_x e = (dQ)_x Q(x)e + Q(x)(dQ)_x e = (dQ)_x e + Q(x)(dQ)_x e$$

from where we conclude that $Q(x)(dQ)_x e = 0$ for all $e \in \operatorname{im} A(x)^*$. In explicit terms this shows that

$$0 = A(x)^*A(x)d\big(A(\cdot)^*A(\cdot)e\big)_x = A(x)^*A(x)d(\bar{A}^e)_x$$

where $\bar{A}^e = \iota_* A^e$. Since by Eq. (1.5.4), $\nabla A^e = A\, d(\bar{A}^e)$, we finally obtain

$$(\nabla A^e)_v = \nabla_v A^e = 0, \quad v \in T_xM,$$

which completes the proof. □

REMARK 1.5.23. On a differentiable manifold M consider an SDE of the type

$$dX = A_0(X)\,dt + A(X) \circ dZ \tag{1.5.12}$$

with $A_0 \in \Gamma(TM)$, $A \in \Gamma\big(\mathrm{Hom}(M \times \mathbb{R}^r, TM)\big)$, and Z a BM on $\mathbb{R}^r$. We assume that the SDE (1.5.12) is *elliptic* in the sense that

$$A(x)\colon \mathbb{R}^r \to T_xM \quad \text{is surjective for each } x \in M.$$

Then there exists a Riemannian metric g on M such that $A(x)^*\colon T_xM \to \mathbb{R}^r$ is an isometric embedding for each $x \in M$. Indeed, let $Q \in \Gamma(T^*M \otimes \mathbb{R}^r)$ be a right-inverse to A, i.e. for each $x \in M$,

$$Q(x)\colon T_xM \to \mathbb{R}^r \ \text{ is linear and } A(x)Q(x) = \mathrm{id}_{T_xM}\,.$$

We define

$$g(u, v) := \langle Q(x)u, Q(x)v\rangle_{\mathbb{R}^r}, \quad u, v \in T_xM.$$

It's easy to see that $Q(x) = A(x)^*$. Now let ∇ be the Levi-Civita connection on (M, g). As in the proof of Theorem 1.5.20 we have

$$\sum_{i=1}^r (\nabla df)(A_i, A_i) = \Delta f$$

where $A_i = A(\cdot)e_i$ for $i = 1, \dots, r$. Since $A_i^2 f = (\nabla df)(A_i, A_i) + \nabla_{A_i} A_i$, we observe that solutions to (1.5.12) define L-diffusions for

$$L = \frac{1}{2}\sum_{i=1}^r A_i^2 + A_0 = \frac{1}{2}\Delta + (A_0 + \sum_{i=1}^r \nabla_{A_i} A_i) = \frac{1}{2}\Delta + V$$

where $V := A_0 + \sum_{i=1}^r \nabla_{A_i} A_i \in \Gamma(TM)$. In other words, with respect to an appropriately chosen Riemannian metric g on M, maximal solutions to (1.5.12) are Brownian motions on (M, g) with drift V.

1.6. Parallel Transport and Stochastically Moving Frames

In the last Section we have shown that Brownian motion on a Riemannian manifold M can be constructed as solution to an appropriate stochastic differential equation on M (driven by a standard Euclidean Wiener process). These constructions are however not canonical which is due to the fact that in general, unless the tangent bundle of M is trivial (i.e. for M parallelizable), the Laplace-Beltrami operator does not have a natural representation in Hörmander form as a sum of squares of vector fields.

The fundamental observation that Brownian motions on Riemannian manifolds can be horizontally lifted via a Riemannian connection to semimartingales on the orthonormal frame bundle $\mathrm{O}(TM) \to M$ over M and satisfy there globally defined canonical stochastic differential equations (SDEs) goes back to the pioneering work of Malliavin, Eells and Elworthy. Conversely, solving SDEs on the frame bundle and projecting the solutions down to the manifold M allows canonical constructions of diffusion processes on M (see [**7, 8, 9, 10, 31, 32, 33, 42**]).

Intuitively the procedure of constructing M-valued processes X from continuous $\mathbb{R}^n$-valued semimartingales Z corresponds to a "*rolling without slipping*" of the manifold M along the trajectories of Z in $\mathbb{R}^n$. It allows to construct to each continuous semimartingale Z in $T_xM \equiv \mathbb{R}^n$ a *stochastic development* X on M, together with a notion of parallel transport along the paths of X on M. Brownian motion X on M starting at $x \in M$ can be thought as the trace printed on M by the paths of an Euclidean Brownian motion Z in $T_xM \cong \mathbb{R}^n$ when "M is rolled along the trajectories of the flat process". The obvious difficulty that paths of Brownian motion are non-differentiable almost surely requires to work with stochastic differential equations instead of pathwise ordinary differential equations.

We begin the discussion with necessary prerequisites on principal bundles and connection forms. Apart from the already studied vector bundles (with a finite-dimensional vector space $V \cong \mathbb{R}^n$ as typical fiber) a further type of fiber bundles is needed, that is principal bundles with a Lie group G as typical fiber. The most important examples will be the frame bundle $\mathrm{L}(TM)$ with $G = \mathrm{GL}(n;\mathbb{R})$ and, in the Riemannian case, the orthonormal frame bundle $\mathrm{O}(TM)$ with the orthogonal group $G = \mathrm{O}(n)$. Vector bundles and principal bundles belong both to the common category of fiber bundles with structure group.

NOTATION 1.6.1. Let G be a Lie group and F a manifold. A *left action* of G on F ("G operates on F from the left") is a differentiable map

$$G \times F \to F, \quad (g,v) \to gv =: L_g v$$

with the properties:

(a) $ev = v$ for $v \in F$ where e is the neutral element in G,
(b) $g_2(g_1 v) = (g_2 g_1)v$ for all $g_1, g_2 \in G$ and $v \in F$.

A left action of G on F is hence a group homomorphism $G \to \mathrm{Diff}(F)$ with the property that the operation $G \times F \to F$ is differentiable. A left action of G on F is called *effective* if $G \to \mathrm{Diff}(F)$, $g \mapsto L_g$ is injective, and *free* if $gv = v$ for some $v \in F$ implies $g = e$.

If $F = V$ is a finite dimensional real vector space, then left actions of G on V given by a differentiable group homomorphism $G \to \mathrm{Aut}(V)$ are called *linear*, respectively *representations of* G, if they are in addition effective.

These concepts carry over correspondingly to right actions of G ("G operates on F from the right"). Note that if $F \times G \to F$, $(v,g) \mapsto vg$ is a right action of G on F, then $G \times F \to F$, $(g,v) \mapsto vg^{-1}$ defines a left action of G on F.

DEFINITION 1.6.2 (Fibre bundle with structure group). Let $\pi\colon E \to M$ be a fiber bundle with typical fiber F and G a Lie group with an effective left action of G on F. A bundle atlas for $\pi\colon E \to M$ is called *G-atlas* if all transition functions are given by differentiable maps taking values in $G \subset \mathrm{Diff}(F)$. The change of two charts (φ_i, U_i), (φ_j, U_j) is thus given by a differentiable map $\phi_{ij}\colon U_i \cap U_j \to G$ such that

$$\begin{tikzcd} & E/(U_i \cap U_j) \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ (U_i \cap U_j) \times F \arrow[rr, dashed] & & (U_i \cap U_j) \times F \\ (x,v) \arrow[rr, mapsto] & & \big(x, \phi_{ij}(x)v\big) \end{tikzcd}$$

The bundle $\pi\colon E \to M$ equipped with a G-Atlas is called *fiber bundle with typical fiber F and structure group G*.

As usual, differentiable right inverses of $\pi\colon E \to M$ are called sections of E. Global resp. local sections of E are denoted by $\Gamma(E)$ resp. $\Gamma(E/U)$.

REMARK 1.6.3. The m-dimensional vector bundles over a manifold M are the fiber bundles over M with typical fiber $\mathbb{R}^m$ and $\mathrm{GL}(m;\mathbb{R})$ as structure group; see Definition 1.0.22. In particular, the tangent bundle $TM \to M$ of a differentiable n-dimensional manifold is a fiber bundle with typical fiber $\mathbb{R}^n$ and structure group $\mathrm{GL}(n;\mathbb{R})$ where the transition functions of the canonical $\mathrm{GL}(n;\mathbb{R})$-atlas take the form $x \mapsto J_x(k \circ h^{-1})$.

DEFINITION 1.6.4 (Principal bundle). Let G be a Lie group. A *principal G-bundle* is a fiber bundle $\pi\colon P \to M$ with typical fiber G and structure group G which operates on G (effectively) from the left by the group multiplication.

Let $\pi\colon P \to M$ be a principal G-bundle. Then G operates in a natural way from the right on P. At first, G operates on G itself on the right by group multiplication, and via charts this action extends to a differentiable free right action on P: an element $u \in P/U$ reads in a chart $P/U \to U \times G$ as (x,g), so that $u\,a$ for $a \in G$ corresponds to (x, ga),

$$\begin{aligned} P/U \longrightarrow U \times G\,, \quad u &\longleftrightarrow (x,g) \\ u\,a &\leftrightarrow (x, ga) \end{aligned}$$

and since this assignment is independent of charts, it gives a well-defined global right action of G on P. The orbits of this action are the fibers of P. Bundle charts (φ, U) from a G-atlas

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

are automatically *G-compatible* (or *equivariant*) in the sense of

$$\varphi(u\,a) = \big(\pi(u), (\mathrm{pr}_G \circ \varphi)(u\,a)\big), \quad u \in \pi^{-1}U,\ a \in G.$$

DEFINITION 1.6.5 (Reduction of the structure group). If the G-Atlas of a fiber bundle $\pi\colon E \to M$ with structure group G contains a G'-Atlas for E where $G' \subset G$ is a Lie subgroup, then $\pi\colon E \to M$ can also be considered as fiber bundle with structure group G'. This procedure is called *reduction of the structure group G to G'*.

REMARK 1.6.6. Reductions of the structure group appear naturally in case of an additional structure on the fibers of E corresponding in suitable charts to a G'-invariant structure on the typical fiber. For example, if a vector bundle $\pi\colon E \to M$ of rank m carries a Riemannian metric, then the fiberwise isometric linear bundle charts provide an $\mathrm{O}(m)$-atlas and hence a reduction of the structure group $\mathrm{GL}(m;\mathbb{R})$ to $\mathrm{O}(n)$. Such bundle charts can be constructed from a $\mathrm{GL}(m;\mathbb{R})$-atlas for $\pi\colon E \to M$ via Gram-Schmidt orthogonalization. In this sense the tangent bundle of an n-dimensional Riemannian manifold becomes a vector bundle with structure group $\mathrm{O}(n)$.

REMARK 1.6.7. (a) (Associated fiber bundles) Let $\pi\colon P \to M$ be a principal G-bundle and F a manifold with an effective left action of G on F. We may consider the right action of G (diagonal action) on $P \times F$ given by

$$(P \times F) \times G \longrightarrow P \times F, \quad (p, v)\, g := (p\, g, g^{-1} v).$$

The projection $P \times F \to P \to M$ is invariant under the diagonal action, and $E \equiv P \times_G F := (P \times F)/G \to M$ defines a fiber bundle with typical fiber F and structure group G. Each chart (φ, U) for P gives a chart for E via

$$\begin{array}{ccccc} E/U = (P/U) \times_G F & \longrightarrow & (U \times G) \times_G F & \xrightarrow{\sim} & U \times F\,, \\ [u, v] & \longmapsto & [\varphi(u), v] & & \end{array}$$

where the second bijection is given through

$$\begin{array}{ccccc} (U \times G) \times_G F \longrightarrow U \times F\,, & [(x, g), v] & \longmapsto & (x, gv) \\ & [(x, 1), w] & \longleftarrow & (x, w). \end{array}$$

The charts for E then have the same transition functions as the G-atlas for P. We call $P \times_G F \to M$ a *fiber bundle associated to P*.

(b) (Associated principal bundles) Let $\pi\colon E \to M$ be a fiber bundle with typical fiber F and structure group G. For $x \in M$ let

$$P_x \equiv \mathrm{Iso}_G(F; E_x) := \big\{ (\varphi|E_x)^{-1} \circ L_g \colon F \to E_x \;\big|\; g \in G \big\}$$

denote the entity of maps $F \to E_x$ induced with respect to one (and then every) bundle chart (φ, U) of E at x by group elements in G. Then $P \equiv \mathrm{Iso}_G(F; E) := \dot{\bigcup}_{x \in M} P_x$ is naturally a principal G-bundle. Bundle charts for P are obtained by assigning to each chart (φ, U) for E the bijection

$$\begin{array}{ccc} P/U \equiv \pi^{-1} U & \longleftarrow & U \times G \\ (\varphi|E_x)^{-1} \circ L_g & \longleftarrow & (x, g). \end{array}$$

We call $P \to M$ the *principal G-bundle associated to E*.

REMARK 1.6.8. The two procedures described in Remark 1.6.7 are functorial constructions inverting each other. More precisely, for a principal G-bundle $P \to M$ and a fiber bundle $E \to M$ with typical fiber F and structure group G, functorial bundle isomorphisms are given as follows:

$$\begin{array}{llcl} P \longrightarrow \mathrm{Iso}_G(F; P \times_G F)\,, & u & \longmapsto & \big(v \mapsto [u, v]\big) \\ E \longleftarrow \mathrm{Iso}_G(F; E) \times_G F\,, & u(v) & \longleftarrow & [u, v]. \end{array}$$

Remark 1.6.7 allows to construct from a principal G-bundle P and a representation $G \to \mathrm{Aut}(V)$ of G the vector bundle associated to P with fiber V, respectively to pass from a vector bundle $\pi\colon E \to M$ of rank m to the principal $\mathrm{GL}(m;\mathbb{R})$-bundle associated to E (or from a Riemannian vector bundle of rank m to the associated principal $\mathrm{O}(m)$-bundle).

EXAMPLE 1.6.9. Let M be a differentiable manifold of dimension n and $TM \to M$ its tangent bundle considered as fiber bundle with typical fiber $\mathbb{R}^n$ and structure group $\mathrm{GL}(n;\mathbb{R})$. The associated principal $\mathrm{GL}(n;\mathbb{R})$-bundle

$$\mathrm{L}(TM) := \mathrm{Iso}_{\mathrm{GL}(n;\mathbb{R})}(\mathbb{R}^n; TM) \to M$$

is called *frame bundle* over M. If TM carries a Riemannian metric g then TM is a vector bundle with structure group $\mathrm{O}(n)$. The associated principal $\mathrm{O}(n)$-bundle

$$\mathrm{O}(TM) := \mathrm{Iso}_{\mathrm{O}(n)}(\mathbb{R}^n; TM) \to M$$

is called *orthonormal frame bundle* over M.

1. The frame bundle $P = \mathrm{L}(TM)$ over M is the principal $\mathrm{GL}(n;\mathbb{R})$-bundle associated to the vector bundle TM. By construction $u \in P_x$ is a linear isomorphism $u\colon \mathbb{R}^n \to T_xM$ which may be identified with the $\mathbb{R}$-basis
$$(u_1, \ldots, u_n) := (ue_1, \ldots, ue_n)$$
for T_xM where e_i denotes the ith standard coordinate vector of $\mathbb{R}^n$. The general linear group $G = \mathrm{GL}(n;\mathbb{R})$ operates on $\mathrm{L}(TM)$ from the right via
$$u\,g\colon\ \mathbb{R}^n \xrightarrow{g} \mathbb{R}^n \xrightarrow{u} T_xM.$$
Thus $ug \in \mathrm{L}(TM)$ for $g = (g_{ij}) \in G$ is given by $(ug)_j = \sum_i g_{ij}u_i$. Bundle charts for $\mathrm{L}(TM)$ are obtained from charts (h, U) for M. Indeed, $\big(\frac{\partial}{\partial h^1}, \ldots, \frac{\partial}{\partial h^n}\big)$ defines a local section $\mathrm{L}(TM)$ over U so that each $u \in \mathrm{L}(TM)$ with $\pi(u) = x \in U$ writes as $u_j = \sum_i a_{ij}(u)\big(\frac{\partial}{\partial h^i}\big)_x$ where $a(u) := \big(a_{ij}(u)\big) \in \mathrm{GL}(n;\mathbb{R})$. Then
$$\pi^{-1}U \xrightarrow{\sim} U \times \mathrm{GL}(n;\mathbb{R}), \quad u \longmapsto \big(\pi(u), a(u)\big),$$
is a bundle chart for $P = \mathrm{L}(TM)$.
2. The orthonormal frame bundle $\mathrm{O}(TM)$ over a Riemannian manifold (M, g) is the principal $\mathrm{O}(n)$-bundle associated to TM (now TM considered as vector bundle with structure group $\mathrm{O}(n)$). To each chart (h, U) for M, Gram-Schmidt orthogonalization of $\big(\frac{\partial}{\partial h^1}, \ldots, \frac{\partial}{\partial h^n}\big)$ gives a local section of $\mathrm{O}(TM)$ over U. Bundle charts for $\mathrm{O}(TM)$ are constructed as above.

Bundle charts (local trivializations) allow to identify neighbouring fibers of principal bundles, but not in a canonical way. As for vector bundles, to relate fibers intrinsically to each others, a connection as additional structure is required. We will see that for principal $\mathrm{GL}(n;\mathbb{R})$-bundles connections correspond canonically to linear connections in associated vector bundles.

DEFINITION 1.6.10 (Connection in a principal bundle). Let $\pi\colon P \to M$ be a principal bundle over M with structure group G. A *G-connection* in P is a differentiable G-invariant splitting h of the following exact sequence of vector bundles over P:

$$\begin{tikzcd} 0 \arrow[r] & \ker d\pi \arrow[r] & TP \arrow[r, "d\pi"] & \pi^*TM \arrow[r] \arrow[l, bend left, dashed, "h"] & 0 \end{tikzcd}$$

where $d\pi \circ h = \mathrm{id}$. This splitting induces a decomposition of TP:

$$TP = V \oplus H := \ker d\pi \oplus h(\pi^*TM).$$

The G-invariance of the splitting means that $H_{ug} = (dR_g)H_u$ for each $u \in P$, where $R_g u := u\,g$ denotes the right action of $g \in G$. For $u \in P$, we call H_u the *horizontal space at* u and $V_u = \{v \in T_uP\colon\ (d\pi)v = 0\}$ the *vertical space at* u. The bundle isomorphism

$$h\colon\ \pi^*TM \xrightarrow{\sim} H \hookrightarrow TP \tag{1.6.1}$$

is called *horizontal lift* of the G-connection; fiberwise it reads as $h_u\colon\ T_{\pi(u)}M \xrightarrow{\sim} H_u$.

REMARK 1.6.11. For each $u \in P$ the vertical space V_u is canonically given and G-invariant. However there is no canonical choice of a complement H_u: A G-connection in P corresponds exactly to a G-invariant choice of a horizontal space H_u for each $u \in P$. By means of the G-connection in P each vector field $X \in \Gamma(TP)$ decomposes in a horizontal and a vertical part:

$$X = \operatorname{hor} X + \operatorname{vert} X.$$

DEFINITION 1.6.12 (Standard-vertical vector field). Let $\pi\colon\ P \to M$ be a principal G-bundle over M. Each $u \in P$ defines an embedding

$$I_u\colon\ G \hookrightarrow P, \quad g \mapsto ug.$$

Its differential at the unit element $e \in G$,

$$\iota_u \equiv (dI_u)_e\colon T_eG \to T_uP, \quad A \longmapsto \hat{A}(u), \tag{1.6.2}$$

gives an identification $\kappa_u\colon\ \mathfrak{g} \xrightarrow{\sim} V_u$ of the Lie algebra $\mathfrak{g} = T_eG$ of G with the vertical fiber V_u at u. The vertical vector field $\hat{A} \in \Gamma(TP)$ on P defined by (1.6.2) is called *standard-vertical vector field* to $A \in \mathfrak{g}$.

DEFINITION 1.6.13 (Connection form). Let $\pi\colon\ P \to M$ be a principal G-bundle over M equipped with a G-connection. The $\mathfrak{g}$-valued one-form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ on P,

$$\omega_u(X_u) := \kappa_u^{-1}(\operatorname{vert} X)_u, \quad X \in \Gamma(TP), \tag{1.6.3}$$

is called *connection form* of the G-connection.

By definition, the connection form ω of a G-connection is *horizontal*, i.e., $\omega(X) = 0$ for a vector field X on P if and only if X is horizontal.

REMARK 1.6.14. Let $\pi\colon\ P \to M$ be a principal G-bundle over M and let $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ be the connection form of a G-connection in P. Then:

(i) $\omega(\hat{A}) = A$ for $A \in \mathfrak{g}$;
(ii) ω is equivariant, i.e.,

$$R_g^*\omega = \mathrm{Ad}(g^{-1})\omega, \quad g \in G,$$

where Ad is the *adjoint representation* of G in $\mathfrak{g}$. Recall that $\mathrm{Ad}(g)\colon\ \mathfrak{g} \to \mathfrak{g}$ for $g \in G$ is defined as the differential of the automorphism $I(g)\colon G \to G$, $h \longmapsto R_{g^{-1}}L_g h := ghg^{-1}$ at $e \in G$. Identifying $\mathfrak{g}$ and left-invariant vector fields on G, and taking into account that $(dL_g)A \equiv A$, i.e., $(dL_g)_h A_h \equiv A_{gh}$, we get $\mathrm{Ad}(g) = dR_{g^{-1}}$.

PROOF. Item (i) is a direct consequence of Definition 1.6.12 and 1.6.13. We want to verify item (ii) which reads as

$$\omega\big((dR_g)X\big) = \mathrm{Ad}(g^{-1})\omega(X), \quad X \in T_uP,\ g \in G. \tag{1.6.4}$$

It is obviously sufficient to consider the following two cases:

(1) X is horizontal, i.e. $X \in H_u$. Since $(dR_g)H_u = H_{ug}$, however also $(dR_g)X$ is horizontal, so that both sides of (1.6.4) vanish.

(2) X is vertical, i.e. $X \in V_u$, and hence $X = \hat{A}(u)$ for some $A \in \mathfrak{g}$. But then $(dR_g)_u\, \hat{A}(u) = \big(\mathrm{Ad}(g^{-1})A\big)\hat{}\ (ug)$ with $\big(\mathrm{Ad}(g^{-1})A\big)\hat{} \in \Gamma(TP)$ the standard-vertical vector field to $\mathrm{Ad}(g^{-1})A \in \mathfrak{g}$, and one obtains

$$\begin{aligned}
\big((R_g)^*\omega\big)_u(X) &= \omega_{ug}\big((dR_g)_u X\big) = \omega_{ug}\big((dR_g)_u \hat{A}(u)\big) \\
&= \omega_{ug}\big(\big(\mathrm{Ad}(g^{-1})A\big)\hat{}\ (u\cdot g)\big) = \mathrm{Ad}(g^{-1})\, A \\
&= \mathrm{Ad}(g^{-1})\, \omega_u\big(\hat{A}(u)\big) = \mathrm{Ad}(g^{-1})\, \omega_u(X)
\end{aligned}$$

which shows the claim. □

REMARKS 1.6.15. (i) A G-connection in P is uniquely determined by its connection form ω: the map $\omega_u\colon T_uP \to \mathfrak{g}$ is linear for each $u \in P$ with $\ker \omega_u = H_u$. Conversely, every equivariant differential form

$$\omega \in \Gamma(T^*P \otimes \mathfrak{g}) \quad \text{with } \omega(\hat{A}) = A \text{ for } A \in \mathfrak{g}$$

defines a G-connection in P whose connection form is given by ω.

(ii) G-invariant splittings of the exact sequence (1.6.10), and hence connections in principal bundles, can thus be described in different ways, for instance,

- as horizontal lifts h, $d\pi \circ h = \mathrm{id}$, with the property that $(dR_g)h = h$ for each $g \in G$;
- as $\mathfrak{g}$-valued differential forms $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$, $\bar{\omega} \circ \iota = \mathrm{id}$, where $\bar{\omega}_u = (u, \omega_u)$ for $u \in P$, satisfying the condition $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$ for $g \in G$;
- as horizontal subbundles H, $V \oplus H = TP$, such that $(dR_g)H_u = H_{ug}$ for each $u \in P$.

$$\begin{tikzcd}
 & P\times\mathfrak{g} \arrow[d, equal] & V\oplus H \arrow[d, equal] & & \\
0 \arrow[r] & \ker d\pi \arrow[r, "\iota"] & TP \arrow[r, "d\pi"] \arrow[l, bend left, dashed, "\bar{\omega}"] & \pi^*TM \arrow[r] \arrow[l, bend left, dashed, "h"] & 0.
\end{tikzcd} \tag{1.6.5}$$

Fiberwise diagram (1.6.5) reads as

$$\begin{tikzcd}
 & & V_u\oplus H_u \arrow[d, equal] & & \\
0 \arrow[r] & \mathfrak{g} \arrow[r, "\iota_u"] & T_uP \arrow[r, "(d\pi)_u"] \arrow[l, bend left, dashed, "\omega_u"] & T_{\pi(u)}M \arrow[r] \arrow[l, bend left, dashed, "h_u"] & 0.
\end{tikzcd} \tag{1.6.6}$$

As already mentioned, there is a one-to-one correspondence between connections in principal bundles and linear connections in associated vector bundles. More precisely we have the following situation.

REMARK 1.6.16. (a) Let $\pi\colon P \to M$ be a principal G-bundle, V a real vector space, $G \to \mathrm{Aut}(V)$ a representation of G and $E = P \times_G V$ the associated vector bundle with fiber V. Each G-connection in P induces a linear connection in E as follows. Denote by

$$P \times V \longrightarrow E = (P \times V)/G, \quad (u,\xi) \longmapsto (u,\xi)\, G,$$

the canonical projection and consider elements $\xi \in V$ as

$$\xi\colon P \to E, \quad u \longmapsto u\,\xi := (u,\xi)\, G.$$

By definition we have $(ug)\xi = u(g\xi)$. We fix $e \in E$ and choose an arbitrary $u \in P_{\pi_E(e)}$. Then there exists exactly one $\xi \in V$ such that $u\xi = e$, and one obtains:

$$\begin{array}{ccccc} u & P & \xrightarrow{\ \xi\ } & E & e \\ \big\downarrow & \big\downarrow{\scriptstyle\pi} & & {\scriptstyle\pi_E}\big\downarrow & \big\downarrow \\ x & M & \xrightarrow{\ \mathrm{id}\ } & M & x \end{array}$$

The differential $(d\xi)_u$ of $\xi\colon P \to E$ at u gives a map $T_uP \to T_eE$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(d\pi)_u & \longrightarrow & T_uP & \xrightarrow{(d\pi)_u} & T_{\pi(u)}M & \longrightarrow & 0 \\ & & & & {\scriptstyle (d\xi)_u}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \ker(d\pi_E)_e & \longrightarrow & T_eE & \xrightarrow{(d\pi_E)_e} & T_{\pi_E(e)}M & \longrightarrow & 0 \end{array}$$

By assumption we have a decomposition $T_uP = V_u(P) \oplus H_u(P)$ which induces a decomposition $T_eE = V_e(E) \oplus H_e(E)$ with $V_e(E) = \ker(d\pi_E)_e$ and $H_e(E)$ the image of $H_u(P)$ under $(d\xi)_u$. One verifies that $H_e(E)$ is well-defined, i.e., independent of the choice of u. Indeed taking ug instead of u leads to $(u\cdot g)\,(g^{-1}\xi) = e$, but by assumption $(dR_g)H_u(P) = H_{ug}(P)$ and consequently

$$d(g^{-1}\xi)_{ug}\, H_{ug}(P) = d\big((g^{-1}\xi) \circ R_g\big)_u\, H_u(P) = (d\xi)_u H_u(P).$$

Hence $H(E)$ defines a subbundle of TE which determines a linear connection in E.

(b) Conversely let E be a vector bundle over M with structure group G and $\mathrm{Iso}_G(V;E)$ the associated principal G-bundle over M; without restrictions we may assume $V = \mathbb{R}^m$. Then each linear connection in E induces a G-connection in P as follows. According to Remark 1.4.18 (i) the horizontal space at $e \in E$ (with $\pi_E(e) = x$) induced by the connection in E writes as

$$H_e(E) = \big\{(dX)_x v\colon v \in T_xM,\ X \in \Gamma(E) \text{ with } X(x) = e \text{ and } \nabla_v X = 0\big\}.$$

Note that each section $\hat X \in \Gamma(P)$ is of the form $\hat X = (X_1, \dots, X_n)$ with $X_i := \hat X e_i \in \Gamma(E)$ and e_i the i-th standard coordinate vector of $\mathbb{R}^n$. For $u \in P$ with $\pi(u) = x$ let now

$$\begin{aligned} H_u(P) := \big\{(d\hat X)_x v : v \in T_xM,\ &\hat X \in \Gamma(P/U) \text{ with } \hat X(x) = u \\ &\text{and } \nabla_v \hat X := (\nabla_v X_1, \dots, \nabla_v X_n) = 0\big\}. \end{aligned}$$

This determines horizontal subbundle $H(P)$ of TP which satisfies $(dR_g)H_u = H_{ug}$ for $u \in P$. Hence it defines a G-connection in P.

One verifies that (a) and (b) are inverse to each other when passing from frame bundles to associated vector bundles, resp., from vector bundles to the associated frame bundles (see Remark 1.6.7). In particular, we have one-to-one correspondences:

$$\text{linear connections in } TM \longleftrightarrow \mathrm{GL}(n;\mathbb{R})\text{-connections in } \mathrm{L}(TM);$$
$$\text{Riemannian connections in } TM \longleftrightarrow \mathrm{O}(n)\text{-connections in } \mathrm{O}(TM).$$

In the sequel we call $\mathrm{GL}(n;\mathbb{R})$-connections in the frame bundle $\mathrm{L}(TM)$ briefly *linear connections on M*, and $\mathrm{O}(n)$-connections in the orthonormal frame bundle $\mathrm{O}(TM)$ *Riemannian connections on M*.

THEOREM 1.6.17 (Horizontal lifts in principal bundles). *Let $\pi\colon P \to M$ be a principal G-bundle over M equipped with a G-connection. Furthermore, let $x\colon I \to M$, $t \mapsto x(t)$, be a differentiable curve and $t_0 \in I$. Then, to each $u_0 \in P$ with $\pi(u_0) = x(t_0)$,* *there exists exactly one horizontal curve* $u\colon I \to P$ *with* $u(t_0) = u_0$ *which is above* $t \mapsto x(t)$*, i.e., such that* $(\pi \circ u)(t) = x(t)$ *and* $\dot u(t) \in H_{u(t)}$ *for each* $t \in I$.

PROOF. It is obviously sufficient to verify existence and uniqueness of the horizontal lift locally about t_0. By means of the G-invariance, along with $t \mapsto u(t)$ and $g \in G$ also $t \mapsto u(t)g$ is a horizontal curve above $t \mapsto x(t)$. Hence pieces of local horizontal lifts can be patched together to obtain the horizontal lift defined on all of I.

Let $\Phi\colon P \times G \to P$, $(u,g) \mapsto ug$, denote the right action of G on P. For fixed $g \in G$ then $R_g \equiv \Phi(\cdot, g)\colon P \to P$ is right multiplication by g, and for $u \in P$ we have the already considered embedding $I_u \equiv \Phi(u,\cdot)\colon G \to P$. Recall that we used its differential $\iota_u \equiv (dI_u)_e\colon \mathfrak{g} \to T_uP$ at $e \in G$ to identify $\mathfrak{g}$ and the vertical fiber V_u, in particular $\omega \circ \iota_u = \mathrm{id}_{\mathfrak{g}}$. Let now $t \mapsto x(t) =: x_t$ be the curve in M where without restrictions we may assume that $x(\cdot)$ takes values in the domain U of a bundle chart (φ, U). By means of $\varphi\colon \pi^{-1}(U) \xrightarrow{\sim} U \times G$ we first procure some differentiable curve $t \mapsto v_t$ in P which lies above $t \mapsto x_t$, for instance $v_t := \varphi^{-1}(x_t, e)$ with e the unit element in G. Next we search a curve $t \mapsto \tilde g_t$ in G such that $t \mapsto u_t := v_t\,\tilde g_t$ becomes horizontal, i.e. $\omega(\dot u) \equiv 0$). Writing $v_t = u_t\, g_t$ with $g_t := \tilde g_t^{-1}$, we get $\dot v_t = (R_{g_t})_* \dot u_t + (I_{u_t})_* \dot g_t$. By means of the equivariance of ω (Remark 1.6.14) along with the relation $I_{u_t} = I_{u_t g_t} \circ L_{g_t^{-1}}$, we then have

$$\omega(\dot v_t) = \mathrm{Ad}(g_t^{-1})\,\omega(\dot u_t) + \big(L_{g_t^{-1}}\big)_* \dot g_t. \tag{1.6.7}$$

To make $t \mapsto u_t$ horizontal the condition $\dot g_t = (L_{g_t})_* \omega(\dot v_t)$ is required. This is a differential equation for $t \mapsto g_t$ which we write as

$$\dot g_t = A(t, g_t) \tag{1.6.8}$$

where $A(t,\cdot) \in \Gamma(TG)$ defined as $A(t,h) = (L_h)_* \omega(\dot v_t)$ is just the left-invariant vector field on G associated to $\omega(\dot v_t) \in \mathfrak{g}$. The proof is completed by the fact that Eq. (1.6.8) has a unique local solution to each given initial condition. $\square$

REMARK 1.6.18. Note that uniqueness in Theorem 1.6.17 comes from the unique solvability of differential equation (1.6.8) for a given initial condition. Uniqueness can also be seen directly: If $t \mapsto u_t$ and $t \mapsto v_t$ are two horizontal lifts, we may write $v_t = u_t\, g_t$. However, since $\omega(\dot v) = \omega(\dot u) \equiv 0$ we conclude from (1.6.7) that $g_t \equiv$ constant. Thus if $u_{t_0} = v_{t_0}$ for one t_0, then necessarily $g_t \equiv e$.

REMARK 1.6.19. If in the situation of Theorem 1.6.17 there is a representation $G \to \mathrm{Aut}(V)$ of G and $E = P \times_G V$ the vector bundle associated to P, then $P \cong \mathrm{Iso}_G(V;E)$, and Theorem 1.6.17 be reduced to the already established existence and uniqueness of horizontal lifts in vector bundles. Indeed, by Remark 1.6.16 (a) one obtains a linear connection in E, and by Theorem 1.4.19 to $\xi \in V$ there exists exactly one horizontal curve $I \to E$, $t \mapsto e_\xi(t)$ over $t \mapsto x(t)$ such that $e_\xi(t_0) = u_0\,\xi \equiv (u_0, \xi)\cdot G$. Then the curve

$$t \mapsto u(t) \in \mathrm{Iso}_G(V; E_{x(t)}), \quad u(t) := \big(\xi \mapsto e_\xi(t)\big),$$

in P is horizontal and has the wanted properties.

COROLLARY 1.6.20. *Let P be a principal G-bundle over a manifold M.*

(1) *Every G-connection in P defines canonically a parallel transport in P along differentiable curves $t \mapsto x(t)$ in M, namely for $t_0, t_1 \in I$ as*

$$/\!/_{t_0,t_1}\colon P_{x(t_0)} \xrightarrow{\sim} P_{x(t_1)}, \quad u_0 \longmapsto u(t_1),$$

where $t \mapsto u(t)$ is the according to Theorem 1.6.17 *uniquely determined horizontal lift of $t \mapsto x(t)$ to P such that $u(t_0) = u_0$.*

(2) *If E a vector bundle associated to P with fiber V, then this parallel transport induces a parallel transport in E, namely as*

$$//_{t_0,t_1}\colon E_{x(t_0)} \xrightarrow{\sim} E_{x(t_1)}, \quad e_0 \longmapsto u(t_1)\,\xi,$$

where as above $t \mapsto u(t)$ with $u(t_0) = u_0$ is horizontal lift of $t \mapsto x(t)$ to P, and $\xi \in V$ is chosen such that $u(t_0)\,\xi \equiv u_0\,\xi = e_0$.

For the remainder of this section we restrict ourselves to the principal bundle $P = \mathrm{L}(TM)$ over a differentiable manifold M with structure group $G = \mathrm{GL}(n;\mathbb{R})$, respectively, $P = \mathrm{O}(TM)$ over a Riemannian manifold M with $G = \mathrm{O}(n)$. The corresponding Lie algebras are then the matrix algebras

$$\mathfrak{g} = \mathrm{M}(n \times n;\mathbb{R}), \quad \text{resp., } \mathfrak{g} = \{A \in \mathrm{M}(n \times n;\mathbb{R})\colon A \text{ skew symmetric}\}.$$

Fixing a G-connection in P, we have the $\mathfrak{g}$-valued connection form (see Definition 1.6.13)

$$\omega \in \Gamma(T^*P \otimes \mathfrak{g}), \quad \omega_u(X_u) = \kappa_u^{-1}(\mathrm{vert}\,X)_u,\ u \in P \text{ and } X \in \Gamma(TP). \tag{1.6.9}$$

In addition to the connection form ω we have the *canonical one-form* of the principal bundle $\pi\colon P \to M$,

$$\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n), \quad \vartheta_u(X_u) := u^{-1}(d\pi X_u),\ u \in P \text{ and } X \in \Gamma(TP), \tag{1.6.10}$$

where as usual $u \in P$ is read as linear isomorphism, resp. isometry, $u\colon\ \mathbb{R}^n \xrightarrow{\sim} T_{\pi(u)}M$. Note that contrary to the connection form the canonical one-form ϑ does not depend on the chosen G-connection.

THEOREM 1.6.21. *The frame bundles $P = \mathrm{L}(TM)$ (M manifold), resp. $P = \mathrm{O}(TM)$ (M Riemannian manifold), considered as manifolds, are parallelizable, i.e., the tangent bundles $T\,\mathrm{L}(TM) \to \mathrm{L}(TM)$ and $T\,\mathrm{O}(TM) \to \mathrm{O}(TM)$ are trivial.*

PROOF. Indeed a G-connection in P decomposes $TP = V \oplus H$. A canonical trivialization for TP is given as follows: the vertical subbundle V is trivialized by the standard-vertical vector fields $\hat{A}$ to A, where A runs through a basis of $\mathfrak{g}$; the horizontal subbundle H is trivialized by the *standard-horizontal vector fields* $L_1, \ldots, L_n$ in $\Gamma(TP)$ defined by

$$L_i(u) := h_u(ue_i).$$

For every $u \in P$,

$$\big(\hat{A}(u), L_i(u):\ A \in \text{basis for } \mathfrak{g},\ i = 1, \ldots, n\big)$$

is a basis of $T_uP = V_u \oplus H_u$. This is obvious from the isomorphisms $\mathfrak{g} \xrightarrow{\sim} V_u$, $A \mapsto \hat{A}(u)$ and $h_u\colon\ T_{\pi(u)}M \xrightarrow{\sim} H_u$. □

REMARK 1.6.22. The standard-vertical, respectively standard-horizontal vector fields are determined by the relations

$$\vartheta(\hat{A}) = 0 \ \text{ and } \ \vartheta(L_i) = e_i,$$
$$\omega(\hat{A}) = A \ \text{ and } \ \omega(L_i) = 0.$$

The canonical second order partial differential operator

$$\Delta^{\mathrm{hor}} := \sum_{i=1}^{n} L_i^2$$

is called *horizontal Laplacian* on $\mathrm{L}(TM)$, resp. $\mathrm{O}(TM)$.

NOTATION 1.6.23. Let $\pi\colon P \to M$ be a principal G-bundle over M equipped with a G-connection. For a vector field $X \in \Gamma(TM)$ we denote by

$$\bar{X} \in \Gamma(TP), \quad \bar{X}_u = h_u(X_{\pi(u)}),\ u \in P,$$

the corresponding horizontal lift to P.

LEMMA 1.6.24. *Let M be a differentiable manifold with a linear connection in TM. If $X, Y \in \Gamma(TM)$ and $\alpha \in \Gamma(T^*M)$, then*

$$\begin{aligned}(\nabla_X Y)_x &= \lim_{\varepsilon\downarrow 0} \frac{/\!/_{0,\varepsilon}^{-1} Y_{\gamma(\varepsilon)} - Y_{\gamma(0)}}{\varepsilon},\\ (\nabla_X \alpha)_x &= \lim_{\varepsilon\downarrow 0} \frac{/\!/_{0,\varepsilon}^{-1} \alpha_{\gamma(\varepsilon)} - \alpha_{\gamma(0)}}{\varepsilon},\end{aligned}$$

where $/\!/_{0,\varepsilon}\colon T_{\gamma(0)}M \to T_{\gamma(\varepsilon)}M$ is the parallel transport along a curve γ on M with the properties that $\gamma(0) = x$ and $\dot\gamma(0) = X_x$.

(Note that for $Y \in \Gamma(TM)$ by definition $(/\!/_{0,\varepsilon}^{-1}\,\alpha_{\gamma(\varepsilon)})(Y_{\gamma(0)}) = \alpha_{\gamma(\varepsilon)}(/\!/_{0,\varepsilon}\,Y_{\gamma(0)})$.

PROOF. The first formula has been shown in Remark 1.4.14, to verify the second one first note that

$$\begin{aligned}\frac{/\!/_{0,\varepsilon}^{-1}\,\alpha_{\gamma(\varepsilon)} - \alpha_{\gamma(0)}}{\varepsilon}\,(Y_{\gamma(0)}) &= \frac{\alpha_{\gamma(\varepsilon)}(/\!/_{0,\varepsilon}\,Y_{\gamma(0)}) - \alpha_{\gamma(0)}\,Y_{\gamma(0)}}{\varepsilon}\\ &= \frac{\alpha_{\gamma(\varepsilon)}\big(/\!/_{0,\varepsilon}\,Y_{\gamma(0)} - Y_{\gamma(\varepsilon)}\big)}{\varepsilon} + \frac{\alpha_{\gamma(\varepsilon)}Y_{\gamma(\varepsilon)} - \alpha_{\gamma(0)}Y_{\gamma(0)}}{\varepsilon}.\end{aligned}$$

Taking the limit as $\varepsilon \downarrow 0$, the right-hand side converges to

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \alpha_{\gamma(0)} \frac{/\!/_{0,\varepsilon}^{-1}\big(/\!/_{0,\varepsilon}Y_{\gamma(0)} - Y_{\gamma(\varepsilon)}\big)}{\varepsilon} + \lim_{\varepsilon\downarrow 0} \frac{\alpha_{\gamma(\varepsilon)}Y_{\gamma(\varepsilon)} - \alpha_{\gamma(0)}Y_{\gamma(0)}}{\varepsilon}\\ &\quad= \alpha_{\gamma(0)}\Big(\lim_{\varepsilon\downarrow 0} \frac{Y_{\gamma(0)} - /\!/_{\varepsilon,0}Y_{\gamma(\varepsilon)}}{\varepsilon}\Big) + \lim_{\varepsilon\downarrow 0} \frac{\alpha_{\gamma(\varepsilon)}Y_{\gamma(\varepsilon)} - \alpha_{\gamma(0)}Y_{\gamma(0)}}{\varepsilon}\\ &\quad= -\alpha_x\big(\nabla_{X_x}Y\big) + X_x(\alpha Y) = (\nabla_X\alpha)_x(Y_x),\end{aligned}$$

which gives the claim. $\square$

NOTATION 1.6.25. Let M be a differentiable manifold and $P = \mathrm{L}(TM)$ be the frame bundle over M, respectively, M a Riemannian manifold and $P = \mathrm{O}(TM)$ the orthonormal frame bundle over M. It is convenient to write vector fields $Y \in \Gamma(TM)$ and differential forms $\alpha \in \Gamma(T^*M)$ as equivariant functions on the frame bundle P,

$$\begin{aligned}&f_Y\colon P \to \mathbb{R}^n, \quad f_Y(u) := u^{-1}Y_{\pi(u)}\\ &F_\alpha\colon P \to \mathbb{R}^n, \quad F_\alpha^i(u) := \alpha_{\pi(u)}(ue_i), \quad i = 1, \dots, n.\end{aligned}$$

Equivariance means that for $g \in G = \mathrm{GL}(n;\mathbb{R})$, respectively $\mathrm{O}(n)$,

$$\begin{aligned}f_Y(ug) &= g^{-1}f_Y(u),\\ F_\alpha(ug) &= g^*F_\alpha(u),\end{aligned}$$

where g^{-1} and g^* are the inverse, resp. dual linear map to g.

THEOREM 1.6.26. *Let M be a differentiable manifold and $P = \mathrm{L}(TM)$ be the frame bundle over M endowed with a $\mathrm{GL}(n;\mathbb{R})$-connection, respectively, M a Riemannian manifold and $P = \mathrm{O}(TM)$ the associated orthonormal frame bundle over M endowed with a $\mathrm{O}(n)$-connection. Then, for vector fields $X, Y \in \Gamma(TM)$, the covariant derivative $\nabla_X Y \in \Gamma(TM)$ with respect to the induced linear connection in TM is given by*

$$(\nabla_X Y)_x = u\big(\bar{X}_u \vartheta(\bar{Y})\big) \quad \textit{for } u \in P \textit{ with } \pi(u) = x. \tag{1.6.11}$$

Writing $Y \in \Gamma(TM)$ as equivariant function f_Y on P, this formula reads as

$$(\nabla_X Y)_x = u(\bar{X}_u f_Y),$$

or equivalently:

$$\langle u^{-1}(\nabla_X Y)_x, e_i\rangle = \bar{X}_u(f_Y^i), \quad i = 1, \ldots, n. \tag{1.6.12}$$

PROOF. We choose a curve γ on M such that $\gamma(0) = x$ and $\dot\gamma(0) = X_x$. Let $t \mapsto u(t)$ be a horizontal lift of $t \mapsto \gamma(t)$ to P. Then, by Corollary 1.6.20, the parallel transport $//_{0,\varepsilon}\colon T_{\gamma(0)}M \to T_{\gamma(\varepsilon)}M$ along γ is given by $//_{0,\varepsilon} = u(\varepsilon)u(0)^{-1}$. By Lemma 1.6.24 we get

$$\begin{aligned}
(\nabla_X Y)_{\pi(u)} &= \lim_{\varepsilon \downarrow 0} \frac{u(0)u(\varepsilon)^{-1} Y_{\pi(u(\varepsilon))} - Y_{\pi(u(0))}}{\varepsilon} \\
&= \lim_{\varepsilon \downarrow 0} \frac{u\ \big(u(\varepsilon)^{-1} Y_{\pi(u(\varepsilon))} - u(0)^{-1} Y_{\pi(u(0))}\big)}{\varepsilon} \\
&= u\Big(\lim_{\varepsilon \downarrow 0} \frac{f\big(u(\varepsilon)\big) - f\big(u(0)\big)}{\varepsilon}\Big) = u\big(\bar{X}_u(f)\big).
\end{aligned}$$

In the last equality we used $\dot u(t) = h_{u(t)}\dot\gamma(t)$ which implies $\dot u(0) = h_u(X_x) = \bar{X}_u$. □

We can give formulas analogous to Eq. (1.6.12) also for the covariant derivative of differential forms can be described. We note the result for later reference.

THEOREM 1.6.27. *Let M be a differentiable manifold and $P = \mathrm{L}(TM)$ be the frame bundle over M endowed with a $\mathrm{GL}(n;\mathbb{R})$-connection, respectively, M a Riemannian manifold and $P = \mathrm{O}(TM)$ the associated orthonormal frame bundle over M endowed with a $\mathrm{O}(n)$-connection. Furthermore, let $\alpha \in \Gamma(T^*M)$ be a differential form on M, according to Notation 1.6.25, read as equivariant function on P. Then, for $X \in \Gamma(TM)$ with horizontal lift $\bar{X} \in \Gamma(TP)$ and $u \in P$, the following formula holds:*

$$(\nabla_X \alpha)_{\pi(u)}(ue_i) = \bar{X}_u F^i, \quad i = 1, \ldots, d. \tag{1.6.13}$$

PROOF. We may proceed as in the proof of Theorem 1.6.26. Let $u \in P$ and $t \mapsto \gamma(t)$ be a curve on M such that $\gamma(0) = \pi(u)$ and $\dot\gamma(0) = X_{\pi(u)}$. Furthermore, let $t \mapsto u(t) \in P$ be the horizontal lift of γ with $u(0) = u$. By Lemma 1.6.24 we obtain:

$$\begin{aligned}
(\nabla_X \alpha)_{\pi(u)}(ue_i) &= \lim_{\varepsilon \downarrow 0} \frac{\big(//_{0,\varepsilon}^{-1}\, \alpha_{\gamma(\varepsilon)}\big)(ue_i) - \alpha_{\gamma(0)}(ue_i)}{\varepsilon} \\
&= \lim_{\varepsilon \downarrow 0} \frac{\alpha_{\gamma(\varepsilon)}\ \big(//_{0,\varepsilon}(ue_i)\big) - \alpha_{\gamma(0)}(ue_i)}{\varepsilon} \\
&= \lim_{\varepsilon \downarrow 0} \frac{\alpha_{\pi \circ u(\varepsilon)}\ \big(u(\varepsilon)e_i\big) - \alpha_{\pi \circ u(0)}\big(u(0)e_i\big)}{\varepsilon} \\
&= \lim_{\varepsilon \downarrow 0} \frac{F^i\big(u(\varepsilon)\big) - F^i\big(u(0)\big)}{\varepsilon} = \bar{X}_u F^i.
\end{aligned}$$

□

DEFINITION 1.6.28 (Horizontal lift of an M-valued semimartingale). Let P a principal G-bundle over a differentiable manifold M and $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ be the connection form of a G-connection in P. For a P-valued semimartingale U, the process $\int_U \omega$ takes values by definition in the Lie algebra $\mathfrak{g}$ and is defined with respect to a basis of $\mathfrak{g}$ as

$$\int_U \omega \equiv \left(\int_U \omega^1, \ldots, \int_U \omega^r\right), \quad \omega = (\omega^1, \ldots, \omega^r).$$

(a) The process U is called *horizontal* if $\int_U \omega = 0$ a.s.
(b) For an M-valued semimartingale X, a semimartingale U taking values in P is called *horizontal lift of* X if $\pi \circ U = X$ a.s. and if in addition U is horizontal.

Definition 1.6.28 generalizes the classical notion of horizontal lift for differentiable curves in M (see Theorem 1.6.17): a curve $t \mapsto u(t)$ over $t \mapsto x(t)$ is called horizontal if $\pi \circ u = x$ and $\omega(\dot{u}) = 0$ (see Example 1.3.11). Existence of horizontal lifts for semimartingales will be proved in Theorem 1.6.35 below.

For the remainder of this Section we deal with the following situation: Let M be an n-dimensional manifold equipped with a torsion-free linear connection, respectively, a Riemannian manifold with the Levi-Civita connection. Above M we consider the frame bundle $P = \mathrm{L}(TM)$ with structure group $\mathrm{GL}(n;\mathbb{R})$, respectively, the orthonormal frame bundle $P = \mathrm{O}(TM)$ with structure group $\mathrm{O}(n)$, each with the induced G-connection on P. In addition to the connection form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ we have the canonical one-form $\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n)$, see (1.6.10). The induced decomposition $TP = V \oplus H$ is then given by $V_u = \ker \vartheta_u$ and $H_u = \ker \omega_u$ for $u \in P$.

DEFINITION 1.6.29 (Anti-development of an M-valued semimartingale). Let X be an M-valued semimartingale and U a horizontal lift of X taking values in $P = \mathrm{L}(TM)$, resp. $\mathrm{O}(TM)$. The $\mathbb{R}^n$-valued semimartingale

$$Z = \int_U \vartheta \equiv \int \vartheta(\circ\, dU)$$

is called *anti-development* of X into $\mathbb{R}^n$ (with respect to the initial frame U_0). In terms of the standard basis of $\mathbb{R}^n$ we have $Z \equiv (Z^1, \ldots, Z^n)$ where $Z^i = \int_U \vartheta^i$. We call

$$\mathscr{A}(X) = U_0 \int_U \vartheta \equiv \int \vartheta(\circ\, dU)$$

anti-development of X into $T_{X_0}M$, or briefly *anti-development* of X. Note that $\mathscr{A}(X)$ is independent of the choice of U_0.

THEOREM 1.6.30. *Let X be an M-valued semimartingale, U a horizontal lift of X to $P = \mathrm{L}(TM)$ resp. $\mathrm{O}(TM)$, and Z an anti-development of X into $\mathbb{R}^n$. The following statements hold:*

(i) $\displaystyle\int_U \sigma = \sum_{i=1}^n \int \sigma(U)\, L_i(U) \circ dZ^i$ *for each differential form* $\sigma \in \Gamma(T^*P)$*;*

(ii) $\displaystyle\int_X \alpha = \sum_{i=1}^n \int \alpha(X)\, Ue_i \circ dZ^i$ *for each differential form* $\alpha \in \Gamma(T^*M)$.

In particular, $d\big(f(U)\big) = \sum_{i=1}^n (L_i f)(U) \circ dZ^i$ *for each function* $f \in C^\infty(P)$*, in short-terms*

$$dU = \sum_{i=1}^n L_i(U) \circ dZ^i, \tag{1.6.14}$$

as well as $d\big(f(X)\big)=\sum\limits_{i=1}^{n}(Ue_i)(f)\circ dZ^i$ *for each function* $f\in C^\infty(M)$*, or in short-terms*

$$dX=U\circ dZ. \tag{1.6.15}$$

PROOF. The additional claims follow from (i) and (ii) with $\sigma=df$ for $f\in C^\infty(P)$, resp. $\alpha=df$ for $f\in C^\infty(M)$.

To (i): According to Theorem 1.3.9 it is sufficient that the right-hand side of (i) has the defining properties of $\int_U\sigma$. For $f\in C^\infty(P)$ we have to show that

$$d\big(f(U)\big)=\sum_i(df)(U)\,L_i(U)\circ dZ^i\equiv\sum_i(L_if)(U)\circ dZ^i$$

which is equivalent to

$$f(U)-f(U_0)=\int_U\sigma\quad\text{where }\sigma\in\Gamma(T^*P),\ \sigma_u:=\sum_i(L_if)(u)\,\vartheta_u^i. \tag{1.6.16}$$

However observe that $\sum_i(L_if)(u)\,\vartheta_u^i=(df)_u\circ\mathrm{pr}_{H_u}$, indeed for $A\in T_uP$ we have

$$\begin{aligned}\sum_i(L_if)(u)\,\vartheta_u^i(A)&=\sum_i(df)_u\,L_i(u)\,\vartheta_u^i(A)\\&=\sum_i(df)_u\,h_u(ue_i)\,\big(u^{-1}(d\pi)_uA\big)^i\\&=(df)_u\,h_u(u\,u^{-1}(d\pi)_uA)\\&=(df)_u\,h_u\big((d\pi)_uA\big)\\&=\big((df)_u\circ\mathrm{pr}_{H_u}\big)(A).\end{aligned}$$

On the other side, we have $(df\circ\mathrm{pr}_V)_u=(df)_u\,\kappa_u\,\omega_u=d(f\circ I_u)_e\,\omega_u$. But U is horizontal and hence $\int_U df\circ\mathrm{pr}_V=0$ which shows that

$$f(U)-f(U_0)=\int_U df=\int_U df\circ\mathrm{pr}_H+\int_U df\circ\mathrm{pr}_V=\int_U df\circ\mathrm{pr}_H=\int_U\sigma.$$

The second defining property of the Stratonovich integral is obvious.

To (ii): It is sufficient to show that

$$d\big(f(X)\big)=\sum_i(df)(X)\,Ue_i\circ dZ^i\equiv\sum_i(Ue_i)(f)\circ dZ^i$$

holds for each function $f\in C^\infty(M)$. With part (i) using that $(d\pi)_u\,L_i(u)=ue_i$, we obtain

$$\begin{aligned}d\big((f\circ\pi)(U)\big)&=\sum_i d(f\circ\pi)(U)\,L_i(U)\circ dZ^i\\&=\sum_i(df)\big(\pi(U)\big)\,(d\pi)(U)\,L_i(U)\circ dZ^i\\&=\sum_i(df)(X)\,Ue_i\circ dZ^i,\end{aligned}$$

which shows the claim. □

THEOREM 1.6.31. *Let X be an M-valued semimartingale, U a horizontal lift of X to $P=\mathrm{L}(TM)$ resp. $\mathrm{O}(TM)$, and Z an anti-development of X into $\mathbb{R}^n$. Then*

(i) $\displaystyle\int a(dU,dU)=\sum_{i,j=1}^{n}\int a(U)\,\big(L_i(U),L_j(U)\big)\,d[Z^i,Z^j]$ *for* $a\in\Gamma(T^*P\otimes T^*P)$*;*

(ii) $\displaystyle\int b(dX,dX) = \sum_{i,j=1}^{n} \int b(X)\big(Ue_i, Ue_j\big)\, d[Z^i, Z^j]$ *for* $b \in \Gamma(T^*M \otimes T^*M)$.

PROOF. It is again sufficient to consider the special case $a = d\varphi_1 \otimes d\varphi_2$ where $\varphi_1, \varphi_2 \in C^\infty(P)$, resp. $b = df_1 \otimes df_2$ where $f_1, f_2 \in C^\infty(M)$. The statements then follow directly from the properties of the quadratic variation or from Theorem 1.6.30 with formula (1.3.10). □

PROPOSITION 1.6.32 (Left-invariant SDE on a Lie group). *Let G be a Lie group and $\mathfrak{g}$ the corresponding Lie algebra. We identify*

$$\mathfrak{g} \xrightarrow{\sim} \{\textit{left-invariant vector fields on } G\}, \quad A \longmapsto A(\cdot),$$

where $A(g) = (L_g)_* A(e) \equiv (L_g)_* A$ and $(L_g)_* \colon \mathfrak{g} \xrightarrow{\sim} T_g G$ is the differential of the left multiplication L_g. Let $A_1, \ldots, A_r \in \mathfrak{g}$ and $A_1(\cdot), \ldots, A_r(\cdot) \in \Gamma(TG)$ the corresponding left-invariant vector fields. Let γ be a continuous $\mathbb{R}^r$-valued semimartingale. Then each maximal solution of the Stratonovich SDE

$$dg = \sum_{i=1}^{r} A_i(g) \circ d\gamma^i \tag{1.6.17}$$

has infinite lifetime. If $(g_t)_{t\geq 0}$ is a solution to SDE (1.6.17)*, then $\tilde g_t := g_t^{-1}$ satisfies the SDE*

$$d\tilde g = -\sum_{i=1}^{r} \big(\mathrm{Ad}(\tilde g^{-1}) A_i\big)(\tilde g) \circ d\gamma^i, \quad \tilde g_0 = g_0^{-1}. \tag{1.6.18}$$

PROOF. (a) Note that SDE (1.6.18) is equivalent to

$$d\tilde g = -\textstyle\sum_{i=1}^{r} (R_{\tilde g})_* A_i(e) \circ d\gamma^i. \tag{1.6.19}$$

Let now $(\tilde g_t)_{t\geq 0}$ be a semimartingale satisfying (1.6.19) and $(g_t)_{t\geq 0}$ be a solution to (1.6.17). Then we have:

$$d\big(f(g\tilde g)\big) = f_*(L_g)_* \circ d\tilde g + f_*(R_{\tilde g})_* \circ dg = 0, \quad f \in C^\infty(G). \tag{1.6.20}$$

Indeed letting $Q\colon G \times G \to G$, $(g, \tilde g) \mapsto g\tilde g = L_g \tilde g = R_{\tilde g} g$ denote multiplication on G, by Remark 1.2.24, to verify the first equality in (1.6.20), it is sufficient to show that

$$(f \circ Q)_* \colon T_{(g,\tilde g)}(G \times G) \cong T_g G \times T_{\tilde g} G \to \mathbb{R}$$

satisfies the formula:

$$(f \circ Q)_*(v, w) = f_*(L_g)_* w + f_*(R_{\tilde g})_* v. \tag{1.6.21}$$

This is however easy to see by curve transport. Let v be represented by the curve α: $\alpha(0) = g$, $\dot\alpha(0) = v$, and analogously w by β: $\beta(0) = \tilde g$, $\dot\beta(0) = w$, then $(f \circ Q)_*(v, w)$ is represented by the $t \mapsto f\big(\alpha(t)\,\beta(t)\big)$ at 0. For this we have

$$\tfrac{d}{dt} f(\alpha\beta) = f_*\,(L_\alpha)_* \dot\beta + f_*(R_\beta)_* \dot\alpha,$$

and hence

$$\tfrac{d}{dt}\big|_{t=0} f(\alpha\,\beta) = f_*(L_g)_* w + f_*(R_{\tilde g})_* v.$$

The second equality in (1.6.20) is then immediate from (1.6.17) and (1.6.19). From (1.6.20) we then conclude that $(\tilde g_t)_{t\geq 0} \equiv (g_t^{-1})_{t\geq 0}$ modulo indistinguishability.

(b) Note that if $(g_t)_{t\geq 0}$ solves SDE (1.6.17) with initial condition $g_0 = e$ and if ξ_0 is an $\mathscr{F}_0$-measurable G-valued random variable, then $(g'_t)_{t\geq 0}$ where $g'_t := \xi_0 g_t$ is the solution with initial condition $g'_0 = \xi_0$.

(c) It remains to verify that the maximal solution to

$$dg = \sum_{i=1}^{r} A_i(g) \circ d\gamma^i, \quad g_0 = e, \tag{1.6.22}$$

has infinite lifetime. To this end, we fix a relatively compact open coordinate neighbourhood V of the unit element e in G and construct inductively an increasing sequence $(\tau_n)_{n\geq 0}$ of stopping times:

$$\tau_0 = 0, \text{ and } \tau_{n+1} = \inf\{t \geq \tau_n\colon\ g^n_t \notin V\} \wedge (n+1), \quad n \geq 0,$$

where g^n denotes the solution to (1.6.17) on $[\tau_n, \tau_{n+1}]$ satisfying $g^n_{\tau_n} = e$. A global solution $g \equiv (g_t)_{t\geq 0}$ to (1.6.22) is then inductively put together by $g|[\tau_n, \tau_{n+1}] := g_{\tau_n} g^n$. It remains to show that it has infinite lifetime which means that $\mathbb{P}\{\sup \tau_n < \infty\} = 0$. Let $\gamma = \mu + \beta$ be the Doob-Meyer decomposition of γ. Possibly after a time transformation, we may assume without restrictions that $[\mu, \mu]_t + \sum_i \int_0^t |d\beta^i| \leq \text{const} \times t$. We want to show that $\mathbb{P}\{\sup \tau_n < N\} = 0$ for each $N \in \mathbb{N}$. To this end, we first note that for any $f \in C^\infty(G)$,

$$\int_{\tau_n}^{\tau_{n+1}} \sum_{i=1}^{r} (A_i f)(g^n) \circ d\gamma^i = \int_{\tau_n}^{\tau_{n+1}} \sum_{i=1}^{r} df A_i(g^n) \circ d\gamma^i = f(g^n_{\tau_{n+1}}) - f(e). \tag{1.6.23}$$

On the other hand, since the functions $A_i(f) \in C^\infty(G)$ are bounded on $\bar{V}$ and since $\tau_{n+1} \wedge N - \tau_n \wedge N \to 0$ for $n \to \infty$, we get that a.s.

$$\int_0^N 1_{]\tau_n, \tau_{n+1}]} \sum_{i=1}^{r} (A_i f)(g^n) \circ d\gamma^i \to 0, \quad n \to \infty. \tag{1.6.24}$$

Since the left-hand sides of (1.6.23) and (1.6.24) agree on $\{\sup \tau_n < N\}$, we conclude that $\mathbb{P}\{\sup \tau_n < N\} = 0$. $\square$

DEFINITION 1.6.33 (Canonical one-form of a Lie group). Let G be a Lie group with Lie algebra $\mathfrak{g}$. The one-form $\theta \in \Gamma(T^*G \otimes \mathfrak{g})$ taking values in $\mathfrak{g}$ and defined by $\theta_g(A_g) := (L_g)_*^{-1} A_g$ is called the *canonical one-form on G*.

REMARK 1.6.34. Let $g = (g_t)_{t\geq 0}$ be a continuous semimartingale taking values in a Lie group G, then $\gamma := \int_g \theta$ defines a $\mathfrak{g}$-valued semimartingale which writes as $\gamma = \sum_{i=1}^r \gamma^i A_i$ after fixing a basis $A_1, \ldots, A_r$ for $\mathfrak{g}$. Note that the semimartingale (g_t) can be recovered from γ as solution to the SDE

$$dX = \sum_{i=1}^{r} A_i(X) \circ d\gamma^i, \quad X_0 = g_0. \tag{1.6.25}$$

This shows that each continuous semimartingale (g_t) on G is solution to an SDE of the form (1.6.25) driven by $\mathfrak{g}$-valued semimartingale (γ_t). In particular, according to Proposition 1.6.32, the inverse process $(g_t^{-1})_{t\geq 0}$ satisfies the SDE

$$d\tilde{X} = -\sum_{i=1}^{r} (R_{\tilde{X}})_* \, A_i \circ d\gamma^i, \quad \tilde{X}_0 = g_0^{-1}.$$

THEOREM 1.6.35 (Horizontal lifts of M-valued semimartingales). *Let P be principal G-bundle over a differentiable manifold M endowed with a G-connection. Furthermore, let x_0 be an M-valued random variable and u_0 a P-valued random variable above x_0, i.e. $\pi(u_0) = x_0$ a.s. Then, to each M-valued semimartingale X with $X_0 = x_0$, there exists a unique horizontal lift U of X onto P such that $U_0 = u_0$ a.s.*

PROOF. We follow the proof in the deterministic case of differentiable curves (Theorem 1.6.17). Without restrictions we may assume that X has infinite lifetime. Choosing a countable covering $(V_k)_{k\geq 0}$ of M by bundle chart domains, Lemma 1.3.1 allows inductively by means of the bundle charts $\varphi\colon \pi^{-1}(V_k) \xrightarrow{\sim} V_k \times G$ to lift X first in some way to P, that is to find a P-valued semimartingale $\tilde{U}$ such that $\pi(\tilde{U}) = X$ and $\tilde{U}_0 = u_0$. The problem is now reduced to determine a G-valued semimartingale $(\tilde{g}_t)_{t\geq 0}$ in such a way that $U := \tilde{U}\,\tilde{g}$ satisfies the wanted properties. First the connection form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ provides a $\mathfrak{g}$-valued semimartingale $\gamma := \int_{\tilde{U}} \omega \equiv \int \omega(\circ\, d\tilde{U})$ which we write as $\gamma = \sum_{i=1}^r A_i \gamma^i$ with respect to a fixed basis basis $A_1, \ldots, A_r$ of $\mathfrak{g}$. With it we define $(g_t)_{t\geq 0}$ as the maximal solution of the SDE

$$dg = \sum_{i=1}^{r} A_i(g) \circ d\gamma^i, \quad g_0 = e,$$

for which Remark 1.6.32 guarantees that (g_t) has infinite lifetime. Letting $\tilde{g}_t := g_t^{-1}$ we want to verify that $U_t = \tilde{U}_t\,\tilde{g}_t$ is horizontal. According to Remark 1.6.32, the inverted process $(\tilde{g}_t)_{t\geq 0} \equiv (g_t^{-1})_{t\geq 0}$ solves the SDE

$$d\tilde{g} = -\sum_{i=1}^{r} \big(\mathrm{Ad}(\tilde{g}^{-1})A_i\big)(\tilde{g}) \circ d\gamma^i, \quad \tilde{g}_0 = e.$$

Letting again $\Phi\colon P \times G \to P$, $(u,g) \mapsto u\cdot g$, furthermore $R_g \equiv \Phi(\cdot, g)$ and $I_u \equiv \Phi(u,\cdot)$, then

$$(\Phi^*\omega)_{(u,g)} = (R_g^*\omega)_u + (I_u^*\omega)_g = (R_g^*\omega)_u + \theta_g$$

where $\theta \in \Gamma(T^*G \otimes \mathfrak{g})$ is the canonical one-form on G given in Definition 1.6.33. By means of the pullback formula (1.3.9) for Stratonovich integrals of differential forms along semimartingales and Remark 1.6.14 (ii) one then obtains

$$\begin{aligned}\int_U \omega = \int_{\Phi(\tilde{U},\tilde{g})} \omega = \int_{(\tilde{U},\tilde{g})} \Phi^*\omega &= \int_{(\tilde{U},\tilde{g})} (R^*\omega + \theta)\\ &= \int (R_{\tilde{g}}^*\omega)(\circ\, d\tilde{U}) + \int \theta(\circ\, d\tilde{g})\\ &= \int \mathrm{Ad}(\tilde{g}^{-1})\,\omega(\circ\, d\tilde{U}) + \int (L_{\tilde{g}^{-1}})_*(\circ\, d\tilde{g}) = 0,\end{aligned}$$

since $\omega(\circ\, d\tilde{U}) = \sum A_i \circ d\gamma^i$ and $d\tilde{g} = -\sum \mathrm{Ad}(\tilde{g}^{-1})A_i(\tilde{g}) \circ d\gamma^i = -\sum (R_{\tilde{g}})_* A_i \circ d\gamma^i$. This shows that U is indeed a horizontal process.

Uniqueness of U is immediate, since given two lifts U and $\tilde{U}$ with the wanted properties, then $U = \tilde{U}g$ where $g \equiv (g_t)_{t\geq 0}$ is a G-valued semimartingale with $g_0 = e$, almost surely. By the calculation above we obtain

$$\omega(\circ\, dU) = \mathrm{Ad}(g^{-1})\,\omega(\circ\, d\tilde{U}) + \theta(\circ\, dg).$$

But U and $\tilde{U}$ are horizontal by assumption, hence $\theta(\circ\, dg) \equiv (L_{g^{-1}})_*(\circ\, dg) = 0$ which implies $dg = 0$ and thus $g_t \equiv g_0 = e$, almost surely. □

The proof of Theorem 1.6.35 provides a structural statement for semimartingales in P which we state in the case of frame bundles in explicit form.

COROLLARY 1.6.36. *Let M be a differentiable manifold and $P = \mathrm{L}(TM)$ with a G-connection where $G = \mathrm{GL}(n;\mathbb{R})$, respectively, let M be a Riemannian manifold and $P = \mathrm{O}(TM)$ with a G-connection where $G = \mathrm{O}(n)$. Assume that $\tilde{U}$ is an arbitrary* *semimartingale taking values in P. Denote its starting value by $\tilde U_0 = u_0$. Integration of the connection form $\omega \in \Gamma(T^*P \otimes \mathfrak{g})$ and the canonical one-form $\vartheta \in \Gamma(T^*P \otimes \mathbb{R}^n)$ along $\tilde U$ gives the semimartingales $\gamma = \int_{\tilde U} \omega$ with values in $\mathfrak{g}$, respectively $Z = \int_{\tilde U} \vartheta$ with values in $\mathbb{R}^n$. Fixing a basis $(A_1, \ldots, A_r)$ for $\mathfrak{g}$ and writing $\gamma = \gamma^1 A_1 + \ldots + \gamma^r A_r$, we define semimartingales g_t taking values in G and U_t taking values in P as solutions to the following SDEs:*

$$dg = \sum_{i=1}^{r} A_i(g) \circ d\gamma^i, \quad g_0 = e, \quad \textit{resp.}$$

$$dU = \sum_{i=1}^{n} L_i(U) \circ dZ^i, \quad U_0 = u_0,$$

where we read $A_1, \ldots, A_r$ as left-invariant vector fields on G and where $L_1, \ldots, L_d$ denote the standard-horizontal vector fields on P. Then, by definition, U is horizontal and $\tilde U = Ug$ holds, modulo indistinguishability.

PROOF. Along with U also $\tilde U g^{-1}$ is a horizontal lift of $\pi(\tilde U)$; since both coincide for $t = 0$ they must be equal. □

REMARK 1.6.37. There is an alternative proof of Theorem 1.6.35 (see [**41**]) which uses the fact that according to Theorem 1.2.23, each semimartingale X on M can be realized as solution to a Stratonovich SDE of the form

$$dX = \sum_{i=1}^{\ell} A_i(X) \circ dZ^i, \quad X_0 = x_0, \tag{1.6.26}$$

where Z is an $\mathbb{R}^\ell$-valued semimartingale for some ℓ. Let $\bar A_i \in \Gamma(TP)$ be the horizontal lift of $A_i \in \Gamma(TM)$, i.e. $\bar A_i(u) = h_u(A_i(\pi u))$ for $u \in P$, and consider the "horizontally lifted SDE" on P:

$$dU = \sum_{i=1}^{\ell} \bar A_i(U) \circ dZ^i, \quad U_0 = u_0. \tag{1.6.27}$$

It is clear that solutions to (1.6.27) are canonical candidates for the wanted horizontal lift. Indeed, we have $d\big(\pi(U)\big) = \sum_i (d\pi)_U \bar A_i(U) \circ dZ^i \equiv \sum_i A_i\big(\pi(U)\big) \circ dZ^i$ with $\pi(U_0) = x_0$, and hence $\pi(U) = X$ by uniqueness of solutions to (1.6.26). On the other hand, we have $\int_U \omega = \sum_i \int \omega(U) \bar A_i(U) \circ dZ^i = 0$. It thus remains to verify that U and X have identical lifetimes which is however not immediately clear from the construction.

We want to summarize the theory developed so far. Let M be a differentiable manifold equipped with a torsion-free connection, or a Riemannian manifold with the Levi-Civita connection. Over M we then have the frame bundle $P = \mathrm{L}(TM)$ with the induced $\mathrm{GL}(n;\mathbb{R})$-connection, respectively the orthonormal frame bundle $P = \mathrm{O}(TM)$ with the induced $\mathrm{O}(n)$-connection.

REMARK 1.6.38. Let u_0 be a P-valued random variable and $x_0 = \pi(u_0)$. If X is a semimartingale on M with starting value $X_0 = x_0$, then by Theorem 1.6.35 there is a unique horizontal lift U of X such that $U_0 = u_0$. By Definition 1.6.29 the anti-development Z of X into $\mathbb{R}^n$ (with initial frame u_0) is given as $Z = \int_U \vartheta$. Modulo choice of initial conditions $X_0 = x$, $U_0 = u$, each of the three processes X, U, Z determines the two others. Indeed, we have:

(a) Z determines U as solution to the SDE

$$dU = \sum_{i=1}^{n} L_i(U) \circ dZ^i, \quad U_0 = u,$$

(b) U determines X via

$$X = \pi(U),$$

(c) X determines Z as

$$Z = \int_U \vartheta$$

where U is the unique horizontal lift of X to P with $U_0 = u$.

Typically, one starts with Z on $\mathbb{R}^n$ (without restrictions $Z_0 = 0$) to determine X on M. We call X the *stochastic development* of Z. Stochastic development provides at the same time the horizontal lift U to P with $U_0 = u_0$. The frame U moves then along X by parallel transport. The process Z is recovered via $Z = \int_U \vartheta$.

REMARKS 1.6.39. (1) The described procedure depends in an obvious way on the choice u_0 above x_0. Choosing instead of u_0 another $\mathscr{F}_0$-measurable P-valued random variable $\tilde{u}_0$ such that $\pi \circ \tilde{u}_0 = x_0$ a.s. leads to $\tilde{u}_0 = u_0 g_0$ for an $\mathscr{F}_0$-measurable random variable g_0 taking values in the Lie group G of invertible, respectively orthogonal $n \times n$-matrices, so that U changes to $\tilde{U} = U g_0$. Since $R_g^* \vartheta = g^{-1}\vartheta$ for $g \in G$, the anti-development Z transforms to

$$\tilde{Z} = \int_{\tilde{U}} \vartheta = \int_U R_{g_0}^* \vartheta = \int_U g_0^{-1} \vartheta = g_0^{-1} Z. \tag{1.6.28}$$

(2) Writing

$$dU = \sum_{i=1}^{n} L_i(U) \circ dZ^i = \sum_{i=1}^{n} h_U(Ue_i) \circ dZ^i \quad \text{and} \quad dX = \sum_{i=1}^{n} Ue_i \circ dZ^i,$$

we arrive at the intrinsic formulas

$$dU = h_U(\circ\, dX) \quad \text{and} \quad dX = U \circ dZ. \tag{1.6.29}$$

(3) Fixing $u \in P$, read as isomorphism (isometry) $u\colon \mathbb{R}^n \xrightarrow{\sim} T_xM$ where $x = \pi(u)$, we may identify Z with the T_xM-valued semimartingale $\tilde{Z} = uZ$. Stochastic development then provides a one-to-one correspondence between continuous semimartingales $\tilde{Z}$ in the tangent space T_xM with $\tilde{Z}_0 = 0$ and semimartingales X on the manifold M with $X_0 = x$, where $\tilde{Z} \longmapsto X = \pi(U)$ and U defined as solution to the SDE

$$dU = \sum_{i=1}^{n} L_i(U) u^{-1} \circ d\tilde{Z}^i, \quad U_0 = u.$$

We want to give a geometric illustration of stochastic development. For instance, let $P = \mathrm{O}(TM)$ the orthonormal frame bundle over a Riemannian manifold M. We fix $u \in \mathrm{O}(TM)$ as isometry $u\colon \mathbb{R}^n \xrightarrow{\sim} T_{\pi(u)}M$ and let $x := \pi(u)$.

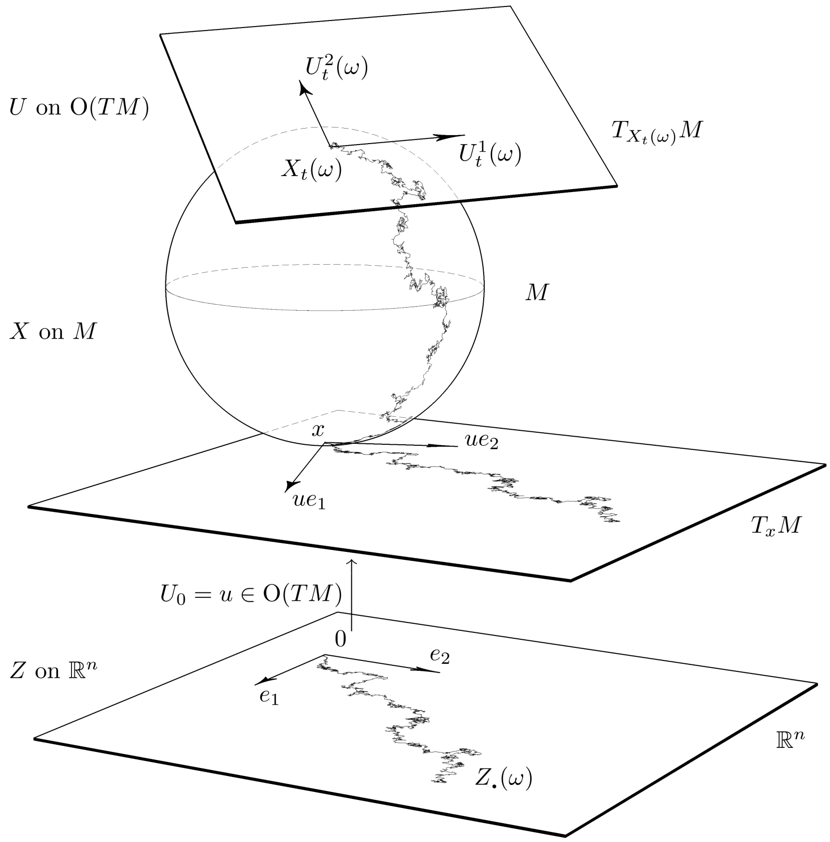


Figure 1.6.1. Stochastic development

One should think of X as the trace which the paths of Z print on the manifold M, under the identification $U\colon \mathbb{R}^n \xrightarrow{\sim} T_X M$, when M is "rolled" along $t \mapsto Z_t$ (*rolling without slipping*). In the probabilistic case however this interpretation requires further explication as in general the trajectories of Z are not differentiable and thus a pathwise procedure does not make immediate sense. Let us thus first have a look at the deterministic case of a differentiable curve $Z\colon t \mapsto z(t)$. We will show that in this case "stochastic development" reduces to the classical Cartan development of the curve $t \mapsto z(t)$.

EXAMPLE 1.6.40 (Cartan development). The *Cartan development* of an $\mathbb{R}^n$-valued curve $t \mapsto z(t)$ is the construction of curves $x\colon t \mapsto x(t) \in M$ and $u\colon t \mapsto u(t) \in P$ (where $P = \mathrm{L}(TM)$, resp. $P = \mathrm{O}(TM)$ in the Riemannian case) such that $u(\cdot)$ lies above $x(\cdot)$ and such that

(i) $\dot{x} = u\,\dot{z}$;
(ii) u is parallel along x.

Condition (i) can be rewritten as

$$dx(t) = u(t)\,dz(t)$$

and "u is parallel along x" is understood in the sense that $\nabla_D u \equiv (\nabla_D u^1, \ldots, \nabla_D u^n) = 0$ where $D = \partial/\partial t$. Condition (ii) means then that $u(\cdot)$ is a horizontal curve; thus $\dot{u} \in H_u \equiv$

$h_u(T_{\pi(u)}M)$, and since $\dot x = (\pi\circ u)^\cdot = \pi_*\dot u = u\,\dot z$ we obtain $\dot u = h_u(\dot x) = h_u(u\dot z)$ by using (i). On the other hand, since $h_u(u\dot z) = \sum_i h_u(ue_i)\dot z^i = \sum_i L_i(u)\dot z^i$, conditions (i) and (ii) are seen to be equivalent to

$$\begin{cases} du = \displaystyle\sum_{i=1}^n L_i(u)\,dz^i, \\ x(\cdot) = (\pi\circ u)(\cdot). \end{cases} \tag{1.6.30}$$

REMARK 1.6.41. (a) Note that Eq. (1.6.30) is the equation introduced above for the procedure of "rolling without slipping" in the special case of a deterministic driving process $z(t)$. In this case stochastic development reduces to classical Cartan development. In the general case of a non-trivial semimartingale Z the ordinary differential equation (1.6.30) for Cartan development needs to be rewritten as a Stratonovich type SDE.

(b) The term $L_i(u_0)$ can be interpreted as infinitesimal direction of the parallel transport of $u_0\in P$ along a curve in M with initial velocity u_0e_i at $\pi(u_0)$, i.e.,

$$\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\, /\!/_{0,\varepsilon}u_0 = L_i(u_0).$$

As already explained, we are mainly interested in the case of frame bundles over M. We distinguished so far the two cases of the frame bundle $\mathrm{L}(TM)$ and the orthonormal frame bundle $P=\mathrm{O}(TM)$ if M carries in addition a Riemannian metric. We want to check quickly that the two points of view are compatible for the procedure of stochastic development.

REMARK 1.6.42. Let M be a Riemannian manifold equipped with the Levi-Civita connection. The inclusion $\big(\mathrm{O}(TM),\mathrm{O}(n)\big) \xhookrightarrow{j} \big(\mathrm{L}(TM),\mathrm{GL}(n;\mathbb{R})\big)$ defines a homomorphism of principal bundles with $\mathfrak{g}=\mathfrak{o}(d) \xhookrightarrow{j_0} \bar{\mathfrak{g}} := \mathfrak{gl}(d;\mathbb{R})$ the inclusion of the corresponding Lie algebras. This gives the following situation:

$$\begin{tikzcd}
 & \mathrm{O}(TM)\times\mathfrak{g} \arrow[d, equal] & V\oplus H \arrow[d, equal] & & \\
0 \arrow[r] & \mathfrak{g} \arrow[r] \arrow[d, "j_0"'] & T\,\mathrm{O}(TM) \arrow[r] \arrow[d, "j_*"] \arrow[l, bend left, dashed, "\omega"] & \pi^*TM \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \bar{\mathfrak{g}} \arrow[r] & T\,\mathrm{L}(TM) \arrow[r] \arrow[l, bend left, dashed, "\bar\omega"] & \pi^*TM \arrow[r] & 0 \\
 & \mathrm{L}(TM)\times\bar{\mathfrak{g}} \arrow[u, equal] & \bar V\oplus\bar H \arrow[u, equal] & &
\end{tikzcd}$$

Let X be an M-valued semimartingale and u_0 an $\mathscr{F}_0$-measurable $\mathrm{O}(TM)$-valued random variable. In addition, let U be the horizontal lift of X to $\mathrm{O}(TM)$ and $\bar U$ the horizontal lift of X to $\mathrm{L}(TM)$ such that $U_0=\bar U_0 = u_0$ a.s. Let $Z=\int_U\vartheta$ and $\bar Z = \int_{\bar U}\bar\vartheta$. Then, modulo indistinguishability, $Z=\bar Z$ and $j(U)=\bar U$ hold.

PROOF. It is straightforward to see that $j_*H = \bar H$, $j^*\bar\omega = j_0\omega$ and $j^*\bar\vartheta = \vartheta$ where ω,ϑ (respectively $\bar\omega,\bar\vartheta$) denote the connection form and canonical one-form on $P=\mathrm{O}(TM)$, respectively on $P=\mathrm{L}(TM)$. This gives

$$\textstyle\int_{j(U)}\bar\omega = \int_U j^*\bar\omega = j_0\int_U\omega = 0$$

which by uniqueness of the horizontal lift implies $j(U)=\bar U$. On the other hand we have

$$\textstyle\int_{j(U)}\bar\vartheta = \int_U j^*\bar\vartheta = \int_U\vartheta$$

which shows $\bar Z = Z$. $\square$

DEFINITION 1.6.43 (Parallel transport along a semimartingale). Let M be a differentiable manifold equipped with a torsion-free connection, or a Riemannian manifold equipped with the Levi-Civita connection. Let X be a semimartingale on M and U an arbitrary horizontal lift of X to $\mathrm{L}(TM)$ resp. $\mathrm{O}(TM)$. For $0 \leq s \leq t$ let $//_{s,t} := U_t \circ U_s^{-1}$ be given by

$$\begin{tikzcd} T_{X_s}M \arrow[rr, "\sim"'] & & T_{X_t}M \\ & \mathbb{R}^n \arrow[ul, "U_s"] \arrow[ur, "U_t"'] & \end{tikzcd}$$

The isomorphisms (resp. isometries in the Riemannian case)

$$//_{0,t}\colon T_{X_0}M \to T_{X_t}M$$

are called *stochastic parallel transport along X*.

REMARK 1.6.44. The parallel transports $//_{0,t}$ extend canonically from the tangent bundle TM to tensors of type (p,q), i.e., to $TM^{\otimes p} \otimes (T^*M)^{\otimes q}$, and then to

$$\bigoplus_{p,q\geq 0}^{N} TM^{\otimes p} \otimes (T^*M)^{\otimes q}, \quad N \in \mathbb{N}.$$

Note that, for $\alpha \in \Gamma(T^*M)$ and $A \in \Gamma(TM)$, by definition

$$\big(//_{0,t}\alpha_{X_0}\big)(A_{X_t}) = \alpha_{X_0}(//_{t,0}A_{X_t}).$$

THEOREM 1.6.45 (Geometric Itô formula). *Let M be a differentiable manifold endowed with a linear connection ∇ (without restriction ∇ torsion-free). Let X be an M-valued semimartingale, U a horizontal lift of X to $\mathrm{L}(TM)$ and $Z = \int_U \vartheta$ the corresponding anti-development of X into $\mathbb{R}^n$. For each $f \in C^\infty(M)$ the following formula holds:*

$$d\big(f(X)\big) = \sum_{i=1}^{n} (df)(X)\,(Ue_i)\,dZ^i + \frac{1}{2}\sum_{i,j=1}^{n} \big(\nabla df\big)(X)\,(Ue_i, Ue_j)\,d[Z^i, Z^j], \tag{1.6.31}$$

or in abbreviated form (see Theorem 1.6.31),

$$d\big(f(X)\big) = (df)(U\,dZ) + \frac{1}{2}\nabla df(dX, dX). \tag{1.6.32}$$

PROOF. From $dU = \sum_i L_i(U) \circ dZ^i$ we first see that

$$\begin{aligned} d\big(f(X)\big) = d\big((f\circ\pi)(U)\big) &= \sum_i L_i(f\circ\pi)(U) \circ dZ^i \\ &= \sum_i L_i(f\circ\pi)(U)\,dZ^i + \frac{1}{2}\sum_{i,j} L_iL_j(f\circ\pi)(U)\,d[Z^i, Z^j] \end{aligned}$$

where $L_i(f\circ\pi)(u) = d(f\circ\pi)_u L_i(u) = (df)_{\pi(u)}(d\pi)_u h_u(ue_i) = (df)_{\pi(u)}(ue_i)$. Hence we have $L_i(f\circ\pi)(u) = F^i(u)$ where $F \equiv F_{df}\colon \mathrm{L}(TM) \to \mathbb{R}^n$ is the equivariant function $F^i(u) = (df)_{\pi(u)}(ue_i)$ associated to df (see Notation 1.6.25). Denoting $\overline{ue_i} := h_u(ue_i)$, then by means of Eq. (1.6.13),

$$L_iL_j(f\circ\pi)(u) = (L_iF^j)(u) = \overline{ue_i}\,F^j = \big(\nabla_{ue_i}df\big)_{\pi(u)}(ue_j) = \nabla df(ue_i, ue_j),$$

from where formula (1.6.31) results. $\square$

REMARK 1.6.46. Let M be a Riemannian manifold with its Levi-Civita connection. Denoting by $\Delta^{\text{hor}} = \sum_i L_i^2$ the horizontal Laplacian on $\mathrm{O}(TM)$ and by Δ the Laplace-Beltrami operator on M, then for each $f \in C^\infty(M)$ the following relation holds:

$$\Delta^{\text{hor}}(f \circ \pi) = (\Delta f) \circ \pi.$$

PROOF. Indeed, for $u \in \mathrm{O}(TM)$, we have

$$\sum_i {L_i}^2(f \circ \pi)(u) = \sum_i \nabla df(ue_i, ue_i) = (\operatorname{trace} \nabla df)\pi(u) = (\Delta f) \circ \pi(u). \qquad \square$$

NOTATION 1.6.47. In terms of the *Itô integral of the one-form df along X*, defined as

$$(\nabla)\int_X df := \int df(U dZ), \tag{1.6.33}$$

Eq. (1.6.32) writes as

$$\int_X df = (\nabla)\int_X df + \frac{1}{2}\int \nabla df(dX, dX). \tag{1.6.34}$$

Note that (1.6.33) extends naturally to differential forms $\alpha \in \Gamma(T^*M)$ as

$$(\nabla)\int_X \alpha := \int \alpha(U dZ).$$

Stochastic development of $\mathbb{R}^n$-valued semimartingales (along with the anti-development of M-valued semimartingales into $\mathbb{R}^n$ as inverse operation) allows to construct to each class of $\mathbb{R}^n$-valued semimartingales a corresponding class of M-valued semimartingales. We want to verify next that under the procedure of stochastic development local martingales on $\mathbb{R}^n$ correspond to ∇-martingales on M, as well as on Riemannian manifolds BM($\mathbb{R}^n$) and BM(M, g) correspond to each other via stochastic development.

THEOREM 1.6.48. *Let M be a differentiable manifold equipped with a torsion-free linear connection ∇. Let X be an M-valued semimartingale and U_0 an $\mathrm{L}(TM)$-valued $\mathscr{F}_0$-measurable random variable such that $\pi(U_0) = X_0$ a.s. Furthermore let $Z = \int_U \vartheta$ be the anti-development of X into $\mathbb{R}^n$ with respect to the initial frame U_0. Then*

(i) *X is a ∇-martingale on M if and only if Z is a local martingale on $\mathbb{R}^n$.*
(ii) *If ∇ is the Levi-Civita connection to some Riemannian metric g on M and if U_0 takes its values in $\mathrm{O}(TM)$, then X is a Brownian motion on (M, g) if and only if Z is a Brownian motion on $\mathbb{R}^n$ (more precisely, a Brownian motion on $\mathbb{R}^n$ stopped at the lifetime ζ of X).*

PROOF. (i) According to Definition 1.4.32 X is a ∇-martingale, if

$$d\big(f(X)\big) - \tfrac{1}{2}(\nabla df)(dX, dX) \overset{\mathrm{m}}{=} 0.$$

for functions $f \in C^\infty(M)$. By means of the Geometric Itô formula 1.6.45 this means that

$$\sum_i (df)(X)(Ue_i)\, dZ^i \overset{\mathrm{m}}{=} 0$$

for any $f \in C^\infty(M)$ which is easily seen (with the help of Lemma 1.3.1) to be equivalent to the condition that Z is a local martingale.

(ii) According to Definition 1.5.17, the semimartingale X is a Brownian motion on (M, g) if

$$d\big(f(X)\big) - \frac{1}{2}(\Delta f)(X)\, dt \overset{\mathrm{m}}{=} 0$$

for all $f \in C^\infty(M)$. By formula (1.6.31), clearly if Z is a Brownian motion $\mathbb{R}^n$, then X will be Brownian motion on (M, g). Conversely, if X is Brownian motion on (M, g) then by Lévy's characterization of M-valued Brownian motions (Theorem 1.5.18) X is a ∇-martingale, and thus Z a local martingale by part (i). On the other hand, we have $Z^i = \int_U \vartheta^i$ where $\vartheta_u^i = \langle d\pi(\,\cdot\,), ue_i\rangle = \pi^*\langle\cdot, ue_i\rangle$. We may calculate the quadratic variation of Z using Remark 1.3.14 as follows:

$$\begin{aligned} d[Z^i, Z^j] &= d\big[\textstyle\int_U \vartheta^i, \int_U \vartheta^j\big] = (\vartheta^i \otimes \vartheta^j)\,(dU, dU) \\ &= \pi^*\big(\langle\cdot, Ue_i\rangle \otimes \langle\cdot, Ue_j\rangle\big)\,(dU, dU) \\ &= \big(\langle\cdot, Ue_i\rangle \otimes \langle\cdot, Ue_j\rangle\big)\,(dX, dX) \\ &= \operatorname{trace}\big(\langle\cdot, Ue_i\rangle \otimes \langle\cdot, Ue_j\rangle\big)(X)\,dt = \delta_{ij}\,dt. \end{aligned}$$

By means of Lévy's characterization for Brownian motions on $\mathbb{R}^n$ we see that Z is a Brownian motion. □

REMARK 1.6.49. 1) Theorem 1.6.48 provides a canonical way to construct Brownian motions on Riemannian manifolds. One obtains Brownian motions on (M, g) with starting point $x \in M$ as stochastic development of a Euclidean Brownian motion B on $\mathbb{R}^n$. To this end we choose $u \in \mathrm{O}(TM)$ such that $\pi(u) = x$ and solve the SDE

$$dU = \sum_{i=1}^n L_i(U) \circ dB^i, \quad U_0 = u. \tag{1.6.35}$$

According to Theorem 1.6.48, then $X = \pi(U)$ is a Brownian motion on (M, g) starting from $X_0 = x$. Note that choosing a different initial frame $u \in \pi^{-1}\{x\}$ in (1.6.35) only changes the underlying Euclidean Brownian motion, in particular, the law of X will be independent of these choices. Indeed, for any $g \in \mathrm{O}(TM)$, along with B also gB is a BM($\mathbb{R}^n$), and hence X constructed by means of ug and B coincides with X constructed by means of u and gB.

2) More generally we have the following observation: For an arbitrary $\mathscr{F}_0$-measurable $\mathrm{O}(n)$-valued random variable g_0 along with B also $g_0 B$ is an $\mathbb{R}^n$-valued Brownian motion. Hence if U is the solution to $dU = \sum_i L_i(U) \circ d(g_0 B)^i$ with $U_0 = u_0$, then $\tilde U := U g_0$ solves the SDE $d\tilde U = \sum_i L_i(\tilde U) \circ dB^i$ with initial value $\tilde U_0 = u_0 g_0$. Indeed, as a consequence of $(R_g)_* h_u = h_{ug}$ for $g \in \mathrm{O}(n)$, we have

$$\begin{aligned} d(Ug_0) &= d(R_{g_0} U) = (dR_{g_0})_U \circ dU = \textstyle\sum_i (R_{g_0})_*\, L_i(U) \circ d(g_0 B)^i \\ &= (R_{g_0})_*\, h_U\big(U \circ d(g_0 B)\big) = h_{Ug_0}(Ug_0 \circ dB) = \textstyle\sum_i L_i(Ug_0) \circ dB^i; \end{aligned}$$

see also the argumentation related to formula (1.6.28).

REMARK 1.6.50. Let X be an M-valued semimartingale with starting point $x \in M$. The anti-development Z of X into $\mathbb{R}^n$ (see Definition 1.6.29) requires the choice of a frame u above x,

$$Z = \int_U \vartheta, \quad U_0 = u.$$

Considering the anti-development of X into $T_x M$, i.e.

$$\mathscr{A}(X) = U_0 \int_U \vartheta,$$

makes the notion intrinsic. Note that $d(\mathscr{A}(X)) = U_0 U_t^{-1} \circ dX$. Our formulas then read as

$$d(\mathscr{A}(X)) = /\!/_{0,t}^{-1} \circ dX, \quad \text{respectively} \;\; dX = /\!/_{0,t} \circ d(\mathscr{A}(X)).$$

In the same way we have

$$dU = h_U(/\!/_{0,t} \circ d\mathscr{A}(X)) \equiv h_U(\circ\, dX). \tag{1.6.36}$$

The intrinsic version of the Geometric Itô formula (Theorem 1.6.45) takes the form

$$d\big(f(X)\big) = (df)\big(/\!/_{0,t} d(\mathscr{A}(X))\big) + \frac{1}{2}\nabla df(dX, dX), \tag{1.6.37}$$

or in integrated form

$$\int_X df = (\nabla)\int_X df + \frac{1}{2}\int \nabla df(dX, dX), \tag{1.6.38}$$

where now

$$(\nabla)\int_X df = \int (df)\big(/\!/_{0,t} d(\mathscr{A}(X))\big), \tag{1.6.39}$$

see Eq. (1.6.33) for the definition of the Itô integral of df along X.

We want to come back briefly to the deterministic case of development of differentiable curves by pointing out that via development and anti-development geodesics on M correspond to straight lines passing through the origin in $\mathbb{R}^n$.

REMARK 1.6.51. Let M be a differentiable manifold, ∇ a linear connection on M and $u_0 \in \mathrm{L}(TM)$ fixed. To each curve $t \mapsto z(t)$ in $\mathbb{R}^n$ with $z(0) = 0$ we consider its development $t \mapsto \gamma(t)$ on M with $\dot\gamma(0) = u_0 \dot z(0)$. (Or conversely: to a curve $t \mapsto \gamma(t)$ in M with $\gamma(0) = \pi(u_0)$ we consider its "anti-development" $z(\cdot) = \int_u \vartheta$ where $t \mapsto u(t)$ is the horizontal lift of γ to $\mathrm{L}(TM)$ with initial value $u(0) = u_0$). Then $t \mapsto \gamma(t)$ is a geodesic on M if and only if $z(t) = \dot z(0)t$ for each t.

PROOF. Suppose first that $t \mapsto \gamma(t)$ is a geodesic on M. Then both $\dot\gamma$ and $u(\cdot)\dot z(0)$ are parallel sections along γ satisfying $\dot\gamma(0) = u(0)\dot z(0)$. By Theorem 1.4.11 hence $\dot\gamma(s) = u(s)\dot z(0)$, and we have

$$\begin{aligned}\big(\textstyle\int_u \vartheta\big)(t) &= \textstyle\int_0^t \vartheta\big(\dot u(s)\big)\, ds = \int_0^t u(s)^{-1}\pi_* \dot u(s)\, ds \\ &= \textstyle\int_0^t u(s)^{-1}\dot\gamma(s)\, ds = \int_0^t \dot z(0)\, ds = \dot z(0)\, t.\end{aligned}$$

Conversely, if $z(t) = \dot z(0)\, t$ then $\dot u(t) = h_{u(t)}\big(u(t)\dot z(t)\big) = h_{u(t)}\big(u(t)\dot z(0)\big)$ and hence $\dot\gamma(t) = (\pi \circ u)^{\cdot}(t) = \pi_* h_{u(t)}\big(u(t)\dot z(0)\big) = u(t)\dot z(0) \equiv /\!/_{0,t}\dot\gamma(0)$. This shows that $\dot\gamma$ is parallel along γ. □

DEFINITION 1.6.52. Let M be a differentiable manifold, ∇ a torsion-free linear connection on M and $x \in M$ a point in M. Furthermore let X be an M-valued semimartingale with $X_0 = x$ and U be a horizontal lift of X to $\mathrm{L}(TM)$ such that $U_0 = u_0 \in \pi^{-1}\{x\}$. The semimartingale X is called *one-dimensional* if there exists a real-valued semimartingale Z^1 and a vector $a \in \mathbb{R}^n$ such that the anti-development $Z = \int_U \vartheta$ of X into $\mathbb{R}^n$ takes the form $Z = Z^1 a$. In addition, X is called *one-dimensional martingale*, respectively *one-dimensional Brownian motion*, if Z^1 is even a real local martingale, respectively $\mathrm{BM}(\mathbb{R})$.

The properties above obviously do not depend on the choice of $u \in \pi^{-1}\{x\}$.

THEOREM 1.6.53 (One-dimensional semimartingales move along geodesics). *Let M be a differentiable manifold, ∇ a torsion-free linear connection on M and X a semimartingale taking values in M with $X_0 = x \in M$. Then:*

(i) *X is a one-dimensional semimartingale if and only if there exist a geodesic $\gamma\colon I \to M$ (defined on some open interval $I \subset \mathbb{R}$) and a real semimartingale X' taking values in I such that $X'_0 = \text{const}$ and $X = \gamma(X')$.*

(ii) *X is a one-dimensional martingale (one-dimensional Brownian motion) if and only if $X = \gamma(X')$ as in* (i) *and X' is in addition a continuous local martingale (Brownian motion).*

PROOF. Let X be an M-valued semimartingale with $X_0 = x$ and U a horizontal lift of X to $\mathrm{L}(TM)$ with $U_0 = u_0$ for some $u_0 \in \pi^{-1}\{x\}$. Furthermore let $Z = \int_U \vartheta$ be the anti-development of X in $\mathbb{R}^n$.

(1) First assume $Z = Z'a$ (where $Z'_0 = 0$). Then U satisfies the SDE

$$dU = \sum_{i=1}^{n} L_i(U) \circ dZ^i = L_a(U) \circ dZ', \quad U_0 = u_0, \tag{1.6.40}$$

where the horizontal vector field L_a on $\mathrm{L}(TM)$ is given by $L_a(u) = h_u(ua)$. Let $t \mapsto u(t)$ be the maximal flow curve to L_a with initial value $u(0) = u_0$, i.e., $\dot u(t) = L_a\big(u(t)\big)$ with $u(0) = u_0$. Then the projection $\gamma := \pi(u)$ defines a geodesic on M: indeed $\dot\gamma = (d\pi)_u \dot u = (d\pi)_u h_u(ua) = ua$ shows that $\dot\gamma$ is parallel along γ. On the other hand, we have

$$d\big(u(Z')\big) = \dot u(Z') \circ dZ' = L_a\big(u(Z')\big) \circ dZ', \quad (u(Z'))_0 = u(0) = u_0,$$

so that by uniqueness of solutions to Eq. (1.6.40) we get $U = u(Z')$ modulo indistinguishability. This implies $X = (\pi \circ u)(Z') = \gamma(Z')$. With $X' := Z'$ we get the claim.

(2) Conversely, suppose that $X = \gamma(X')$ for some geodesic γ and a real semimartingale X' where by assumption $X'_0 = \text{const}$. Without restrictions we may assume $X'_0 = 0$. Letting $t \mapsto u(t)$ be the horizontal lift of γ to $\mathrm{L}(TM)$ with $u(0) = u_0$, we get by $\tilde U := u(X')$ a semimartingale on $\mathrm{L}(TM)$ which projects to X and satisfies $\int_{\tilde U} \omega = 0$ by the pullback formula (1.3.9) since trivially $\omega(\dot u) \equiv 0$. Hence $\tilde U$ is a horizontal lift of X with $U_0 = \tilde U_0$ a.s. and thus by uniqueness $U = \tilde U$ modulo indistinguishability. On the other hand, $\dot\gamma$ is parallel along γ and hence $\dot\gamma(\,\cdot\,) = u(\,\cdot\,)a$ for some $a \in \mathbb{R}^n$ from where we get $\dot\gamma(X') = Ua$. The last equality implies

$$dX = \dot\gamma(X') \circ dX' = Ua \circ dX' = U \circ d(X'a) = \sum_i (Ue_i) \circ d(X'a)^i$$

and hence $dU = \sum_i L_i(U) \circ d(X'a)^i$ from where $Z = X'a$ follows. Hence X is a one-dimensional semimartingale.

(3) Part (ii) of the Theorem is obvious since according to (1) and (2) we may choose $X' = Z^1$. $\square$

1.7. Morphisms of Martingales and Brownian Motions

In Section 1.6 we have seen in great generality how to construct martingales and Brownian motions on manifolds. In this Section we are going to give functional characterizations of martingales and Brownian motions, in terms of their behaviour under transformations by maps between manifolds. It will turn out that only very specific maps, so-called harmonic morphisms, map Brownian motions to Brownian motions. Harmonic morphisms in higher dimensions are difficult to find. If however it is only required that Brownian motions are transformed to martingales, then there is the larger class of harmonic maps. Conversely, harmonic maps are completely characterized by this property. This point of view leads to the general goal in this Section of studying maps between manifolds by analyzing how they change the stochastic behaviour of certain classes of manifold-valued stochastic processes.

Before introducing the necessary vocabulary, we want to briefly summarize how linear connections in vector bundles canonically induce connections in new vector bundles obtained by vector bundle operations.

REMARK 1.7.1. Let $\pi\colon E \to M$ be a vector bundle over a differentiable manifold M and

$$\Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (V, A) \mapsto \nabla_V A,$$

a linear connection in E. Then, according to Leibniz rule, ∇ extends to a linear linear connection on

$$\bigoplus_{r,s=0}^{N} (\underbrace{E \otimes \ldots \otimes E}_{r} \otimes \underbrace{E^* \otimes \ldots \otimes E^*}_{s}).$$

More specifically, if ∇^E, ∇^F are linear connections in E, respectively F (both vector bundles over M), then we have

- the *direct sum of the connections* ∇ in $E \oplus F$ defined by
$$\nabla_V(A \oplus B) = \nabla_V^E A \oplus \nabla_V^F B, \quad V \in \Gamma(TM),\ A \in \Gamma(E),\ B \in \Gamma(F);$$
- the *product connection* ∇ in $E \otimes F$ defined by
$$\nabla_V(A \otimes B) = \big(\nabla_V^E A\big) \otimes B + A \otimes \big(\nabla_V^F B\big), \quad V \in \Gamma(TM),\ A \in \Gamma(E),\ B \in \Gamma(F);$$
- the *dual connection* ∇^{E^*} in E^* (see Definition 1.4.28) defined by
$$\big(\nabla_V^{E^*}\alpha\big)(A) = V(\alpha A) - \alpha(\nabla_V A), \quad V \in \Gamma(TM),\ \alpha \in \Gamma(E^*),\ A \in \Gamma(E);$$
- the *pullback connection* $\nabla = \nabla^{f^*E}$ in f^*E (see Definition 1.4.7) for a differentiable map $f\colon M \to N$ and E a vector bundle over N, determined by
$$\nabla_v f^*A = \nabla_{f_*v}^E A, \quad v \in TM,\ A \in \Gamma(E),$$
where $f^*A = A \circ f \in \Gamma(f^*E)$.

It is easy to see that pullback of connections is compatible with the other operations for vector bundles.

LEMMA 1.7.2. *Let E, F be vector bundle over a differentiable manifold M and ∇^E, ∇^F linear connections in E respectively F. Furthermore let $\phi \in \Gamma(E^* \otimes F)$, i.e., a homomorphism of vector bundles $\phi\colon E \to F$ over M. For $B \in \Gamma(E)$, let $\phi B \in \Gamma(F)$ where $(\phi B)_x := \phi_x B_x$. Then:*

$$\nabla_A^F(\phi B) = (\nabla_A^{E^*\otimes F}\phi)B + \phi\nabla_A^E B, \quad A \in \Gamma(TM).$$

PROOF. By linearity we can restrict ourselves to the case $\phi = e \otimes \varphi$ where $e \in \Gamma(E^*)$ and $\varphi \in \Gamma(F)$, but then

$$\begin{aligned}
\nabla_A^F\big((e \otimes \varphi)B\big) &= \nabla_A^F\big((eB)\varphi\big) = A(eB)\varphi + (eB)\nabla_A^F\varphi \\
&= \Big((\nabla_A^{E^*}e)B + e\nabla_A^E B\Big)\,\varphi + (eB)\nabla_A^F\varphi \\
&= \Big(\big((\nabla_A^{E^*}e)B\big)\varphi + (eB)\nabla_A^F\varphi\Big) + e(\nabla_A^E B)\varphi \\
&= \big(\nabla_A^{E^*\otimes F}(e \otimes \varphi)\big)B + (e \otimes \varphi)(\nabla_A^E B).
\end{aligned}$$
$\square$

DEFINITION 1.7.3 (Affine and convex mappings). Let M and N be differentiable manifolds, endowed with a torsion-free linear connection in TM, respectively TN, and $f\colon M \to N$ be a differentiable map. For each $x \in M$ the differential $(df)_x\colon T_xM \to T_{f(x)}N$ of f at x is a linear map. The covariant derivative of the section

$$df \in \Gamma(T^*M \otimes f^*TN)$$

gives the *Hessian* or *second fundamental form* of f,

$$\nabla df \in \Gamma(T^*M \otimes T^*M \otimes f^*TN), \quad (\nabla df)(A,B) = \big(\nabla_A df\big)B \in \Gamma(f^*TN).$$

For each $x \in M$ this gives a bilinear form $(\nabla df)_x\colon T_xM \times T_xM \to T_{f(x)}N$. The map f is called *affine* or *totally geodesic* if $\nabla df \equiv 0$.

In the special case $N = \mathbb{R}$, the map f is called *convex at* x if $(\nabla df)_x \geq 0$ (i.e., positively semidefinite), and *strictly convex at* x if $(\nabla df)_x > 0$ (i.e., positively definite). Finally, f is called *convex*, respectively *strictly convex*, if f is convex, respectively strictly convex at each $x \in M$.

REMARK 1.7.4. Let $\pi\colon \mathrm{L}(TM) \to M$ be the frame bundle over a differentiable manifold M. Then $L_iL_j(f\circ\pi)(u) = \nabla df(ue_i, ue_j)$ (see the proof of Theorem 1.6.45), and hence f is convex at x if and only if $\big(L_iL_j(f\circ\pi)\big)_{1\leq i,j\leq n}$ is positively semidefinite along the fiber $\pi^{-1}\{x\}$.

DEFINITION 1.7.5 (Energy density, tension field, harmonic map). Let (M,g) and (N,h) be Riemannian manifolds, endowed with the Levi-Civita connection. To a differentiable map $f\colon M \to N$ we have the two fundamental forms, namely

(i) the *first fundamental form* of f defined as pullback f^*h of the metric h under f, i.e., $f^*h \in \Gamma(T^*M \otimes T^*M)$ where $(f^*h)_x(u,v) = h_{f(x)}(f_*u, f_*v)$ for $u,v \in T_xM$;
(ii) the *second fundamental form* $\nabla df \in \Gamma(T^*M \otimes T^*M \otimes f^*TN)$ of f defined as covariant derivative of $df \in \Gamma(T^*M \otimes f^*TN)$.

Taking trace with respect to the given metrics gives

$$\begin{aligned}
&\operatorname{trace} f^*h = |df|^2 \in C^\infty(M) \quad \text{(the \textit{energy density of} } f\text{)},\\
&\operatorname{trace} \nabla df = \tau(f) \in \Gamma(f^*TN) \quad \text{(the \textit{tension field of} } f\text{)}.
\end{aligned}$$

Mappings $f \in C^\infty(M,N)$ with vanishing tension field $\tau(f) = 0$ are called *harmonic*. In the special case $N = \mathbb{R}$, the map f is called *subharmonic* if $\tau(f) = \Delta f \geq 0$.

LEMMA 1.7.6. *Let M and N be differentiable manifolds, endowed with a torsion-free linear connection in TM, respectively TN, and $f\colon M \to N$ a differentiable map. For $B \in \Gamma(TM)$ let $df B \in \Gamma(f^*TN)$ be defined by $(df B)_x = (df)_x B_x \in T_{f(x)}N$. Then for $A, B \in \Gamma(TM)$:*

$$\nabla_A^{f^*TN}(df B) = \big(\nabla_A df\big)B + df\, \nabla_A B, \tag{1.7.1}$$

or equivalently: $(\nabla df)(A,B) = \nabla_A^{f^*TN}(df B) - df\,\nabla_A B$.

PROOF. The claim is a direct consequence of Lemma 1.7.2 with $E = TM$, $F = f^*TN$ and $\phi = df$. $\square$

COROLLARY 1.7.7. *In the situation of Lemma* 1.7.6 *the bilinear form ∇df is symmetric, i.e.,*

$$\nabla df(A,B) = \nabla df(B,A) \quad \textit{for } A, B \in \Gamma(TM).$$

PROOF. Since the connection on N is torsion-free, we get from the first of Cartan's structural equations (see Theorem 1.4.27) the relation

$$\nabla_A^{f^*TN}(df B) - \nabla_B^{f^*TN}(df A) = df\,[A,B].$$

Since also the connection on M is torsion-free, i.e., $\nabla_A B - \nabla_B A = [A,B]$, the claim follows from Eq. (1.7.1):

$$\begin{aligned}\nabla df(A,B) - \nabla df(B,A) &= \nabla_A^{f^*TN}(df\,B) - \nabla_B^{f^*TN}(df\,A) - df\,\big(\nabla_A B - \nabla_B A\big)\\ &= df\,[A,B] - df\,[A,B] = 0.\end{aligned}$$

□

THEOREM 1.7.8 (Composition formula). *Let $M \xrightarrow{f} N \xrightarrow{\varphi} N'$ be smooth maps between differentiable manifolds, each manifold endowed with a torsion-free connection. For the Hessian of $\varphi \circ f$ it holds:*

$$\nabla d(\varphi \circ f) = \varphi_* \nabla df + f^* \nabla d\varphi. \tag{1.7.2}$$

In the case of Riemannian manifolds this gives

$$\tau(\varphi \circ f) = \varphi_* \tau(f) + \operatorname{trace}(f^* \nabla d\varphi). \tag{1.7.3}$$

PROOF. For the verification of Eq. (1.7.2) we use Lemma 1.7.2 with $E = f^*TN$, $F = (\varphi \circ f)^*TN' \equiv f^*(\varphi^*TN')$ and $\phi = f^*d\varphi$ (then $\phi_x = (d\varphi)_{f(x)}$ for $x \in M$; see Example 1.0.30). For vector fields $A, B \in \Gamma(TM)$ this gives the formula

$$\begin{aligned}\nabla_A^F\big(d(\varphi \circ f)B\big) &= \nabla_A^F\big((f^*d\varphi)df B\big)\\ &= \big(\nabla_A^{E^*\otimes F}(f^*d\varphi)\big)(df B) + (f^*d\varphi)\nabla_A^E(df B)\\ &= (f^*\nabla d\varphi)(A,B) + (f^*d\varphi)\nabla_A^E(df B),\end{aligned}$$

where the last equality comes from the definition of the pullback connection on $E^* \otimes F \cong f^*(T^*N \otimes \varphi^*TN')$. Altogether this gives

$$\begin{aligned}\nabla d(\varphi \circ f)(A,B) = \big(\nabla_A d(\varphi \circ f)\big)B &= \nabla_A\big(d(\varphi \circ f)B\big) - d(\varphi \circ f)\nabla_A B\\ &= (f^*\nabla d\varphi)(A,B) + (f^*d\varphi)\nabla_A^E(df B) - (f^*d\varphi)df\,\nabla_A B\\ &= (f^*\nabla d\varphi)(A,B) + (f^*d\varphi)\nabla df(A,B).\end{aligned}$$

Eq. (1.7.3) follows from Eq. (1.7.2) by taking trace. □

REMARK 1.7.9. Theorem 1.7.8 shows in particular that also the composition $\varphi \circ f$ is affine if f and φ are affine. In case of f harmonic and φ affine, also $\varphi \circ f$ is harmonic. However, in general, the composition of harmonic maps is not again harmonic.

COROLLARY 1.7.10. *Let M be a manifold endowed with a torsion-free linear connection and $f\colon M \to \mathbb{R}$ be a differentiable function. The following characterizations hold:*

(i) *f is affine if and only if the composition $f \circ \gamma$ is affine, i.e., $(f \circ \gamma)'' \equiv 0$ for any geodesic $\gamma\colon I \to M$ ($I \subset \mathbb{R}$ interval).*
(ii) *f is convex (resp. strictly convex) if and only if for each geodesic curve $\gamma\colon I \to M$ the composition $f \circ \gamma$ is convex (resp. strictly convex), i.e., $(f \circ \gamma)'' \geq 0$ (resp. > 0).*

PROOF. First observe that for a smooth curve γ on M by Eq. (1.7.2)

$$(f \circ \gamma)'' = f_* \nabla d\gamma + \gamma^* \nabla df.$$

Since $(\nabla d\gamma)(D,D) = \nabla_D \dot\gamma$, a curve γ is a geodesic if and only if γ is affine. On the other hand, $(\gamma^*\nabla df)(D,D) = (\nabla df)(\dot\gamma,\dot\gamma)$ so that for geodesic curves $\gamma\colon I \to M$ the equation $(f \circ \gamma)''(t) = (\nabla df)\big(\dot\gamma(t),\dot\gamma(t)\big)$ holds from where all claims are immediate. □

COROLLARY 1.7.11. *Let M, N be differentiable manifolds endowed with torsion-free linear connections. A differentiable map $f\colon M \to N$ is affine if and only if f transfers geodesics on M to geodesics on N.*

PROOF. As already noted, for curves γ on M, "affine" has the same meaning "geodesic" so that the claim follows from $\nabla d(f \circ \gamma) = f_* \nabla d\gamma + \gamma^* \nabla df$. $\square$

We now return to random motions on manifolds with the goal to investigate maps between manifolds under the aspect of how they transform classes of processes such as Brownian motions or ∇-martingales. To motivate this procedure we consider first the example of Brownian motions on (M, g).

Let (M, g) and (N, h) be Riemannian manifolds, each endowed with the Levi-Civita connection, and $f\colon M \to N$ a differentiable map. Let X be a $\mathrm{BM}(M, g)$ starting in $x \in M$ and fix $u \in \mathrm{O}(TM)$ over x. There is a unique horizontal lift U of X to $\mathrm{O}(TM)$ such that $U_0 = u$. The lifted Brownian motion U is a flow process to $\frac{1}{2}\Delta^{\mathrm{hor}}$ where

$$\Delta^{\mathrm{hor}} \equiv \sum_i L_i^2$$

is the horizontal Laplacian on $\mathrm{O}(TM)$, and U is called *horizontal Brownian motion* on the orthonormal frame bundle $\mathrm{O}(TM)$. On the other hand, X comes by stochastic development from the Euclidean Brownian motion $Z = \int_U \vartheta$ in $\mathbb{R}^n$. Recall that the anti-development $\mathscr{A}(X) = u \int_U \vartheta$ of X takes values in $T_x M$ and is independent of the choice of $U_0 = u$.

$$\begin{tikzcd}
 & U \arrow[dd, mapsto] & \mathrm{O}(TM) \arrow[d, "\pi_M"] & \mathrm{O}(TN) \arrow[d, "\pi_N"'] & \tilde{U} \arrow[dd, mapsto] & \\
 & & M \arrow[r, "f"] & N & & \\
\mathrm{BM}(M,g) & X \arrow[rrr, mapsto] & & & \tilde{X} & \text{semimartingale on } N \\
\mathrm{BM}(T_x M) & \mathscr{A}(X) & & & \mathscr{A}(\tilde{X}) & \text{semimartingale on } T_{f(x)}N
\end{tikzcd}$$

The process $\tilde{X} := f(X)$ on the target manifold N is in general no longer a Brownian motion. By definition, it is however a semimartingale on N (with $f(x)$ as starting point). We may take a horizontal lift $\tilde{U}$ of $\tilde{X}$ to $\mathrm{O}(TN)$ where $\tilde{U}_0 = \tilde{u}$ for some $\tilde{u} \in \mathrm{O}(TN)$ above $f(x)$. In addition , we have the anti-development $\mathscr{A}(\tilde{X})$ of $\tilde{X}$ which by definition is a semimartingale taking values in $T_{f(x)}N$.

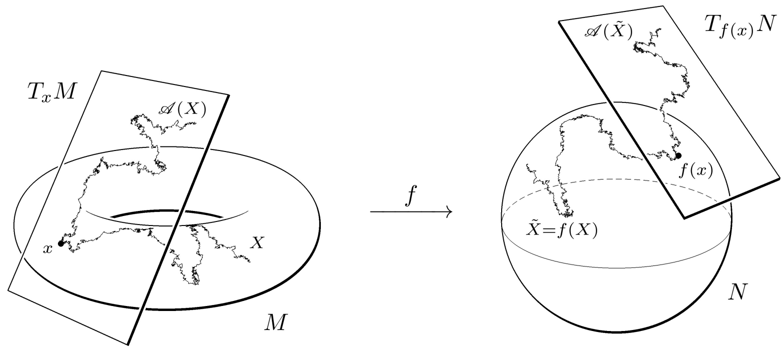


Figure 1.7.1. Anti-development of the target process

The idea is now to use the Doob-Meyer decomposition $d\tilde{Z} = d\tilde{Z}^{\text{Mart}} + d\tilde{Z}^{\text{drift}}$ of $\tilde{Z}$ to gain information about f. In particular, we shall see that the energy density $|df|^2$ and the tension field $\tau(f)$ of f can be recovered from the knowledge of $\tilde{Z}$, respectively $\mathscr{A}(\tilde{X})$. Before treating the case of a Brownian motion X we want first consider the general situation.

THEOREM 1.7.12. *Let M and N be differentiable manifolds, each endowed with a torsion-free linear connection, and $f\colon M \to N$ be a differentiable map. Furthermore, let X be a semimartingale on M and $\mathscr{A}(X)$ its anti-development to $T_{X_0}M$; correspondingly let $\mathscr{A}(\tilde{X})$ be the anti-development of $\tilde{X} := f(X)$ taking values in $T_{f(X_0)}M$. Finally, let U be a horizontal lift of X to $\mathrm{L}(TM)$, respectively $\tilde{U}$ a horizontal lift of $\tilde{X}$ to $\mathrm{L}(TN)$. Then it holds*

$$d\mathscr{A}(\tilde{X}) = \widetilde{/\!/}_{0,\bullet}^{-1}(df)_X \, /\!/_{0,\bullet} \, d\mathscr{A}(X) + \frac{1}{2}\widetilde{/\!/}_{0,\bullet}^{-1}\nabla df(dX, dX), \tag{1.7.4}$$

where $/\!/_{0,t} = U_t \circ U_0^{-1}$ denotes parallel transport along X, respectively $\widetilde{/\!/}_{0,t} = \tilde{U}_t \circ \tilde{U}_0^{-1}$ along $\tilde{X}$. Here $df \equiv f_*$ is the tangent map to f, i.e., $df_x\colon T_xM \to T_{f(x)}N$ for $x \in M$.

REMARK 1.7.13. In terms of the processes $Z = \int_U \vartheta$ in $\mathbb{R}^{\dim M}$, respectively $\tilde{Z} = \int_{\tilde{U}} \vartheta$ in $\mathbb{R}^{\dim N}$, formula (1.7.4) writes as

$$d\tilde{Z} = \tilde{U}^{-1}(df)_X \, U \, dZ + \tfrac{1}{2}\tilde{U}^{-1}\nabla df(dX, dX) \tag{1.7.5}$$

where $\tilde{U}^{-1}(df)_X \, U \, dZ = \sum_i \tilde{U}^{-1}(df)_X \, Ue_i \, dZ^i$ and

$$\tilde{U}^{-1}\nabla df(dX, dX) = \sum_{i,j} \tilde{U}^{-1}\nabla df(Ue_i, Ue_j) \, dZ^i dZ^j.$$

PROOF OF THEOREM 1.7.12. Let $\varphi \in C^\infty(N)$. On one hand, we have by the geometric Itô formula (1.6.32)

$$d\big(\varphi(\tilde{X})\big) = \varphi_* \widetilde{/\!/}_{0,t} \, d\mathscr{A}(\tilde{X}) + \frac{1}{2}\nabla^N d\varphi(d\tilde{X}, d\tilde{X})$$

where $\nabla^N d\varphi(d\tilde{X}, d\tilde{X}) = (f^*\nabla^M d\varphi)(dX, dX)$ by the pullback formula (Theorem 1.3.8). On the other hand, we can equally write

$$d\big(\varphi(\tilde{X})\big) = d\big((\varphi \circ f)(X)\big) = (\varphi \circ f)_* /\!/_{0,t} \, d\mathscr{A}(X) + \frac{1}{2}\nabla^M d(\varphi \circ f)(dX, dX),$$

where $\nabla^M d(\varphi\circ f)(dX, dX) = (\varphi_*\nabla df + f^*\nabla d\varphi)(dX, dX)$ according to the composition formula (Theorem 1.7.8). Comparing the two formulae shows that for each $\varphi \in C^\infty(N)$ it holds

$$\varphi_* \widetilde{/\!/}_{0,t}\, d\mathscr{A}(\tilde X) = \varphi_* f_* /\!/_{0,t}\, d\mathscr{A}(X) + \frac{1}{2}\varphi_*\nabla df(dX, dX),$$

and thus

$$\widetilde{/\!/}_{0,t}\, d\mathscr{A}(\tilde X) = f_* /\!/_{0,t}\, d\mathscr{A}(X) + \frac{1}{2}\nabla df(dX, dX)$$

which gives the claim. □

COROLLARY 1.7.14. *Let (M, g) and (N, h) be Riemannian manifolds, endowed with the Levi-Civita connection, and $f\colon M \to N$ a differentiable map. Let now X be a Brownian motion on (M, g) starting at $X_0 = x \in M$. Then $\mathscr{A}(X)$ is a Brownian motion in T_xM, and for $\tilde X = f(X)$ on N it holds*

$$d\mathscr{A}(\tilde X) = \widetilde{/\!/}_{0,\bullet}^{-1}(df)_X\, /\!/_{0,\bullet}\, d\mathscr{A}(X) + \frac{1}{2}\widetilde{/\!/}_{0,\bullet}^{-1}\tau(f)\, dt. \tag{1.7.6}$$

In addition, we have

$$h(d\tilde X, d\tilde X) = |df|^2(X)\, dt. \tag{1.7.7}$$

PROOF. We now work with the orthonormal frame bundles O(TM), respectively O(TN). Let U and $\tilde U$ be horizontal lifts of X to L(TM), respectively of $\tilde X$ to L(TN). We shall show that

$$h(d\tilde X, d\tilde X) = d[\tilde Z, \tilde Z] = |df|^2(X)\, dt$$

where $\tilde Z = \int_{\tilde U}\vartheta$. Note that, by assumption, $Z = \int_U \vartheta$ is a Brownian motion on $\mathbb{R}^n$ where $n = \dim M$. Furthermore, we have $dX = \sum_i Ue_i \circ dZ^i$ and $d\tilde X = \sum_i (df)_X Ue_i \circ dZ^i$. Hence we obtain

$$\begin{aligned} h(d\tilde X, d\tilde X) &= (f^*h)(dX, dX) \\ &= \sum_{i,j} (f^*h)_X(Ue_i, Ue_j)\, dZ^i dZ^j \\ &= \sum_i (f^*h)_X(Ue_i, Ue_i)\, dt \\ &= \sum_i h_{f(X)}\big((df)_X Ue_i, (df)_X Ue_i\big)\, dt = |df|^2(X)\, dt, \end{aligned}$$

as well as $d[\tilde Z, \tilde Z] = \langle \tilde U^{-1} d\tilde X, \tilde U^{-1} d\tilde X\rangle = h(d\tilde X, d\tilde X)$. □

THEOREM 1.7.15 (Stochastic characterization of affine and harmonic maps). *Let M and N be smooth manifolds and $f\colon M \to N$ be a differentiable map.*

(i) *Let ∇^M on M, respectively ∇^N on N, be torsion-free linear connections. Then f is affine if and only if f maps ∇^M-martingales on M to ∇^N-martingales on N.*
(ii) *Let (M, g) and (N, h) be Riemannian manifolds and ∇^M, respectively ∇^N the corresponding Levi-Civita connections. Then f is harmonic if and only if f maps* BM(M, g) *to the class* Mart(N, h) *of ∇^N-martingales on N.*

PROOF. (i) By Theorem 1.6.48, X is a ∇^M-martingale on M if and only if $\mathscr{A}(X)$ is a local martingale. Hence, by Theorem 1.7.12, $\tilde X = f(X)$ is a ∇^N-martingale on N (equivalently $\mathscr{A}(\tilde X)$ a local martingale) for each ∇^M-martingale X if and only if $\nabla df(dX, dX) = 0$ for each ∇^M-martingale X which in turn is equivalent to $\nabla df = 0$.

Indeed, by stochastic development, ∇^M-martingales X on M are of the form $dX = \sum_i Ue_i \circ dZ^i$ for some local martingale Z in $\mathbb{R}^n$ ($n = \dim M$). Taking $Z = (B, 0, \ldots, 0)$ where B a real Brownian motion thus gives $\nabla df(dX, dX) = \nabla df(Ue_1, Ue_1,)\, dt = 0$. For a constant starting point $X_0 = x \in M$, the frame $U_0 = u \in \mathrm{L}(TM)$ with $\pi(u) = x$ can be chosen arbitrarily, so that necessarily $\nabla df = 0$ must hold.

(ii) According to formula (1.7.6), f maps BM(M, g) to Mart(N, h) if and only if $\tau(f)(X) = 0$ along each Brownian motion X on M. Since the starting point of X can be chosen arbitrarily, this however means $\tau(f) = 0$. □

The proof of Theorem 1.7.15 shows that a map f is already affine if it transfers one-dimensional martingales on M to one-dimensional martingales on N.

COROLLARY 1.7.16. *Let M, N be two differentiable manifolds, endowed with torsion-free linear connections, and $f\colon M \to N$ a differentiable map. The following items are equivalent:*

(i) *f is affine;*
(ii) *f maps one-dimensional martingales to one-dimensional martingales;*
(iii) *f maps one-dimensional Brownian motions to one-dimensional Brownian motions.*

PROOF. By Corollary 1.7.10, f is affine if and only if f maps geodesics to geodesics. On the other hand, by Theorem 1.6.53, one-dimensional martingales and one-dimensional Brownian motions move on geodesics. Affine maps f hence transfer one-dimensional martingales (resp. one-dimensional Brownian motions) to one-dimensional martingales (resp. one-dimensional Brownian motions). Conversely, if f is a differentiable map with this property, then for each geodesic curve γ on M, the composition $f \circ \gamma$ maps continuous real local martingales (resp. real Brownian motions) to Mart(N, h); for each geodesic curve γ on M, by Theorem 1.7.15, the composition $f \circ \gamma$ is thus affine ($\equiv$ harmonic), and hence f itself affine. □

DEFINITION 1.7.17 (Horizontally conformal map, harmonic morphism). Let (M, g) and (N, h) be Riemannian manifolds and $f\colon M \to N$ a differentiable map. Then f is said to be *horizontally conformal*, if

(a) at each point $x \in M$ at which $(df)_x \neq 0$ the linear map $(df)_x\colon T_xM \to T_{f(x)}N$ is surjective;
(b) there exists a function $\lambda\colon M \to \mathbb{R}_+$ such that, for all $v, w \in \big(\ker(df)_x\big)^\perp$,

$$h_{f(x)}(f_*v, f_*w) = \lambda^2(x)\, g_x(v, w).$$

The function $\lambda\colon M \to \mathbb{R}_+$ is called *dilatation* of f where $\lambda(x) := 0$ if $(df)_x = 0$. A map $f\colon M \to N$ is called *harmonic morphism* (*with dilatation* λ) if f is harmonic and horizontally conformal (with dilatation λ).

LEMMA 1.7.18. *Let $f\colon (M, g) \to (N, h)$ be a differentiable map between Riemannian manifolds and $\lambda\colon M \to \mathbb{R}_+$ a function. The following items are equivalent:*

(i) *f is horizontally conformal with dilatation λ;*
(ii) *$df \circ (df)^{\mathrm{ad}} = \lambda^2 \,\mathrm{id}\,|f^*TN$ where $(df)^{\mathrm{ad}}\colon f^*TN \to TM$ is the homomorphism of vector bundles fiberwise adjoint to df;*
(iii) *$g\big(\mathrm{grad}(\varphi \circ f), \mathrm{grad}(\psi \circ f)\big) = \lambda^2\, h(\mathrm{grad}\,\varphi, \mathrm{grad}\,\psi) \circ f$ for all $\varphi, \psi \in C^\infty(N)$.*

Then necessarily $\lambda^2 = |df|^2_{\mathrm{op}}$ where $|df|_{\mathrm{op}}$ is the operator norm of df. Note that $k|df|^2_{\mathrm{op}} = |df|^2$ where $k = \dim N$.

PROOF. (i) $\Leftrightarrow$ (ii) The adjoint $(df)_x^{\mathrm{ad}}\colon T_{f(x)}N \to T_xM$ to $(df)_x$ is determined by

$$h_{f(x)}\big((df)_x v, u\big) = g_x\big(v, (df)_x^{\mathrm{ad}} u\big), \quad v \in T_xM,\ u \in T_{f(x)}N,$$

and f is hence horizontally conformal if and only if for all $x \in M$,

$$(df)_x \circ (df)_x^{\mathrm{ad}} = \lambda^2(x)\ \mathrm{id}\,|T_{f(x)}N.$$

(ii) $\Leftrightarrow$ (iii) Since $g\big(A, (df)^{\mathrm{ad}}(f^*\operatorname{grad}\varphi)\big) = A(\varphi \circ f)$ for $A \in \Gamma(TM)$, we have $(df)^{\mathrm{ad}}(f^*\operatorname{grad}\varphi) = \operatorname{grad}(\varphi \circ f)$ for $\varphi \in C^\infty(N)$ from where the equivalence follows. The additional claim is obvious. $\square$

THEOREM 1.7.19 (Analytic characterization of harmonic morphisms). *Let $f\colon (M,g) \to (N,h)$ be a differentiable map between Riemannian manifolds and let $\lambda\colon M \to \mathbb{R}_+$ be a function. The following conditions are equivalent:*

(i) *f is a harmonic morphism (with dilatation λ);*
(ii) *$\Delta_M(\varphi \circ f) = \lambda^2\,(\Delta_N\varphi \circ f)$ for $\varphi \in C^\infty(N)$.*

PROOF. (i) $\Rightarrow$ (ii) Since f is harmonic, by composition formula (1.7.3) it holds

$$\Delta_M(\varphi \circ f) = \varphi_*\tau(f) + \operatorname{trace}(f^*\nabla d\varphi) = \operatorname{trace}(f^*\nabla d\varphi).$$

We have to show that $\Delta_M(\varphi \circ f)(x) = \lambda^2(x)\,(\Delta_N\varphi \circ f)(x)$ for $x \in M$. To this end, without restrictions, we may assume that $(df)_x \neq 0$. If then $(a_1, \ldots, a_\ell)$ is an orthonormal basis of $\big(\ker(df)_x\big)^\perp$, then by the horizontal conformality of f

$$\Big(\tfrac{1}{\lambda(x)}\,(df)_x\,a_i\colon\ 1 \le i \le \ell\Big)$$

defines an orthonormal basis of $T_{f(x)}N$, and hence

$$\Delta_M(\varphi \circ f)(x) = \sum_{i=1}^{\ell}(\nabla d\varphi)(f_*a_i, f_*a_i) = \lambda^2(x)\,(\Delta_N\varphi \circ f)(x).$$

(ii) $\Rightarrow$ (i) For $\varphi, \psi \in C^\infty(N)$ we have on one hand

$$\begin{aligned}\Delta_M\big((\varphi\psi) \circ f\big) &= (\varphi \circ f)\,\Delta_M(\psi \circ f) + (\psi \circ f)\,\Delta_M(\varphi \circ f)\\ &\quad + 2\,g\big(\operatorname{grad}(\varphi \circ f), \operatorname{grad}(\psi \circ f)\big),\end{aligned}$$

on the other hands it holds $\Delta_N(\varphi\psi) = \varphi\,\Delta_N\psi + \psi\,\Delta_N\varphi + 2\,h(\operatorname{grad}\varphi, \operatorname{grad}\psi)$. Composing the last equation with f and multiplying by λ^2, then subtraction from the first equation gives

$$g\big(\operatorname{grad}(\varphi \circ f), \operatorname{grad}(\psi \circ f)\big) = \lambda^2\,h(\operatorname{grad}\varphi, \operatorname{grad}\psi) \circ f$$

which shows that f is horizontally conformal. It remains to verify $\tau(f) = 0$. To this end, we conclude again as above from horizontal conformality of f that for $\varphi \in C^\infty(N)$

$$\operatorname{trace}(f^*\nabla d\varphi) = \lambda^2\,(\Delta_N\varphi \circ f).$$

But since $\varphi_*\tau(f) + \operatorname{trace}(f^*\nabla d\varphi) = \Delta_M(\varphi \circ f) = \lambda^2 \cdot (\Delta_N\varphi \circ f)$, we have $\varphi_*\tau(f) = 0$ for any $\varphi \in C^\infty(N)$, and thus $\tau(f) = 0$. $\square$

THEOREM 1.7.20. *Let $f\colon (M,g) \to (N,h)$ be a differentiable map between Riemannian manifolds. The following conditions are equivalent:*

(i) *f is a harmonic morphism (with dilatation λ);*
(ii) *f maps $\mathrm{BM}(M,g)$ to $\mathrm{BM}(N,h)$ modulo time change, more precisely: to each Brownian motion X on (M,g) there exists a Brownian motion $\tilde{X}$ on (N,h) such that $f(X_t) = \tilde{X}_{T_t}$ a.s. where $T_t = \int_0^t \lambda^2(X_s)\,ds$.*

REMARK 1.7.21. Note that the N-valued Brownian motion $\tilde{X}$ in (ii) is determined through the condition $f(X_t) = \tilde{X}_{T_t}$ only up to time T_∞; it may however always be extended to maximal lifetime by "piecing on" an independent Brownian motion: the anti-development $\int_{\tilde{U}} \vartheta$ of $\tilde{X}$ gives first a stopped Brownian motion on $\mathbb{R}^{\dim N}$ which can be extended to all of $\mathbb{R}_+$. Stochastic development of this Brownian motion then gives the wanted prolongation of $\tilde{X}$. In this case the equality $f(X_t) = \tilde{X}_{T_t}$ then holds on an enlarged probability space.

PROOF OF THEOREM 1.7.20. By Theorem 1.7.15 (ii) the map f is harmonic if and only if for each Brownian motion X on M, the target process $f(X)$ is a ∇-martingale on N (with respect to the Levi-Civita connection ∇) which according to Theorem 1.6.48 means that all anti-developments $\tilde{Z}$ of $f(X)$ in $\mathbb{R}^{\dim N}$ are local martingales. Since, modulo time change, $f(X)$ is a BM(N, h) if and only if $\tilde{Z}$ is a BM$(\mathbb{R}^{\dim N})$, it remains to show that if f is in addition horizontally conformal then all anti-developments $\tilde{Z}$ of $f(X)$ are Brownian motions on $\mathbb{R}^{\dim N}$ modulo time change. Using the notations of Remark 1.7.13, we have

$$\begin{aligned} d\tilde{Z}^k d\tilde{Z}^\ell &= \sum_i \langle \tilde{U}^{-1}(df)_X U e_i, e_k\rangle \, \langle \tilde{U}^{-1}(df)_X U e_i, e_\ell\rangle \, dt \\ &= \sum_i g\big(Ue_i, (df)^{\mathrm{ad}} \tilde{U} e_k\big) \, g\big(Ue_i, (df)^{\mathrm{ad}} \tilde{U} e_\ell\big) \, dt \\ &= g\big((df)^{\mathrm{ad}} \tilde{U} e_k, (df)^{\mathrm{ad}} \tilde{U} e_\ell\big) \, dt. \end{aligned}$$

Hence it remains to observe that $g\big((df)^{\mathrm{ad}}\tilde{U}e_k, (df)^{\mathrm{ad}}\tilde{U}e_\ell\big)\, dt = \lambda^2(X)\, \delta_{k\ell}\, dt$ holds for all k, ℓ and all horizontal lifts $\tilde{U}$ of semimartingales of the form $\tilde{X} = f(X)$ with X in BM(M, g) if and only if f is horizontally conformal with dilatation λ. $\square$

Theorem 1.7.15 (i) says in particular that the composition $\varphi(X)$ of an M-valued martingale X with an affine function $\varphi \in C^\infty(M)$ gives a real local martingale. However, to use affine functions as "martingale testers" and to characterize M-valued martingales by this property usually fails due to the lacking richness of affine functions: in general, non-constant real-valued affine functions may even not exist locally. A suitable substitute for affine functions are convex functions. There are typically also obstructions of topological and geometric nature for existence of globally defined non-trivial convex functions, but locally convex functions provide a rich class of functions.

LEMMA 1.7.22. *Let M be a manifold endowed with a torsion-free linear connection. To each $x \in M$ there exists an open neighbourhood U and a strictly convex function $\varphi \in C^\infty(U)$ with prescribed 2-jet, i.e., given $a \in \mathbb{R}$, $b \in T_x^*M$ and $C \in T_x^*M \otimes T_x^*M$ positive definite, there is an open neighbourhood U of x and a function $\varphi \in C^\infty(U)$ such that $\varphi(x) = a$, $(d\varphi)_x = b$, $(\nabla d\varphi)_x = C$ and $\nabla d\varphi > 0$ on U.*

PROOF. We choose normal coordinates h about x as follows. The exponential map

$$\exp_x\colon (T_xM, 0) \to (M, x), \quad v \mapsto \gamma_v(1),$$

where γ_v is the geodesic curve determined by $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$, is well-defined locally about 0 and has full rank at 0, as can be seen from

$$(d\exp_x)_0 v = \tfrac{d}{dt}\big|_{t=0} \exp_x(tv) = \tfrac{d}{dt}\big|_{t=0} \gamma_{tv}(1) = \tfrac{d}{dt}\big|_{t=0} \gamma_v(t) = v.$$

Fixing a linear isomorphism $\iota\colon \mathbb{R}^n \xrightarrow{\sim} T_xM$, then by the local inverse theorem, $\exp_x \circ\, \iota$ maps an open ε-ball V_ε in $\mathbb{R}^n$ about 0 diffeomorphically onto an open neighbourhood of x

in M and $h := (\exp_x \circ \iota\,|V_\varepsilon)^{-1}$ defines a local chart at x. Now let $b = \sum_i b_i\,(dh^i)_x$ and $C = \sum_{i,j} C_{ij}\,(dh^i \otimes dh^j)_x$, and define

$$\varphi = a + \sum_i b_i\,h^i + \sum_{i,j} C_{ij}\,h^i\,h^j,$$

then $\varphi|U$ has the wanted properties for some sufficiently small open neighbourhood U of x. Indeed, letting $\partial_i = \frac{\partial}{\partial h^i}$ and $(\nabla d\varphi)_{ij} = (\nabla d\varphi)(\partial_i, \partial_j)$, we have

$$\nabla d\varphi|U = \sum_{i,j} (\nabla d\varphi)_{ij}\,dh^i \otimes dh^j = \sum_{i,j} \big(\partial_i\partial_j\varphi - \sum_k \Gamma^k_{ij}\,\partial_k\varphi\big)\,dh^i \otimes dh^j.$$

Note that, by construction of the chart, $\Gamma^k_{ij}(x) = 0$ which can be seen as follows: letting $\gamma_v(t) = \exp_x(tv)$ be again the geodesic curve defined locally about $t = 0$ and determined by the properties $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$, we have for any $v \in T_xM$,

$$0 = \frac{d^2}{dt^2}\Big|_{t=0} \gamma^k_v(t) = \sum_{i,j} \Gamma^k_{ij}(x)\,v^i\,v^j,$$

which implies $\Gamma^k_{ij}(x) = 0$ as ∇ is torsion-free. □

As a result of the richness of germs of convex functions guaranteed by Lemma 1.7.22, affine and harmonic maps can be characterized through their functional behaviour under pullback.

THEOREM 1.7.23 (Pullback properties of affine/harmonic maps).

(i) *Let $f\colon M \to N$ be a differentiable map between manifolds equipped with torsion-free linear connections. The following items are equivalent:*
  (a) *f is affine;*
  (b) *pullbacks $f^*\varphi$ of germs of convex functions on N are convex, i.e., for each convex function $\varphi \in C^\infty(V)$ defined on an open subset V of N, the composition $\varphi \circ f$ is convex on $f^{-1}V$.*

(ii) *Let $f\colon (M,g) \to (N,h)$ be a differentiable map between Riemannian manifolds. The following items are equivalent:*
  (a) *f is harmonic;*
  (b) *pullbacks $f^*\varphi$ of germs of convex functions on N are subharmonic, i.e., for each convex function $\varphi \in C^\infty(V)$ defined on an open subset V of N, the composition $\varphi \circ f$ is subharmonic on $f^{-1}V$.*

PROOF. The implications (a) $\Rightarrow$ (b) are each direct consequences of the composition formulas (1.7.2) and (1.7.3)

$$\nabla d(\varphi \circ f) = \varphi_*\,\nabla df + f^*\nabla d\varphi \quad \text{resp.} \quad \Delta(\varphi \circ f) = \varphi_*\tau(f) + \operatorname{trace}(f^*\nabla d\varphi).$$

The implications (b) $\Rightarrow$ (a) rely on the richness of germs of convex functions as formulated in Lemma 1.7.22. For instance, as in part (i), whenever $\nabla d(\varphi \circ f) \geq 0$ holds if $\nabla d\varphi \geq 0$, then already $\nabla df = 0$ must be satisfied, otherwise there would exist $x \in M$ and $v \in T_xM$ such that $w := (\nabla df)_x(v,v) \neq 0$ in $T_{f(x)}N$. By Lemma 1.7.22, there is then an open neighbourhood V of $f(x)$ in N and a convex function $\varphi \in C^\infty(V)$ such that $(d\varphi)_{f(x)}w < -|(df)_xv|^2$ and $(\nabla d\varphi)_{f(x)} = h_{f(x)}$. But this would imply

$$\big(\nabla d(\varphi \circ f)\big)_x(v,v) = (d\varphi)_{f(x)}w + |(df)_xv|^2 < 0,$$

in contradiction to $\nabla d(\varphi \circ f) \geq 0$. The implication (b) $\Rightarrow$ (a) in (ii) can be shown analogously. □

In the stochastic context the richness of germs of convex functions allows a characterization of martingales, which has first been used by Darling [**6**] for the definition of ∇-martingales.

THEOREM 1.7.24 (Darling's characterization of ∇-martingales). *Let M be a differentiable manifold, ∇ a torsion-free linear connection on M and X an M-valued semimartingale. Then X is a ∇-martingale if and only if for each $\varphi \in C^\infty(M)$ and each open $V \subset M$ such that $\varphi|V$ is convex, the following holds true: If*

$$\varphi(X) = \varphi(X_0) + N + A$$

is the Doob-Meyer decomposition of the real semimartingale $\varphi(X)$ and if σ, τ are stopping times such that $\sigma \leq \tau$ and $X|[\sigma, \tau[$ takes values in V, then the process A is monotonically increasing on $[\sigma, \tau[$ a.s.

PROOF. By the Geometric Itô formula (Theorem 1.6.45) and the notations there, we have for each $\varphi \in C^\infty(M)$ the formula

$$d\big(\varphi(X)\big) = \sum_i (d\varphi)(X)\,(Ue_i)\,dZ^i + \frac{1}{2}(\nabla d\varphi)(dX, dX).$$

Denoting by $Z = Z^{\mathrm{Mart}} + Z^{\mathrm{drift}}$ the Doob-Meyer decomposition of the $\mathbb{R}^n$-valued semimartingale Z, we obtain for the "drift part" A of $\varphi(X)$ the representation

$$dA = \sum_i (d\varphi)(X)\,(Ue_i)\,d(Z^{\mathrm{drift}})^i + \frac{1}{2}(\nabla d\varphi)(dX, dX). \tag{1.7.8}$$

According to Theorem 1.6.48 (i), the process X is a ∇-martingale on M if and only if $Z^{\mathrm{drift}} \equiv 0$ modulo indistinguishability. Hence necessity of the given condition is obvious. Recall that $1_{[\sigma,\tau[}\,(\nabla d\varphi)(dX, dX)$ is the differential of an increasing process. This is an immediate consequence of the definition of the b-quadratic variation, e.g. formula (1.3.3), since $X|[\sigma, \tau[$ takes values in V and $(\nabla d\varphi)_x$ is positive semidefinite for $x \in V$.

Conversely, suppose now that for each $\varphi \in C^\infty(M)$ and each open subset $V \subset M$ the following condition holds: If $\varphi|V$ is convex and $X|[\sigma, \tau[$ takes values in V then $A|[\sigma, \tau[$ is almost surely monotonically increasing. We have to show that $Z^{\mathrm{drift}} \equiv 0$ under this condition. By means of Lemma 1.3.1 the claim can be localized in space, and without restriction we may assume that X takes its values in a fixed relatively compact open subset V whose closure $\bar{V}$ lies completely in the domain of a chart h for M. We fix a positive definite section g of $T^*M \otimes T^*M$ over V, for instance, $g = \sum_i dh^i \otimes dh^i$, and are going to show that for each $f \in C^\infty(M)$ and $\varepsilon > 0$, the process

$$\int \sum_i (df)_X\,(Ue_i)\,d(Z^{\mathrm{drift}})^i + \frac{1}{2}\varepsilon \int g(dX, dX) \tag{1.7.9}$$

is almost surely isotone. This then gives immediately the claim, since with $\varepsilon \downarrow 0$ in (1.7.9) one obtains that

$$\int (df)_X\,(U dZ^{\mathrm{drift}}) \equiv \int \sum_i (df)_X\,(Ue_i)\,d(Z^{\mathrm{drift}})^i$$

is almost surely isotone. Passing from f to $-f$ thus shows that $\int (df)(X)\,(U\,dZ^{\mathrm{drift}})$ is almost surely constant, and since this holds for all $f \in C^\infty(M)$, we conclude $Z^{\mathrm{drift}} \equiv 0$ modulo indistinguishability.

In the sequel let $f \in C^\infty(M)$ and g be a positively definite section of $T^*M \otimes T^*M$ over V; it remains to show that the process

$$N := \int (df)_X\,(U dZ^{\text{drift}}) + \frac{1}{2}\int g(dX, dX) \tag{1.7.10}$$

is almost surely isotone. To this end, we construct a family $(N^\delta)_{\delta>0}$ of isotone processes with the property that $N^\delta \to N$ almost surely as $\delta \downarrow 0$, uniformly on compact time intervals of the form $[0,t]$. At this place the local richness of convex functions comes into effect, as by Lemma 1.7.22, to each point $a \in V$ there exists an open neighbourhood V_a of a and a strictly convex function φ^a on V_a such that

$$\varphi^a(a) = 0, \quad (d\varphi^a)_a = (df)_a, \quad (\nabla d\varphi^a)_a = g_a,$$

and such that in addition, for fixed $\delta > 0$, possibly after shrinking of V_a, it holds that

$$\begin{aligned} &\sup_{x \in V_a} \big| d(\varphi^a \circ h^{-1}) - d(f \circ h^{-1}) \big| \big(h(x)\big) \le \delta, \\ &\sup_{x \in V_a} \big| (\nabla d\varphi^a)_x(\partial_i, \partial_j) - g_x(\partial_i, \partial_j) \big| \le \delta \end{aligned} \tag{1.7.11}$$

where $\partial_i = \frac{\partial}{\partial h^i}$ with respect to a fixed chart (h, V). For a given $\delta > 0$ then V is already covered by finitely many V_a's, and according to Lemma 1.3.1 we can find a sequence $(\tau_n)_{n\ge 0}$ of stopping times such that

$$0 = \tau_0 \le \tau_1 \le \tau_2 \le \dots \quad \text{and} \quad \sup_n \tau_n = \infty,$$

and such that on each interval $[\tau_n, \tau_{n+1}[$ the process X takes values only in one (of the finitely many) $V_{a(n)}$. Therewith we finally define the process

$$N^\delta = \int \sum_n 1_{[\tau_n, \tau_{n+1}[}\,(d\varphi^{a(n)})(X)\,(U dZ^{\text{drift}}) + \frac{1}{2}\int \sum_n 1_{[\tau_n, \tau_{n+1}[}\,(\nabla d\varphi^{a(n)})(dX, dX)$$

which is almost surely isotone, since by construction $N\delta$ satisfies monotonicity on each subinterval $[\tau_n, \tau_{n+1}[$. We want to verify the convergence $\delta \to 0$ almost surely and uniformly $[0,t]$ as $\delta \downarrow 0$. But now we have

$$\begin{aligned} N_t^\delta - N_t = &\int_0^t \sum_n 1_{[\tau_n, \tau_{n+1}[}\,(d\varphi^{a(n)} - df)(X)\,(U dZ^{\text{drift}}) \\ &+ \frac{1}{2}\int_0^t \sum_n 1_{[\tau_n, \tau_{n+1}[}\,(\nabla d\varphi^{a(n)} - g)(dX, dX) \end{aligned} \tag{1.7.12}$$

and (1.7.11) can be used to estimate. For the first term we have the estimate

$$\begin{aligned} &\Big| \int_0^t \sum_n 1_{[\tau_n, \tau_{n+1}[}\,(d\varphi^{a(n)} - df)(X)\,(U dZ^{\text{drift}}) \Big| \\ &\qquad = \Big| \int_0^t \sum_n 1_{[\tau_n, \tau_{n+1}[}\,\big[d(\varphi^{a(n)} \circ h^{-1}) - d(f \circ h^{-1}) \big]\,(dh)(X)(U dZ^{\text{drift}}) \Big| \\ &\qquad \le \delta \sum_i \sup_{[0,t]} \big| (dh)(X)(U e_i) \big| \int_0^t \big| d(Z^{\text{drift}})^i \big|. \end{aligned}$$

In a similar way, letting $X^i = h^i(X)$, again with (1.7.11), we obtain for the second term in (1.7.12) the estimate

$$\begin{aligned}
&\Big|\int_0^t \sum_n 1_{[\tau_n,\tau_{n+1}[}\,(\nabla d\varphi^{a(n)} - \nabla df)(dX,dX)\Big| \\
&\qquad = \Big|\sum_{i,j}\int_0^t \sum_n 1_{[\tau_n,\tau_{n+1}[}\,\big[\nabla d\varphi^{a(n)}(\partial_i,\partial_j) - \nabla df(\partial_i,\partial_j)\big]\,d[X^i,X^j]\Big| \\
&\qquad \le \delta \sum_{i,j}\int_0^t \big|d[X^i,X^j]\big|
\end{aligned}$$

which completes the proof. □

1.8. Convergence and Confluence of Martingales

In this Section we want to elaborate and develop further the theory of martingales on manifolds. One of the difficulties of the theory relies in the fact that on manifolds there is no counterpart of the linear concept of taking conditional expectations. This apparent drawback is due to the nature of the subject and makes martingales on manifolds to an interesting non-linear instrument. The close connection between the behavior of martingales on manifolds and questions of convex geometry will quickly become apparent, for instance, questions of approximability of the Riemannian distance function on a manifold by convex functions. Such questions are known to be closely linked to the curvature of a Riemannian manifold.

One of basic tools of scalar martingale theory is the martingale convergence theorem which guarantees, for instance, that bounded martingales on $\mathbb{R}^n$ converge, i.e. have an almost sure limit as $t \to \infty$. In this form the convergence theorem obviously does not carry over to manifolds, as martingales on M taking values in a relatively compact subset do not need to converge which can already seen from simple examples, like Brownian motions on compact Riemannian manifolds or one-dimensional Brownian motions $X = \gamma(B)$ where γ is a closed geodesic curve.

As well-known [**38**], for real-valued continuous local martingale X, the following sets coincide modulo nullsets:

$$\Big\{\lim_{t\to\infty} X_t \text{ exists in } \mathbb{R}\Big\},\quad \Big\{[X,X]_\infty < \infty\Big\},\quad \Big\{\sup_{t\in\mathbb{R}_+} X_t < \infty\Big\},\quad \Big\{\inf_{t\in\mathbb{R}_+} X_t > -\infty\Big\}.$$

On the other hand, the concept of quadratic variation of M-valued semimartingales provides a notion to quantify the "oscillation" of M-valued martingales. Since each M-valued martingale X comes via stochastic development from an $\mathbb{R}^n$-valued local martingale Z and since for Riemannian manifolds the Riemannian quadratic variation of X coincides with the quadratic variation of Z,

$$g(dX,dX) = \sum_{i,j} g(Ue_i, Ue_j)\,dZ^i dZ^j = \sum_i d[Z^i,Z^i] = d[Z,Z],$$

it is not surprising that convergence of martingales on manifold can be expressed in terms of finiteness of the quadratic variation as $t \to \infty$.

Before entering into details we collect some notations.

NOTATION 1.8.1. For an adapted continuous process A which is pathwise locally of bounded variation, we call $V_A = \int |dA|$ the variation process of A. For an M-valued continuous semimartingale X and a bilinear form $b \in \Gamma(T^*M \otimes T^*M)$ let $\int b(dX,dX)$

be the b-quadratic variation and $\int |b(dX, dX)|$ its variation process. For $b, g \in \Gamma(T^*M \otimes T^*M)$, we write $b \le g$ if $g - b$ is positive semidefinite which means that the bilinear form $(g - b)_x \in T_x^*M \otimes T_x^*M$ is positive semidefinite for each $x \in M$.

REMARK 1.8.2. Let X be an M-valued semimartingale. If S, T are two $\mathbb{R}_+$-valued random variables (not necessarily stopping times) with the property that $X|[S, T[$ takes its values in an open set U in M, and if $b \in \Gamma(T^*M \otimes T^*M)$ such that $b \ge 0$ on U (i.e. b_x positive semidefinite for $x \in U$), then the process $\int b(dX, dX)$ is almost surely isotone on $[S, T[$. If in addition $-g \le b \le g$ on U where $g \in \Gamma(T^*M \otimes T^*M)$, then $\int |b(dX, dX)| \le \int g(dX, dX)$ a.s. on $[S, T[$.

LEMMA 1.8.3. *Let M be a manifold and ∇ a torsion-free linear connection on M. Every point in M has an open neighbourhood U such that each M-valued ∇-martingale X converges almost surely on the set*

$$\begin{aligned} \Omega_0 &:= \{X_t \in U \textit{ eventually}\} \\ &\equiv \big\{\omega \in \Omega\colon\ \exists\, t(\omega) \in \mathbb{R}_+ \textit{ such that } X_s(\omega) \in U \textit{ for all } s \ge t(\omega)\big\}. \end{aligned}$$

PROOF. For $x \in M$ we choose a sufficiently small open neighbourhood U of x such that by Lemma 1.7.22 we can find a bounded function $\varphi = (\varphi^1, \ldots, \varphi^n) \in C^\infty(M; \mathbb{R}^n)$ with the following properties:

(a) $\varphi^i|U$ is convex for $1 \le i \le n = \dim M$,
(b) $(\varphi|U, U)$ defines a chart for M about x.

Since X is a ∇-martingale, by the Geometric Itô formula (1.6.32), we get

$$\varphi^i(X) - \varphi^i(X_0) = M^i + A^i$$

where $M^i \in \mathscr{M}$ and $dA^i = \frac{1}{2}\nabla d\varphi^i(dX, dX)$. By construction $\varphi^i|U$ is convex, and hence the process A^i is almost surely eventually isotone on Ω_0, and in particular pathwise bounded from below on Ω_0. Since the functions φ^i are bounded, we observe that for each index i the local martingale

$$M^i = \varphi^i(X) - \varphi^i(X_0) - A^i$$

is almost surely pathwise bounded from above on Ω_0, and hence convergent on Ω_0. Conversely this shows however that each A^i is actually almost surely bounded on Ω_0 and (since eventually isotone on Ω_0) also convergent. □

REMARK 1.8.4. The proof above actually shows that $\varphi^i(X)$ is even a *semimartingale up to* ∞ on the set $\Omega_0 = \{X_t \in U$ eventually$\}$, i.e., if $\varphi^i(X) = \varphi^i(X_0) + M^i + A^i$ denotes the Doob-Meyer decomposition of $\varphi^i(X)$, then M^i_∞ and A^i_∞ exist almost surely on Ω_0, and for almost all $\omega \in \Omega_0$, the map $[0, \infty] \to \mathbb{R}$, $t \mapsto A^i_t(\omega)$, is of bounded variation. The last claim comes from the fact that on Ω_0, for sufficiently large s, it holds:

$$\int_0^\infty |dA^i| = \int_0^s |dA^i| + A^i_\infty - A^i_s.$$

We want to explain the notion of a semimartingale up to ∞ also for M-valued semimartingales.

DEFINITION 1.8.5 (M-valued semimartingale up to ∞). Let $\Omega_0 \subset \Omega$ be a measurable subset. An M-valued semimartingale X is called a *semimartingale up to* ∞ *on* Ω_0 if for any $\varphi \in C^\infty(M)$ the composition $\varphi(X)$ is a semimartingale up to ∞ on Ω_0.

REMARK 1.8.6. A real-valued semimartingale Y is obviously a semimartingale up to ∞ on all of Ω if and only if $\lim_{t\to\infty} Y_t$ exists almost surely and the time-changed process $\tilde{Y}$,

$$\tilde{Y}_t := \begin{cases} Y_{t/(1-t)} & \text{for } 0 \le t < 1, \\ Y_\infty & \text{for } t \ge 1, \end{cases}$$

is a semimartingale (with respect to the filtration $\tilde{\mathscr{F}}_t = \mathscr{F}_{t/(1-t)}$ for $0 \le t < 1$ and $\tilde{\mathscr{F}}_t = \mathscr{F}_\infty$ for $t \ge 1$).

LEMMA 1.8.7. *Let M be a differentiable manifold and ∇ a torsion-free linear connection on M. Every M-valued ∇-martingale X is a semimartingale up to ∞ on the set*

$$\Big\{ \lim_{t\to\infty} X_t \text{ exists in } M \Big\}.$$

PROOF. Let $(U_n)_{n\in\mathbb{N}}$ be a covering of M by open subsets U_n with the property as in Lemma 1.8.3. As explained in the proof to Lemma 1.8.3, to each U_n there is then a function $\varphi_n \in C^\infty(M;\mathbb{R}^n)$ such that $(\varphi_n|U_n, U_n)$ defines a chart for M and such that $\varphi_n(X)$ is a semimartingale up to ∞ on the set $\Omega_n := \{X_\infty \text{ exists in } U_n\}$. But then X itself is a semimartingale up to ∞ on Ω_n, and hence also on $\bigcup_n \Omega_n \equiv \{X_\infty \text{ exists in } M\}$. □

THEOREM 1.8.8 (Convergence Theorem of Darling-Zheng). *Let M be a manifold, endowed with a torsion-free linear connection ∇, and X be a ∇-martingale on M. Let g be an arbitrary Riemannian metric on M and $[X,X] = \int g(dX,dX)$ the g-quadratic variation of X. Then (modulo sets of measure 0) the following inclusions hold true:*

$$\big\{X_\infty \text{ exists in } M\big\} \subset \big\{[X,X]_\infty < \infty\big\} \subset \big\{X_\infty \text{ exists in } \hat{M} \equiv M\cup\{\infty\}\big\}.$$

PROOF. The first inclusion is a direct consequence of Lemma 1.8.7 which assures that X is a semimartingale up to ∞ on the subset $\Omega_0 := \big\{X_\infty \text{ exists in } M\big\}$ of Ω. This implies $\int_0^\infty \nabla d\varphi(dX,dX) < \infty$ almost surely on Ω_0 for each $\varphi \in C^\infty(M)$, and then also $\int_0^\infty g(dX,dX) < \infty$ almost surely on Ω_0.

For the verification of the second inclusion we note that modulo nullsets

$$\big\{X_\infty \text{ exists in } \hat{M}\,\big\} = \big\{\varphi(X) \text{ converges in } \mathbb{R} \text{ for each } \varphi \in C_c^\infty(M)\big\}$$

(where $C_c^\infty(M)$ denotes again the space of test functions on M). For a fixed test function $\varphi \in C_c^\infty(M)$, by compactness reasons, there is a constant $c > 0$ such that

$$-c\,g \le \nabla d\varphi \le c\,g \quad \text{and} \quad d\varphi\otimes d\varphi \le c\,g.$$

This allows to estimate:

$$\begin{aligned} &\textstyle\int_0^t |\nabla d\varphi(dX,dX)| \le c\int_0^t g(dX,dX) = c\,[X,X]_t, \quad \text{as well as} \\ &[\varphi(X),\varphi(X)]_t = \textstyle\int_0^t (d\varphi\otimes d\varphi)(dX,dX) \le c\,[X,X]_t. \end{aligned}$$

Let now $\varphi(X) = \varphi(X_0) + N + A$ denote the Doob-Meyer decomposition of $\varphi(X)$. Then both $[N,N] = [\varphi(X),\varphi(X)]$ as well as $A = \frac{1}{2}\int \nabla d\varphi(dX,dX)$ have an almost-sure limit on the set $\big\{[X,X]_\infty < \infty\big\}$ as $t\to\infty$, and consequently also $\varphi(X)$ itself converges on $\big\{[X,X]_\infty < \infty\big\}$ almost surely. □

COROLLARY 1.8.9. *Let (M,g) be a Riemannian manifold and X a Brownian motion* BM(M,g) *of maximal lifetime ζ. For each predictable stopping time ξ such that $\xi \le \zeta$ almost surely, the following inclusions hold modulo $\mathbb{P}$-nullsets:*

$$\big\{X_{\xi-} \textit{ exists in } M\big\} \subset \big\{\xi < \infty\big\} \subset \big\{X_{\xi-} \textit{ exists in } \hat{M}\big\}. \tag{1.8.1}$$

PROOF. By means of a time change (see Remark 1.1.15) which transforms the stochastic interval $[0,\xi[$ to $[0,\infty[$, the Brownian motion $X|[0,\xi[$ transforms to a martingale $\hat{X}$ defined on all of $\mathbb{R}_+$. But then we have $[\hat{X},\hat{X}]_\infty = [X,X]_\xi = n\,\xi$ (where $n = \dim M$), and the Convergence Theorem 1.8.8 of Darling-Zheng gives the claim. □

DEFINITION 1.8.10. Let (M,g) be a Riemannian manifold and ∇ the Levi-Civita connection on M.

(i) (M,g) is called *stochastically complete* if for each M-valued ∇-martingale X,
$$\big\{[X,X]_\infty < \infty\big\} \subset \big\{X_\infty \text{ exists in } M\big\}, \quad \text{modulo } \mathbb{P}\text{-nullsets.}$$

(ii) (M,g) is called BM-*complete* (or *complete for Brownian motions*) if for each predictable stopping time $\xi > 0$ and every M-valued Brownian motion X defined on $[0,\xi[$,
$$\big\{\xi < \infty\big\} \subset \big\{X_{\xi-} \text{ exists in } M\big\}, \quad \text{modulo } \mathbb{P}\text{-nullsets.}$$

(iii) (M,g) is said to be *metrically complete* (or *geodesically complete*) if for any $x \in M$ and $v \in T_xM$ the unique geodesic curve γ with $\gamma(0) = x$ and $\dot\gamma(0) = v$ is defined on all of $\mathbb{R}$.

REMARK 1.8.11. BM-completeness of a Riemannian manifold (M,g) means that each Brownian motion of maximal lifetime on (M,g) has actually infinite lifetime and cannot explode in finite time. Stochastic completeness of (M,g) means that martingales X on (M,g) with finite "intrinsic time" $T_t = \int_0^t g(dX,dX)$ cannot explode.

Compact Riemannian manifolds (M,g) are always metrically complete, and also stochastically complete by the Martingale convergence Theorem 1.8.8 of Darling-Zheng. Trivially, stochastic completeness implies BM-completeness, but not vice versa: for instance, $M = \mathbb{R}^2 \setminus \{(1,0)\}$ is BM-complete but not stochastically complete, as can be seen from the example $X = (X^1, X^2)$ with X^1 a BM($\mathbb{R}$) and $X^2 = 0$.

REMARK 1.8.12. Stochastically complete Riemannian manifolds are metrically complete.

PROOF. Assuming that (M,g) is metrically incomplete, we find a geodesic $\gamma\colon\,]a,b[\, \to M$ where $-\infty \le a < 0 < b \le \infty$ such that its domain $]a,b[$ is a proper subset of $\mathbb{R}$ which can not further be extended. Let now $Y \in \mathscr{M}$ be a convergent $]a,b[$-valued local martingale such that $Y_0 = 0$ and $Y_\infty \in \{a,b\} \cap \mathbb{R}$ almost surely (constructed for instance from a stopped BM($\mathbb{R}$) via time change). In particular, we have then $[Y,Y]_\infty < \infty$ almost surely. The composition $X := \gamma(Y)$ is by Theorem 1.6.53 (ii) a (one-dimensional) M-valued martingale with the property that $\mathbb{P}\{X_t \text{ converges for } t \to \infty\} = 0$. On the other hand, by means of pullback formula (1.3.4), we obtain
$$[X,X]_\infty \equiv \int_0^\infty g(dX,dX) = \int_0^\infty |\dot\gamma(Y_s)|^2\, d[Y,Y] = |\dot\gamma(0)|^2\, [Y,Y]_\infty < \infty,$$
which shows that (M,g) not stochastically complete. □

The converse in Remark 1.8.12 is false in general: metrically complete Riemannian manifolds are not even BM-complete. Brownian motions on metrically complete Riemannian manifolds may explode in finite time, as will be shown in a later section. Also BM-completeness does not imply metric completeness, as can be seen from Brownian motion on $\mathbb{R}^n \backslash \{\text{point}\}$ for $n \ge 2$.

DEFINITION 1.8.13. An *exhaustion function* on a differentiable manifold M is a *proper* map $\varphi \in C^\infty(M; \mathbb{R}_+)$. The map φ is called proper if all sublevel sets $\{\varphi \le c\}$ are compact, or in other words, if $\varphi(x) \to \infty$ as $x \to \infty$ in $\hat{M}$.

THEOREM 1.8.14. *Let (M, g) be a Riemannian manifold which carries an exhaustion function $\varphi \in C^\infty(M; \mathbb{R}_+)$ with bounded gradient (i.e. $|\operatorname{grad} \varphi| \le \mathrm{const}$). Then:*

(i) *(M, g) is metrically complete;*
(ii) *(M, g) BM-complete if in addition $\Delta\varphi$ is bounded from above;*
(iii) *(M, g) is stochastically complete if in addition $\nabla d\varphi$ is bounded from above (i.e. $\nabla d\varphi \le c\, g$ for some $c > 0$).*

PROOF. (i) Assume there is a geodesic curve $\gamma\colon [0, b[\; \to M$ in M which cannot be extended beyond b. Then we have

$$\big|(\varphi \circ \gamma)'\big| = \big|\langle(\operatorname{grad} \varphi)_{\gamma(t)}, \dot\gamma(t)\rangle\big| \le \mathrm{const}\, |\dot\gamma(t)| = \mathrm{const}\, |\dot\gamma(0)| < \infty,$$

so that $(\varphi \circ \gamma)|[0, b[$ is bounded, and since by assumption the function φ is proper, $\gamma\big([0, b[\big)$ will be relatively compact. However this leads immediately to a contradiction: there is a sequence (t_n) in $[0, b[$ such that $t_n \uparrow b$ with the property that $\dot\gamma(t_n)$ has a limit $v_0 \in TM$; but then there exists a neighbourhood V of v_0 in TM and $\varepsilon > 0$ such that each geodesic curve α with $\alpha(0) = \pi(v)$ and $\dot\alpha(0) = v \in V$, is well-defined on the interval $]-\varepsilon, \varepsilon[$; thus choosing $t_{n_0} > b - \varepsilon$ with $\dot\gamma(t_{n_0}) \in V$, we see that γ can be extended beyond b.

(ii) Let $\Delta\varphi$ be bounded from above and let X be a $\mathrm{BM}(M, g)$ of maximal lifetime ζ. We want to show that $\mathbb{P}\{\zeta = \infty\} = 1$. To this end denote by $\varphi(X) = \varphi(X_0) + N + A$ the Doob-Meyer decomposition of $\varphi(X)$. Then, in particular,

$$[N, N] = [\varphi(X), \varphi(X)] = \int |\operatorname{grad} \varphi|^2(X)\, dt, \quad A = \frac{1}{2}\int \Delta\varphi(X)\, dt,$$

from where we conclude that $[N, N]_\zeta \le \mathrm{const} \times \zeta$ and $\limsup_{t\uparrow\zeta} A_t \le \mathrm{const} \times \zeta$. Hence, we have $\mathbb{P}$-a.s. the inclusion

$$\big\{\zeta < \infty\big\} \subset \big\{\varphi(X)_{\zeta-} \text{ exists in } \mathbb{R}\big\}.$$

But ζ is the maximal lifetime of X and thus $X_t \to \infty$ in $\hat{M}$ almost surely on $\{\zeta < \infty\}$ as $t \uparrow \zeta$, and consequently $\varphi(X_t) \to \infty$ from where we conclude that $\mathbb{P}\{\zeta < \infty\} = 0$.

(iii) Assume now $\nabla d\varphi$ to be bounded from above and let X be a martingale on (M, g). By the Convergence Theorem 1.8.8 of Darling-Zheng, it is sufficient to show that almost surely

$$\big\{[X, X]_\infty < \infty\big\} \subset \big\{\varphi(X) \not\to \infty\big\}.$$

However, by assumption, there is a constant $c > 0$ such that $d\varphi \otimes d\varphi \le c\, g$ and $\nabla d\varphi \le c\, g$. Hence, denoting by $\varphi(X) = \varphi(X_0) + N + A$ the Doob-Meyer decomposition of $\varphi(X)$, we conclude

$$[N, N] = \int (d\varphi \otimes d\varphi)\,(dX, dX) \le c\,[X, X], \quad \text{and}$$
$$A = \frac{1}{2}\int \nabla d\varphi(dX, dX) \le \frac{1}{2} c\,[X, X]$$

from where the claim follows. □

We want to note already at this point how Theorem 1.8.14 is usually applied. On a connected Riemannian manifold (M, g) one constructs an exhaustion function φ via a suitable smoothing of the distance function $\varphi_0 = d_M(x_0, \cdot)$ to a given point x_0 in M.

Recall that for two points x_0, x_1 in M the distance $d_M(x_0, x_1)$ is defined as the infimum length of all (piecewise) differentiable curves connecting x_0 and x_1 (cf. Definition 1.5.3). We shall see that $|\operatorname{grad}\varphi_0| = 1$ at points where φ_0 is differentiable, and that there in addition $\Delta\varphi_0$, respectively $\nabla d\varphi_0$, can be controlled by curvature bounds (Hessian Comparison Theorem).

As already explained, the concept of ∇-martingales covers the class of local martingales on the real line; on manifolds however a distinction of local versus true martingales is meaningless. In the scalar theory however this point is by no means only of a technical nature, for instance when it comes to questions of whether the knowledge of the state X_t for a fixed $t > 0$, together with the filtration $(\mathscr{F}_s)_{0\le s\le t}$ up to time t, allows to reconstruct the whole process $X|[0,t]$. In scalar martingale theory, the "size" of a (local) martingale is controlled by the quadratic variational process; this aspect of the theory can be carried over to manifolds through the notion of quadratic variation of a martingale.

DEFINITION 1.8.15 (H^p-martingale). Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection on M and $1 \le p \le \infty$. A ∇-martingale X on M with Riemannian quadratic variation $[X,X] = \int g(dX, dX)$ is called *H^p-martingale* if $[X,X]_\infty^{1/2} \in L^p(\mathbb{P})$, i.e., $\mathbb{E}\big[[X,X]_\infty^{p/2}\big] < \infty$.

Note that for stochastically complete manifolds, by the Convergence Theorem of Darling-Zheng, the condition "$[X,X]_\infty < \infty$ almost surely" characterizes convergent martingales X on M. As already noted, each ∇-martingale X on M comes by stochastic development from an $\mathbb{R}^n$-valued local martingale Z and the Riemannian quadratic variation $[X,X]$ of X coincides with the quadratic variation $[Z,Z] \equiv \sum_i [Z^i, Z^i]$ of Z, hence in particular $\mathbb{E}\big([X,X]_\infty^{p/2}\big) = \mathbb{E}\big([Z,Z]_\infty^{p/2}\big)$. Definition 1.8.15 thus corresponds to the general approach to carry over $\mathbb{R}^n$-valued concepts to manifolds via stochastic development.

EXAMPLE 1.8.16. In the special case $M = \mathbb{R}^n$, a martingale $X \in \mathscr{M}_0(\mathbb{R}^n)$ is an H^2-martingale (i.e., $\mathbb{E}[X,X]_\infty < \infty$) if and only if $X_\infty^* \in L^2(\mathbb{P};\mathbb{R}^n)$ where X^* denotes the (componentwise) maximal process of X. This condition is equivalent to $X_t = \mathbb{E}^{\mathscr{F}_t}[X_\infty]$ for some $X_\infty \in L^2(\mathbb{P};\mathbb{R}^n)$. Thus on $\mathbb{R}^n$, the H^2-martingales coincide with the class of L^2-bounded martingales. Consequently, on an n-dimensional Riemannian manifold, the H^2-martingales are exactly those martingales which come from an L^2-bounded $\mathbb{R}^n$-valued martingale via stochastic development.

REMARK 1.8.17. Since the class of H^p-martingales is invariant under time-change, Definition 1.8.15 extends in an obvious way to martingales that are only defined on a finite time interval $[0,t]$ or up to some predictable stopping time ξ.

THEOREM 1.8.18. *Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection on M. Let $U \subset M$ be an open subset, $\lambda\colon U \to \mathbb{R}_+$ a continuous function and $\varphi \in C^\infty(U)$ such that $\alpha \le \varphi \le \beta$ (for some constants $\alpha, \beta > 0$) such that*

$$\nabla d\varphi + 2\lambda\varphi\, g \le 0.$$

Then, for each ∇-martingale X on M taking values in U, it holds

$$\mathbb{E}\left[\exp\Big(\int_0^\infty \lambda(X)\, d[X,X]\Big)\right] < \infty.$$

If in addition $\lambda \ge \varepsilon > 0$ for some constant ε, then X is an H^p-martingale for any $1 \le p < \infty$, and in particular almost surely convergent.

PROOF. Let X be ∇-martingale taking values in U. Then the real process

$$S := \varphi(X)\exp\left(\int \lambda(X)\,d[X,X]\right)$$

is a local supermartingale, as can be seen directly by Itô's formula:

$$\begin{aligned} dS &= \exp(\dots)\,d\big(\varphi(X)\big) + \varphi(X)\ \exp(\dots)\,\lambda(X)\,d[X,X] \\ &\overset{\mathrm{m}}{=} \exp(\dots)\ \big(\tfrac{1}{2}\,\nabla d\varphi(dX,dX) + \varphi(X)\,\lambda(X)\,g(dX,dX)\big) \\ &= d(\text{decreasing process}). \end{aligned}$$

By means of a localizing sequence of stopping times $\tau_n \uparrow \infty$ for S, we obtain for any $t \geq 0$ the estimate $\mathbb{E}[S_t^{\tau_n}] \leq \mathbb{E}[S_0^{\tau_n}] = \mathbb{E}[S_0 \leq \beta]$, and hence by Fatou's Lemma

$$\beta \geq \liminf_{n\to\infty} \mathbb{E}[S_t^{\tau_n}] \geq \mathbb{E}\big[\liminf_{n\to\infty} S_t^{\tau_n}\big] = \mathbb{E}[S_t] \geq \alpha\,\mathbb{E}\left[\exp\left(\int_0^t \lambda(X)\,d[X,X]\right)\right].$$

This completes the proof. $\square$

THEOREM 1.8.19. *Let (M,g) be a Riemannian manifold and ∇ be the Levi-Civita connection on M. Suppose that K is a compact subset of M such that there is a strictly convex C^∞-function defined on an open neighbourhood of K. Then each ∇-martingale on M taking its values in K is an H^p-martingale for $1 \leq p < \infty$, and hence almost surely convergent.*

PROOF. By assumption there is an open set U containing K and carrying a strictly convex function $\varphi \in C^\infty(U)$. Multiplying φ by -1 we have $\nabla d\varphi < 0$ on U. Without restrictions we may assume that φ is bounded and, if necessary by adding a positive constant, that $\alpha \leq \varphi \leq \beta$ with $\alpha, \beta > 0$. By compactness reasons, we may assume, possibly after reducing the size of U, that even $\nabla d\varphi + 2\varepsilon\varphi\, g \leq 0$ holds on U for some sufficiently small $\varepsilon > 0$. The claim then follows from Theorem 1.8.18 with $\lambda \equiv \varepsilon$. $\square$

Note that Theorem 1.8.19 covers the well-known fact that bounded $\mathbb{R}^n$-valued local martingales converge almost surely. However, as already mentioned, manifold-valued martingales taking values in a compact set are not at all convergent in general.

We want to discuss another well-known property of continuous real martingales in the case of M-valued martingales. For a real martingale X, the knowledge of X_t at a fixed time $t > 0$, together with the filtration $(\mathscr{F}_s)_{0\leq s\leq t}$, already determines the martingale $(X_s)_{0\leq s\leq t}$ up to time t, namely as $X_s = \mathbb{E}^{\mathscr{F}_s}[X_t]$ almost surely. An equivalent formulation of this property is that if X and Y are continuous real martingales adapted to the same filtration and if $X_t = Y_t$ for some $t > 0$, then already $X|[0,t] = Y|[0,t]$ modulo indistinguishability. We call this property *non-confluence* of real martingales. This leads to the question to what extent it is possible to have confluence of non-identical martingales on manifolds at a certain time.

THEOREM 1.8.20 (Minimum principle). *Let (M,g) be a Riemannian manifold and ∇ the Levi-Civita connection on M. Let $\varphi \in C(M;\mathbb{R})$ and $U := \{\varphi > 0\}$, $D := \{\varphi \leq 0\}$. Furthermore, let $\lambda\colon U \to \mathbb{R}_+$ be a continuous function. Suppose that φ is bounded on U and $\varphi|U \in C^\infty(U)$, and in addition $\nabla d\varphi + 2\lambda\varphi\, g \geq 0$ on U. Extending λ to all of M by $d\lambda|D := 0$, the following statement holds: If a ∇-martingale X with the property*

$$\mathbb{E}\left[\exp\Big(\int_0^\infty \lambda(X)\,d[X,X]\Big)\right] < \infty \tag{1.8.2}$$

converges to a D-valued random variable, then X lives completely in D.

PROOF. Let X be a ∇-martingale with the property (1.8.2) such that $X_t \to X_\infty$ almost surely for some D-valued random variable X_∞. As X has continuous paths and U is open, it is sufficient to show that $\mathbb{P}\{X_{t_0} \in U\} = 0$ for each fixed $t_0 \geq 0$.

To this end let $S := \varphi(X)\,\exp\bigl(\int \lambda(X)\,d[X,X]\bigr)$ and $\tau = \inf\{t \geq t_0 \colon X_t \notin U\}$. As in the proof of Theorem 1.8.18 one verifies that

$$Y_t := 1_{\{X_{t_0} \in U\}}\, S_{t_0+(t\wedge\tau)}, \quad t \geq 0,$$

defines a non-negative local submartingale (with respect to the filtration $(\hat{\mathscr{F}}_t)_{t\geq 0}$ where $\hat{\mathscr{F}}_t := \mathscr{F}_{t_0+t}$). By assumption, setting $\alpha := \sup(\varphi|U)$, we have

$$Y_t \leq \alpha\,\exp\left(\int_0^\infty \lambda(X)\,d[X,X]\right) \in L^1(\mathbb{P}),$$

so that the process Y is uniformly integrable. From the fact that $Y_\infty = 0$ almost surely, it follows that $0 \leq \mathbb{E}[Y_0] \leq \mathbb{E}[Y_\infty] = 0$. Since $Y_0|\{X_{t_0} \in U\} > 0$ we conclude that $\mathbb{P}\{X_{t_0} \in U\} = 0$. □

DEFINITION 1.8.21 (Convex geometry). Let M be a manifold equipped with a torsion-free linear connection. An open subset V of M is said to have *convex geometry* if there exists a non-negative convex smooth function

$$\phi\colon V \times V \to \mathbb{R}_+$$

which vanishes exactly on the diagonal $\Delta = \{(x,x)\colon x \in V\}$.

Convexity of ϕ in Definition 1.8.21 is understood with respect to the direct sum connection on $M \times M$, see Remark 1.7.1.

REMARK 1.8.22. Let ∇, ∇' be linear connections on differentiable manifolds M resp. M'. The *direct sum connection* on the product $\bar{M} = M \times M'$ is given as follows: Each vector field $\bar{A} \in \Gamma(\bar{\alpha}^*T\bar{M})$ along a curve $\bar{\alpha} = (\alpha, \alpha')$ on $\bar{M}$ decomposes as $\bar{A} = (A, A')$ with a vector field A along α on M and a vector field A' along α' on M'. The covariant derivative of $\bar{A}$ along $\bar{\alpha}$ is given by

$$\bar{\nabla}_D \bar{A} := (\nabla_D A, \nabla_D A'). \tag{1.8.3}$$

An immediate consequence is that a curve $\bar{\gamma} = (\gamma, \gamma')\colon I \to \bar{M}$ is a geodesic if and only if $\gamma\colon I \to M$ and $\gamma'\colon I \to M'$ are both geodesic curves. By Corollary 1.7.10, a differentiable function $\phi\colon M \times M' \to \mathbb{R}$ is hence convex if for all geodesics γ on M and γ' on M' the curve $t \mapsto \phi\bigl(\gamma(t), \gamma'(t)\bigr)$ is convex, i.e., a curve with non-negative second derivative.

Combined with Corollary 1.7.11 this shows that both the projections $\mathrm{pr}\colon \bar{M} \to M$ and $\mathrm{pr}'\colon \bar{M} \to M'$, as well as the canonical embeddings $\iota_x\colon M' \to \bar{M}$ and $\iota_{x'}\colon M \to \bar{M}$ for $x \in M$ resp. $x' \in M'$ are affine maps. From the probabilistic perspective, according to Theorem 1.7.15 (i), a process $\bar{X} = (X, X')$ is hence a $\bar{\nabla}$-martingale on $\bar{M}$ if X and X' are martingales on M, resp. M' (adapted to the same filtration).

If M and M' are in addition Riemannian manifolds with Riemannian metrics g, resp. g', then canonically also $\bar{M} = M \times M'$ is a Riemannian manifold where the metric $\bar{g}$ is given by

$$\bar{g}_{\bar{x}}(\bar{v}, \bar{w}) := g_x(v,w) + g'_{x'}(v', w'), \quad \bar{x} = (x, x'),\ \bar{v} = (v, v'),\ \bar{w} = (w, w')$$

and (1.8.3) defines the Levi-Civita connection on $\bar{M}$.

REMARK 1.8.23. We want to give a description of the direct sum connection $\bar{\nabla}$ via the corresponding frame bundles. Let $P := \mathrm{Iso}_G(\mathbb{R}^n; E) \to M$ be the principal G-bundle associated to the tangent bundle $E := TM \to M$, i.e., $P = \mathrm{L}(TM)$ with $G = \mathrm{GL}(n;\mathbb{R})$, resp. $P = \mathrm{O}(TM)$ with $G = \mathrm{O}(n)$ in the case of a Riemannian manifold (M, g). With the analogous notations we have the principal G'-bundle $P' := \mathrm{Iso}_{G'}(\mathbb{R}^{n'}; E') \to M'$ over M'. Induced by the linear connections in E and E' we have a G-connection H in P and a G'-connection H' in P'. The product $\tilde{P} := P \times P' \to \bar{M}$ is then a principal $\tilde{G}$-bundle with $\tilde{G} := G \times G'$ and

$$\tilde{H} := H \times H' \subset TP \times TP' = T\tilde{P}$$

gives a $\tilde{G}$-connection in $\tilde{P}$. The corresponding connection form and canonical one-form on $\tilde{P}$ are given by $\tilde{\omega} = (\omega, \omega') \in \Gamma(T^*\tilde{P} \otimes \tilde{\mathfrak{g}})$, respectively $(\vartheta, \vartheta') \in \Gamma(T^*\tilde{P} \otimes \mathbb{R}^{n+n'})$. Letting now $\bar{E} := E \times E'$ and $\bar{n} := n + n'$, and $\bar{G} = \mathrm{GL}(\bar{n};\mathbb{R})$ resp. $\bar{G} = \mathrm{O}(\bar{n})$, there is a canonical homomorphism of Lie groups

$$\alpha\colon \tilde{G} \to \bar{G}, \quad (A_1, A_2) \mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}. \tag{1.8.4}$$

On the other hand, to $(u, u') \in \tilde{P}$ we get a linear isomorphism $\bar{u}\colon \mathbb{R}^{\bar{n}} \to \bar{E}_{\bar{\pi}(u,u')}$ in an obvious way, and the induced map over $\bar{M}$,

$$\sigma\colon \tilde{P} \to \bar{P}, \quad \tilde{u} = (u, u') \mapsto \bar{u},$$

is given in bundle charts by α. Obviously σ is equivariant: $\sigma(\tilde{u}\tilde{g}) = \sigma(\tilde{u})\,\alpha(\tilde{g})$ for $\tilde{u} \in \tilde{P}$ and $\tilde{g} \in \tilde{G}$, more precisely,

$$\sigma(u, u') = (d\iota_{x'})_x \circ u \circ \mathrm{pr} + (d\iota_x)_{x'} \circ u' \circ \mathrm{pr}' \quad \text{for } (u, u') \in \tilde{P}_{(x,x')}.$$

The $\bar{G}$-connection $\bar{H}$ in $\bar{P}$ induced by $\tilde{H}$ is finally given by

$$\bar{H}_{\sigma(\tilde{u})g} = (R_g \circ \sigma)_* \tilde{H}_{\tilde{u}}, \quad g \in \bar{G}.$$

For the connection form $\bar{\omega} \in \Gamma(T^*\bar{P} \otimes \bar{\mathfrak{g}})$ induced by $\bar{H}$ and the canonical one-form $\bar{\vartheta} \in \Gamma(T^*\bar{P} \otimes \mathbb{R}^{\bar{n}})$ on $\bar{P}$ then obviously

$$\sigma^*\bar{\omega} = (d\alpha)_1 \circ (\omega, \omega') \quad \text{and} \quad \sigma^*\bar{\vartheta} = (\vartheta, \vartheta'). \tag{1.8.5}$$

LEMMA 1.8.24. *Let ∇ and ∇' be linear connections on differentiable manifolds M, resp. M'. Suppose that X and X' are semimartingales taking values in M, resp. M' (adapted to the same filtration); let U and U' be the corresponding horizontal lifts in P, resp. P', as well as $Z = \int_U \vartheta$ and $Z' = \int_{U'} \vartheta'$ the anti-developments taking values in $\mathbb{R}^n$, resp. $\mathbb{R}^{n'}$. Suppose that $\bar{M} = M \times M'$ is equipped with the canonical direct sum connection. Then $\bar{U} := \sigma(U, U')$ taking values in $\mathrm{L}(\bar{M})$, resp. $\mathrm{O}(\bar{M})$, is a horizontal lift of the semimartingale $\bar{X} := (X, X')$, and the $\mathbb{R}^{\bar{n}}$-valued anti-development of $\bar{X}$ is given by (Z, Z').*

PROOF. Since $\bar{\pi} \circ \bar{U} = (\pi, \pi') \circ (U, U') = (X, X')$, we calculate by means of (1.8.5)

$$\int_{\bar{U}} \bar{\omega} = \int_{(U,U')} \sigma^*\bar{\omega} = (d\alpha)_1\Big(\int_{(U,U')} (\omega, \omega')\Big) = (d\alpha)_1\Big(\int_U (\omega, 0) + \int_{U'} (0, \omega')\Big) = 0 \quad \text{and}$$

$$\int_{\bar{U}} \bar{\vartheta} = \int_{(U,U')} \sigma^*\bar{\vartheta} = \int_{(U,U')} (\vartheta, \vartheta') = \int_U (\vartheta, 0) + \int_{U'} (0, \vartheta') = (Z, 0) + (0, Z') = (Z, Z')$$

which verifies the claim. $\square$

After these technical remarks we now return to the notion of convex geometry introduced in Definition 1.8.21.

THEOREM 1.8.25 (Non-confluence of ∇-martingales). *Let M be a differentiable manifold endowed with a torsion-free linear connection and $K \subset M$ be a compact subset such that an open neighbourhood V of K has convex geometry. If then, for a given filtration, X and X' are ∇-martingales on M taking values in K, which both converge and such that $X_\infty = X'_\infty$ almost surely, then already $X = X'$ modulo indistinguishability.*

PROOF. Suppose that $\phi\colon V \times V \to \mathbb{R}_+$ describes the convex geometry of V, where without loss of generality we may assume ϕ to be bounded. For Riemannian manifolds the claim follows from the minimum principle (Theorem 1.8.20) with $\lambda = 0$ (applied on the product manifold $V \times V$). In the general case the proof of Theorem 1.8.20 carries over verbatim with $\lambda = 0$. Indeed then (X, X') is a martingale on $M \times M$ taking values in $V \times V$ and $Y := \phi(X, X')$ a bounded non-negative submartingale. As by assumption $Y_\infty = 0$, almost surely, we conclude that for each $t \geq 0$,

$$0 \leq Y_t \leq \mathbb{E}^{\mathscr{F}_t}[Y_\infty] = 0 \quad \text{almost surely,}$$

and hence $X_t = X'_t$ almost surely, since ϕ vanishes only on the diagonal. □

DEFINITION 1.8.26 (Totally geodesic submanifold). Let M_0 and M be differentiable manifolds, equipped with torsion-free linear connections, and let $M_0 \stackrel{\iota}{\hookrightarrow} M$ be an embedding. The manifold M_0 is called a *totally geodesic submanifold* of M if the inclusion map ι is affine.

Without loss of generality we may consider M_0 as subspace of M.

REMARK 1.8.27. An embedding $M_0 \stackrel{\iota}{\hookrightarrow} M$ is obviously a totally geodesic submanifold M_0 of M if and only if ι transfers M_0-geodesics into M-geodesics, or equivalently: if $x_0 \in M_0$ and $v_0 \in T_{x_0}M_0$, as well as γ a geodesic curve in M such that $\gamma(0) = \iota(x_0)$ and $\dot\gamma(0) = \iota_* v_0$, then $\gamma\big(]-\varepsilon, \varepsilon[\big) \subset \iota(M_0)$ for some $\varepsilon > 0$.

EXAMPLE 1.8.28. Let M be a connected differentiable manifold and $\varphi\colon M \to \mathbb{R}$ an affine function. Then $M_0 = \{\varphi = c\}$ defines a totally geodesic submanifold of M for each $c \in \mathbb{R}$.

PROOF. At first, for affine functions φ, we remark that $(d\varphi)_x = 0$ for some $x \in M$ already implies $d\varphi = 0$ locally about x. Indeed, for a geodesic curve γ in M with $\gamma(0) = x$ the composition $\varphi \circ \gamma$ defines a straight line with slope $(\varphi \circ \gamma)'(0) = 0$; hence φ is constant locally about x. Thus, since by assumption M is connected, the existence of a critical point of φ means that $\varphi \equiv \text{const}$. Hence assume now $(d\varphi)_x \neq 0$ for each $x \in M$ and consider $M_0 = \varphi^{-1}\{c\}$. In particular, c is then a regular value for φ. For $x_0 \in M_0$ and $v \in T_{x_0}M_0$ let γ be the geodesic in M determined by $\gamma(0) = x_0$ and $\dot\gamma(0) = v$. We conclude that then $(\varphi \circ \gamma)'(0) = 0$ and consequently $\varphi \circ \gamma \equiv \varphi \circ \gamma(0) = c$ which shows that γ lies in M_0. □

LEMMA 1.8.29. *Let M be a differentiable manifold equipped with a torsion-free linear connection. Let M_0 be a totally geodesic submanifold of M and $x_0 \in M_0$. There exists an open neighbourhood V of x_0 in M and a convex function $\varphi \in C^\infty(V)$ such that*

$$\{\varphi = 0\} = V \cap M_0 \quad \textit{and} \quad \{\varphi > 0\} = V \backslash M_0.$$

PROOF. Denoting by $\text{codim}(M_0) = \dim M - \dim M_0 = n - n_0$ the codimension of M_0, we may assume without loss of generality that $0 < \text{codim}(M_0) < n$. As in the proof of Lemma 1.7.22 we introduce normal coordinates (h, V) for M about x_0 via the exponential map. As M_0 is totally geodesic, h can be chosen such that $h = (\phi, \psi)$ and $V \cap M_0 = \{\psi = 0\}$. Then all Christoffel symbols Γ_{ij}^k vanish at the point x_0. On the other hand, as M_0 is totally geodesic, all geodesic curves $t \mapsto \gamma(t)$ on M with initial condition $\gamma(0) = x \in M_0$ and $\dot\gamma(0) \in T_x M_0$ stay in M_0 for small values of t, which in addition implies for $x \in V \cap M_0$,

$$\Gamma_{ij}^k(x) = 0, \quad 1 \le i, j \le n_0, \quad n_0 + 1 \le k \le n, \tag{1.8.6}$$

as can be seen from the description of geodesic curves in coordinates,

$$\ddot\gamma^k(t) + \sum_{i,j} \Gamma_{ij}^k\big(\gamma(t)\big)\,\dot\gamma^i(t)\,\dot\gamma^j(t) = 0.$$

We are going to show that, with an appropriate choice of the constant $c > 0$, the function

$$\varphi := \tfrac12 \left(c + |\phi|^2\right) |\psi|^2 \tag{1.8.7}$$

satisfies the claim (after possibly shrinking of V). To this end, we have to verify that φ is convex on a neighbourhood of x_0.

We start by calculating $(\nabla d\varphi)_{ij} = \partial_i \partial_j\, \varphi - \sum_k \Gamma_{ij}^k\, \partial_k \varphi$ in the chart (h, V); see (1.5.8). Denoting the components of h by $(\phi^1, \dots, \phi^{n_0}, \psi^{n_0+1}, \dots, \psi^n)$, it holds that

$$\partial_i\partial_j\varphi = \begin{cases} \delta_{ij}\,|\psi|^2 & \text{for } 1 \le i, j \le n_0, \\ 2\,\phi^i\psi^j & \text{for } 1 \le i \le n_0, \quad n_0+1 \le j \le n, \\ \delta_{ij}\big(c + |\phi|^2\big) & \text{for } n_0 + 1 \le i, j \le n\,; \end{cases}$$

as well as

$$\sum_{k=1}^n \Gamma_{ij}^k\, \partial_k\varphi = |\psi|^2 \sum_{k=1}^{n_0} \Gamma_{ij}^k\, \phi^k + \big(c + |\phi|^2\big) \sum_{k=n_0+1}^{n} \Gamma_{ij}^k\, \psi^k.$$

Using the abbreviation $H_{ij} := (\nabla d\varphi)_{ij}$, we have to show that H on $V \backslash M_0 = V \cap \{\psi \ne 0\}$ is positive definite for some sufficiently small open neighbourhood V of the point x_0. In terms of the decomposition of $\{1, \dots, n\}$ into $I = \{1, \dots, n_0\}$ and $J = \{n_0 + 1, \dots, n\}$, we see that H is positive definite on $V \cap \{\psi \ne 0\}$ if and only if

$$H^* := \begin{pmatrix} \frac{1}{|\psi|^2}\, (H_{ij})_{(i,j) \in I \times I} & \frac{1}{|\psi|}\, (H_{ij})_{(i,j) \in I \times J} \\ \frac{1}{|\psi|}\, (H_{ij})_{(i,j) \in J \times I} & (H_{ij})_{(i,j) \in J \times J} \end{pmatrix}$$

is positive definite on $V \cap \{\psi \ne 0\}$. However, as easily seen, it can be achieved that H^* on $V \cap \{\psi \ne 0\}$ is arbitrarily close to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & c\,1 \end{pmatrix}$$

by sufficiently shrinking the neighbourhood V of x_0 and choosing the constant $c > 0$ in (1.8.7) small enough. To see this for the coefficients of the first quadrant of H^*, we use that on a sufficiently small neighbourhood of x_0, as a consequence of (1.8.6), one can estimate

$$|\Gamma_{ij}^k| \le C\, |\psi^k|, \quad 1 \le i, j \le n_0, \quad n_0 + 1 \le k \le n,$$

with a constant $C > 0$. □

COROLLARY 1.8.30. *Let the manifold M be equipped with a torsion-free linear connection ∇ and suppose that M_0 is a totally geodesic submanifold of M. Then every point in M_0 possesses an open neighbourhood V in M with the property: if X is a ∇-martingale on M taking values in V such that for some $t > 0$, the variable X_t takes almost surely its values in M_0, then $X|[0, t]$ lives entirely in M_0.*

PROOF. We choose φ as in Lemma 1.8.29 where without loss of generality we may assume that φ is bounded. Then the composition $\varphi(X)$ defines a bounded non-negative submartingale with the property that $\varphi(X_t) = 0$ almost surely, and thus necessarily already $\varphi(X) \equiv 0$ on $[0, t]$ modulo indistinguishability. □

THEOREM 1.8.31. *Let M be a differentiable manifold and ∇ be a torsion-free linear connection on M. Then each point has a neighbourhood V of convex geometry. In particular, each point has a neighbourhood V with the following property: if X and X' are two V-valued ∇-martingales on M (adapted to the same filtration) such that $X_t = X'_t$ almost surely, for some $t > 0$, then already $X = X'$ on $[0, t]$ modulo indistinguishability.*

PROOF. The first part of the claim follows from Lemma 1.8.29, applied to the diagonal manifold $M \xhookrightarrow{\iota} M \times M$, $x \mapsto (x, x)$, considered as totally geodesic submanifold of the $M \times M$ where the product $M \times M$ is equipped with the direct sum connection. The second part follows from Corollary 1.8.30 or also directly from Theorem 1.8.25. □

The results of this section show that the behavior of martingales on manifolds can be controlled in domains which are "small" in the sense that they support convex functions with specific properties. Aspects of global martingale theory, however, such as interaction with global geometry, are unaffected by this.

The local existence of suitable convex functions is ensured by Theorem 1.8.25 and Theorem 1.8.31, where the question naturally arises as to how large the domains of such "convexity areas" can be chosen in concrete cases. In the next Chapter, among other things, we will connect such questions with the concept of the curvature of a Riemannian manifold.

1.9. Stochastic Differentials and Second Order Tangent Spaces

This Section is not intended for the introduction of new concepts; the aim is rather to regard the methods of Stochastic Analysis as developed so far from a different perspective.

The symbol dX for an M-valued semimartingale X has so far not been considered as a mathematical object by its own; it has been used as a formal notation which received a precise meaning only by composition with scalar-valued functions. The object dX is however interesting as it does not behave as a tangent vector, for instance in the intuitive sense of "$dX_t \in T_{X_t}M$", as one might think, rather it shares the formal properties of a second order tangent vector. In this Section we want to investigate how to interpret dX as a "section of the second order tangent bundle $T^2M \to M$ along X", giving a meaning to $dX_t \in T^2_{X_t}M$; see [**12**], [**36**], [**35**], as well as [**39**], [**40**] for more detailed expositions in this direction.

As differential geometry of second order is not commonly widespread, the use of concepts of second order concepts in Stochastic Analysis gives sometimes the impression that Stochastic Differential Geometry requires a revision of standard differential geometry. We tried so far to counteract this impression by a consequent use of standard geometric notations. Nevertheless it is interesting to note that the simple concept of a second order tangent space permits a geometric description of the transformation behaviour given by Itô's formula.

In this Section, M will always be a differentiable manifold, for $x \in M$, $\mathscr{E}_x(M)$ will denote the ring of germs of C^∞ functions on M at the point x, and $\mathfrak{m}_x(M) \subset \mathscr{E}_x(M)$ the maximal ideal of germs $[\varphi]$ with the property that $\varphi(x) = 0$.

DEFINITION 1.9.1 (Tangent space of order k). Let M be a manifold and $k \in \mathbb{N}$. The finite dimensional real vector space

$$\begin{aligned} T_x^k M :&= \big(\mathfrak{m}_x(M)/\mathfrak{m}_x(M)^{k+1}\big)^* \\ &\equiv \big\{v \in \mathscr{E}_x(M)^*:\ v(1) = 0 \text{ and } v(\varphi^{k+1}) = 0,\ \text{if } \varphi(x) = 0\big\} \end{aligned}$$

is called *tangent space of order* k to M at the point x.

Tangent spaces of first order in the sense of Definition 1.9.1 coincide with the usual tangent spaces.

REMARK 1.9.2. For any manifold M it holds $T_x^1 M = T_x M$ for all $x \in M$.

PROOF. The inclusion $T_x M \subset T_x^1 M$ is obvious, since by definition $T_x^1 M$ is the real vector space of derivations at x, i.e.

$$T_x M = \big\{v \in \mathscr{E}_x(M)^*:\ v(\varphi\psi) = \varphi(x)\, v(\psi) + \psi(x)\, v(\varphi)\big\}.$$

Conversely, assume that $v \in T_x^1 M$; we want to show that $v(\varphi\psi) = \varphi(x)\, v(\psi) + \psi(x)\, v(\varphi)$ for all $\varphi, \psi \in \mathscr{E}_x(M)$. By the equation $2\,\varphi\psi = (\varphi+\psi)^2 - \varphi^2 - \psi^2$ it is sufficient to verify $v(\varphi^2) = 2\,\varphi(x)v(\varphi)$ for $\varphi \in \mathscr{E}_x(M)$. By assumption we have $v\big(\varphi - \varphi(x)\big)^2 = 0$, from where

$$0 = v\big(\varphi^2 - 2\,\varphi(x)\varphi + \varphi(x)^2\big) = v(\varphi^2) - 2\,\varphi(x)v(\varphi) + v\big(\varphi(x)^2\big) = v(\varphi^2) - 2\,\varphi(x)v(\varphi)$$

follows. This shows the inclusion $T_x^1 M \subset T_x M$. □

DEFINITION 1.9.3 (Cotangent space of order k, differential of order k). Let M be a manifold and $k \in \mathbb{N}$. The finite dimensional real vector space

$$T_x^{*k} M := (T_x^k M)^* = \mathfrak{m}_x(M)/\mathfrak{m}_x(M)^{k+1}$$

is called *cotangent space of order* k to M at the point x. For $\varphi \in C^\infty(M)$ resp.,$\varphi \in \mathscr{E}_x(M)$, we denote $(d^k\varphi)_x := [\varphi - \varphi(x)] \in \mathfrak{m}_x(M)/\mathfrak{m}_x(M)^{k+1}$ the *differential of order* k of φ at the point x.

Obviously $(d^1\varphi)_x \equiv d\varphi_x \in T_x^* M$. In addition, it is easy to see that $T^k M \to M$ and $T^{*k} M \to M$ constitute vector bundle over M. Differentiable sections of these bundles, i.e. elements of $\Gamma(T^k M)$ resp. $\Gamma(T^{*k} M)$, are called *vector fields of order* k, resp. *differential forms of order* k.

REMARK 1.9.4 (Push-forward and pull-back). Every differentiable map $f\colon M \to N$ between manifolds induces canonically vector bundle homomorphisms

$$f_*\colon T^k M \to f^* T^k N \quad \text{resp.,} \quad f^*\colon f^*(T^{*k} N) \to T^{*k} M,$$

namely $(f_*)_x\colon T_x^k M \to T_{f(x)}^k N$, $v \longmapsto (f_*)_x v$ where $(f_*)_x v\,(\varphi) := v(\varphi \circ f)$ for $\varphi \in \mathscr{E}_{f(x)}(N)$, and $(f^*)_x\colon T_{f(x)}^{*k} N \to T_x^{*k} M$, $\vartheta \longmapsto \vartheta \circ (f_*)_x$.

In the sequel we focus on the case $k = 2$ and we want first to check that vector fields of second order correspond to differential operators without constant term of order at most two. For $L \in \Gamma(T^2 M)$ and $\varphi \in \mathscr{E}_x(M)$ let $(L\varphi)(x) := L_x\varphi$ where $L_x \in T_x^2 M$, i.e. $L_x \in \mathscr{E}_x(M)^*$ with $L_x 1 = 0$ and $L_x\varphi^3 = 0$ if $\varphi(x) = 0$. Writing $\varphi = \bar\varphi \circ h$ in a chart h at x with $h(x) = 0$ and $\bar\varphi \in \mathscr{E}_0(\mathbb{R}^n)$, then $\bar\varphi$ can be represented by Taylor's formula as

$$\bar\varphi(y) = \bar\varphi(0) + \sum_i (D_i\bar\varphi)(0)\, y^i + \sum_{i,j} \gamma_{ij}(y)\, y^i y^j$$

where $\gamma_{ij} \in \mathscr{E}_0(\mathbb{R}^n)$ is such that $\gamma_{ij}(0) = \frac{1}{2}\,(D_iD_j\bar\varphi)(0)$. Hence it holds

$$\begin{aligned}(L\varphi)(x) &= L_x\big(\sum_i (D_i\bar\varphi)(0)\,h^i + \sum_{i,j}(\gamma_{ij}\circ h)\,h^ih^j\big)\\ &= \sum_i (D_i\bar\varphi)(0)\,L_xh^i + \tfrac12\sum_{i,j}(D_iD_j\bar\varphi)(0)\,L_x(h^ih^j)\\ &\qquad\qquad + \sum_{i,j} L_x\big(\big[\gamma_{ij}\circ h - \tfrac12 (D_iD_j\bar\varphi)(0)\big]\,h^ih^j\big).\end{aligned}$$

But we have $L_x(fgh) = 0$ for functions f, g, h defined locally about x with the property that $f(x) = g(x) = h(x) = 0$ which implies $L_x\big(\big[\gamma_{ij}\circ h - \frac12 (D_iD_j\bar\varphi)(0)\big]\,h^ih^j\big) = 0$. Hence, letting $b^i(x) = L_xh^i$ and $a^{ij}(x) = \frac12\,L_x(h^ih^j)$, we obtain, as wanted, a representation of the form

$$L_x = \sum_i b^i(x)\,\big(\tfrac{\partial}{\partial h^i}\big)_x + \sum_{i,j} a^{ij}(x)\,\big(\tfrac{\partial}{\partial h^i}\big)_x\big(\tfrac{\partial}{\partial h^j}\big)_x. \tag{1.9.1}$$

This also shows that $TM \hookrightarrow T^2M$ is canonically a subbundle of T^2M.

REMARK 1.9.5. Functions $\varphi, \psi \in C^\infty(M)$ on a manifold M induce canonically sections of $T^{*2}M \to M$ through

$$\begin{aligned} d^2\varphi \in \Gamma(T^{*2}M), &\quad L \mapsto (d^2\varphi)L = L\varphi,\\ d\varphi\cdot d\psi \in \Gamma(T^{*2}M), &\quad L \mapsto \Gamma(\varphi,\psi) \equiv \tfrac12\,\big[L(\varphi\psi) - \varphi\,L(\psi) - \psi\,L(\varphi)\big],\end{aligned}$$

where $L \in \Gamma(T^2M)$. In particular, for $A, B \in \Gamma(TM)$, it holds then

$$d^2\varphi(A) = A(\varphi), \quad d^2\varphi(A\cdot B) = AB(\varphi), \tag{1.9.2}$$

$$d\varphi\cdot d\psi(A) = 0, \quad d\varphi\cdot d\psi(A\cdot B) = \tfrac12\,\big[A(\varphi)\,B(\psi) + A(\psi)\,B(\varphi)\big], \tag{1.9.3}$$

where $d^2\varphi$ and $d\varphi\cdot d\psi$ are already determined by (1.9.1). Recall that $A\cdot B \in \Gamma(T^2M)$ denotes the composition of the derivations A, B.

For functions $\varphi, \psi \in C^\infty(M)$ it is easy to see that

$$d^2(\varphi\psi) = \varphi\,d^2\psi + \psi\,d^2\varphi + 2\,d\varphi\cdot d\psi. \tag{1.9.4}$$

Moreover note that $d^2\varphi\,|\Gamma(TM) = d\varphi$ and $d\varphi\cdot d\psi\,|\Gamma(TM) = 0$.

COROLLARY 1.9.6. *Let M be a manifold and (h, U) a chart at x. Then*

$$\begin{aligned}&\big(\tfrac{\partial}{\partial h^i},\ \tfrac{\partial^2}{\partial h^i\partial h^i} : 1 \le i \le n\,;\quad \tfrac{\partial^2}{\partial h^i\partial h^j} : 1 \le i < j \le n\big), \quad \textit{respectively,}\\ &\big(d^2h^i,\ dh^i\cdot dh^i : 1 \le i \le n\,;\quad 2\,dh^i\cdot dh^j : 1 \le i < j \le n\big)\end{aligned}$$

are frame systems for the second order tangent bundle T^2M over U, respectively the second order cotangent bundle $T^{*2}M$ over U, which are dual to each other.

EXAMPLE 1.9.7. Let $L \in \Gamma(T^2M)$ and $\Gamma(\varphi,\psi) = (d\varphi\cdot d\psi)L$ for $\varphi, \psi \in C^\infty(M)$. As noted above, we interpret L as PDO of second order via $(L\varphi)(x) = L_x\varphi$, $\varphi \in C^\infty(M)$.

(a) If $L = A_0 + \sum_i A_i^2$ is a PDO in Hörmander form with $A_0, A_i \in \Gamma(TM)$, then

$$\Gamma(\varphi,\psi) = \sum_i (A_i\varphi)(A_i\psi).$$

(b) If (h, U) is a chart such that $L|U = \sum_i b^i\,\frac{\partial}{\partial h^i} + \sum_{i,j} a^{ij}\,\frac{\partial}{\partial h^i}\frac{\partial}{\partial h^j}$, then

$$\Gamma(\varphi,\psi)|U = \sum_{i,j} a^{ij}\,(\partial_i\varphi)\,(\partial_j\psi) \quad \text{where } \partial_i = \tfrac{\partial}{\partial h^i}.$$

More generally, each section $L \in \Gamma(T^2M)$ has a representation as $L = \sum_{\text{finite}} L^\nu$, where $L^\nu = A \in \Gamma(TM)$ or $L^\nu = A\cdot B$ with $A, B \in \Gamma(TM)$. It holds $\Gamma(\varphi,\psi) = 0$ for all $\varphi,\psi \in C^\infty(M)$ if and only if $L \in \Gamma(TM)$.

NOTATION 1.9.8. For any two vector fields $A, B \in \Gamma(TM)$ we have $A\cdot B \in \Gamma(T^2M)$, defined as composition of the derivation A and B; for two differential forms $\alpha,\beta \in \Gamma(T^*M)$ we have (slightly more general than Remark 1.9.5) $\alpha\cdot\beta \in \Gamma(T^{*2}M)$ well defined through $(\alpha\cdot\beta)_x := (d\varphi\cdot d\psi)_x$ if $\alpha_x = (d\varphi)_x$ and $\beta_x = (d\psi)_x$.

Analogously to Lemma 1.3.2 we have for differential forms of second order the following Lemma.

LEMMA 1.9.9. *On any manifold M there exists a finite number of real-valued functions $\varphi^1, \ldots, \varphi^k \in C^\infty(M)$ such that the following properties hold:*

(i) *Each $\vartheta \in \Gamma(T^{*2}M)$ writes as $\vartheta = \sum_{\nu=1}^k \vartheta_\nu\, d^2\varphi^\nu$ where $\vartheta_\nu \in C^\infty(M)$.*
(ii) *If X is a continuous semimartingale on M, then each continuous adapted $T^{*2}M$-valued process Θ over X (i.e., $\Theta_t \in T^{*2}_{X_t}M$ for $t \in \mathbb{R}_+$) has a representation of the form $\Theta = \sum_{\nu=1}^k \Theta_\nu\, (d^2\varphi^\nu)(X)$ with continuous adapted real-valued processes Θ_ν.*

PROOF. The proof proceeds along the usual scheme, as already exploited in the proof of Lemma 1.3.2. We first realize M via a Whitney embedding $h\colon M \hookrightarrow \mathbb{R}^\ell$ as closed submanifold of a suitable $\mathbb{R}^\ell$. There is a partition $(\phi_\lambda)_{\lambda\in\Lambda}$ of the unity on M and a family $(I_\lambda)_{\lambda\in\Lambda}$ of subsets $I_\lambda \subset \{1,\ldots,\ell\}$ with the following property: For each $\lambda \in \Lambda$ the components $(h^i)_{i\in I_\lambda}$ define a chart for M on an open neighbourhood of $\operatorname{supp}(\phi_\lambda)$.

For (i): By Corollary 1.9.6, we have

$$\phi_\lambda\, \vartheta = \sum_{i=1}^{\ell} \vartheta_i^\lambda\, d^2h^i + \sum_{i,j=1}^{\ell} \vartheta_{ij}^\lambda\, dh^i\cdot dh^j$$

where $\vartheta_i^\lambda, \vartheta_{ij}^\lambda \in C^\infty(M)$ are such that $\operatorname{supp}(\vartheta_i^\lambda), \operatorname{supp}(\vartheta_{ij}^\lambda) \subset \operatorname{supp}(\phi_\lambda)$ where $\vartheta_i^\lambda := 0$ for $i \notin I_\lambda$ and $\vartheta_{ij}^\lambda := 0$ for $\{i,j\} \not\subset I_\lambda$. Letting $\tilde\vartheta_i := \sum_\lambda \vartheta_i^\lambda$ and $\tilde\vartheta_{ij} := \sum_\lambda \vartheta_{ij}^\lambda$, this gives the representation

$$\begin{aligned}\vartheta &= \sum_{i=1}^{\ell} \tilde\vartheta_i\, d^2h^i + \sum_{i,j=1}^{\ell} \tilde\vartheta_{ij}\ dh^i\cdot dh^j \\ &= \sum_{i=1}^{\ell} \tilde\vartheta_i\, d^2h^i + \tfrac12 \sum_{i,j=1}^{\ell} \tilde\vartheta_{ij}\left[d^2(h^ih^j) - h^i\, d^2h^j - h^j\, d^2h^i\right],\end{aligned}$$

which shows that ϑ has a representation of the claimed form. Part (ii) is shown analogously. □

We want to come back now to the initial question concerning the status of differentials of M-valued semimartingales. For an M-valued semimartingale X we first define

$$(dX)(\varphi) := d\big(\varphi(X)\big), \quad \varphi \in C^\infty(M).$$

If (h,U) is a chart for M and σ,τ stopping times with the property that $X|[\sigma,\tau[$ takes only values in U, then with $\partial_i = \frac{\partial}{\partial h^i}$ and $X^i = h^i(X)$ it holds

(1.9.5)
$$1_{[\sigma,\tau[}\, d\big(\varphi(X)\big) = 1_{[\sigma,\tau[} \sum_i (\partial_i\varphi)(X)\, dX^i + 1_{[\sigma,\tau[}\ \tfrac12 \sum_{i,j} (\partial_i\partial_j\varphi)(X)\, d[X^i,X^j].$$

As the left-hand side of (1.9.5) is coordinate invariant, also the right-hand side does not depend on the choice of the chart h on U, and for $\varphi \in C^\infty(M)$ one finds

$$\begin{aligned} d\big(\varphi(X)\big) &= \sum_i (\partial_i\varphi)(X)\,dX^i + \frac{1}{2}\sum_{i,j}(\partial_i\partial_j\varphi)(X)\,d[X^i,X^j] \\ &\equiv \sum_n 1_{[\tau_n,\tau_{n+1}[}\Big(\sum_i(\partial_i\varphi)(X)\,dX^i + \frac{1}{2}\sum_{i,j}(\partial_i\partial_j\varphi)(X)\,d[X^i,X^j]\Big), \end{aligned}$$

where in the last line we choose to a countable covering of coordinate neighbourhoods the sequence $(\tau_n)_{n\geq 0}$ of stopping times according to Lemma 1.3.1.

Formally we may write this as

$$dX = \sum_i (dX^i)\,\partial_i + \frac{1}{2}\sum_{i,j} d[X^i,X^j]\,\partial_i\partial_j,$$

from where we can already read off that, at least in a formal sense, the differential dX behaves as a section of $T^2M \to M$ along X. The precise meaning of this heuristic argument is given by the following Theorem.

THEOREM 1.9.10 (Principle of Laurent Schwartz). *Let X be an M-valued semimartingale. There exists exactly one linear mapping*

$$\Theta \longmapsto \textstyle\int\langle\Theta, dX\rangle$$

from the real vector space of continuous adapted $T^{*2}M$-valued processes Θ over X (i.e., $\Theta_t \in T^{*2}_{X_t}M$ for $t\in\mathbb{R}_+$) to $\mathscr{S}$ with the following properties:

(1.9.6) $\qquad d^2\varphi(X) \mapsto \varphi(X) - \varphi(X_0), \quad \varphi \in C^\infty(M),$

(1.9.7) $\qquad K\,\Theta \mapsto \int K\langle\Theta, dX\rangle,\quad$ *K continuous, adapted, real-valued process.*

where by definition, $\langle\Theta, dX\rangle = d\int\langle\Theta, dX\rangle$.

NOTATION 1.9.11. We call $\int\langle\Theta, dX\rangle$ the *integral of* Θ *along* X. If in particular $\Theta = \vartheta(X)$ for some $\vartheta \in \Gamma(T^{*2}M)$, we write also $\int\langle\vartheta, dX\rangle$ instead of $\int\langle\Theta, dX\rangle$.

PROOF OF THEOREM 1.9.10. By Lemma 1.9.9 (ii)the process Θ has a representation of the form $\Theta = \sum_{\text{finite}}\Theta_\nu(d^2\varphi^\nu)(X)$; hence necessarily

$$\int\langle\Theta, dX\rangle = \sum_\nu \int\Theta_\nu\,d\big(\varphi^\nu(X)\big). \tag{1.9.8}$$

It remains to show that $\int\langle\Theta, dX\rangle$ is well-defined by (1.9.8). Assuming for instance that $\sum_{\text{finite}} K_\nu\,(d^2\varphi^\nu)(X) = 0$, we have to verify that already $\sum K_\nu\,d\big(\varphi^\nu(X)\big) = 0$. Without restrictions we may replace here K_ν by $K_\nu\,1_{[\sigma,\tau[}$ and assume that X takes on $[\sigma,\tau[$ only values in the coordinate neighbourhood U of a fixed chart (h,U). In terms of $\varphi^\nu|U = \bar\varphi^\nu\circ h$ one observes at first that over U

$$\begin{aligned} d^2\varphi^\nu &= d^2(\bar\varphi^\nu\circ h) \\ &= \sum_i(D_i\bar\varphi^\nu\circ h)\,d^2h^i + \sum_i(D_i^2\bar\varphi^\nu\circ h)\,dh^i\cdot dh^i + \sum_{i<j}(D_iD_j\bar\varphi^\nu\circ h)\,2\,dh^i\cdot dh^j \\ &= \sum_i(D_i\bar\varphi^\nu\circ h)\,d^2h^i + \sum_{i,j}(D_iD_j\bar\varphi^\nu\circ h)\,dh^i\cdot dh^j, \end{aligned}$$

and hence for $\bar{X} := h(X)$ by assumption

$$\begin{aligned}0 &= \sum_\nu K_\nu\,(d^2\varphi^\nu)(X)\\ &= \sum_i \sum_\nu K_\nu (D_i\bar{\varphi}^\nu)(\bar{X})\,(d^2h^i)(X) + \sum_{i,j}\sum_\nu K_\nu (D_iD_j\bar{\varphi}^\nu)(\bar{X})\,(dh^i\,dh^j)(X),\end{aligned}$$

from where we get $\sum_\nu K_\nu(D_i\bar{\varphi}^\nu)(\bar{X}) \equiv 0$ and $\sum_\nu K_\nu(D_iD_j\bar{\varphi}^\nu)(\bar{X}) \equiv 0$ almost surely for all i, j. On the other hand, this implies

$$\begin{aligned}\sum_\nu K_\nu\, d\big(\varphi^\nu(X)\big) &= \sum_\nu K_\nu\, d\big(\bar{\varphi}^\nu)(\bar{X})\\ &= \sum_i \left(\sum_\nu K_\nu (D_i\bar{\varphi}^\nu)(\bar{X})\right) d\bar{X}^i + \frac{1}{2}\sum_{i,j}\left(\sum_\nu K_\nu(D_iD_j\bar{\varphi}^\nu)(\bar{X})\right) d[\bar{X}^i, \bar{X}^j] = 0,\end{aligned}$$

which shows the claim. □

THEOREM 1.9.12 (Pullback formula). *Let $\phi\colon M \to N$ be a differentiable map between manifolds and Θ be a continuous adapted $T^{*2}N$-valued process over $\phi(X)$ (i.e., $\Theta_t \in T^{*2}_{\phi\circ X_t}N$ for $t \in \mathbb{R}_+$). Then $\phi^*\Theta$ is a $T^{*2}M$-valued process over X and satisfies*

$$\int \langle \phi^*\Theta, dX\rangle = \int \langle \Theta, d\big(\phi(X)\big)\rangle. \tag{1.9.9}$$

In particular, for $\vartheta \in \Gamma(T^{*2}N)$ then $\int\langle\phi^*\vartheta, dX\rangle = \int\langle\vartheta, d\big(\phi(X)\big)\rangle$.

PROOF. Because of $\phi^*d^2\varphi = d^2(\varphi\circ\phi)$, the left-hand side of (1.9.9) has the defining properties of the integral of Θ along $\phi\circ X$. □

Before putting the integral $\int\langle\vartheta, dX\rangle$ of a second order differential form $\vartheta \in \Gamma(T^{*2}M)$ along X in perspective to the integrals treated in Section 1.3, e.g. $\int b(dX, dX)$ for $b \in \Gamma(T^*M \otimes T^*M)$, respectively $\int_X \alpha$ for $\alpha \in \Gamma(T^*M)$, we want to note some further aspects of second order tangent spaces.

REMARK 1.9.13. Denoting for a manifold M by $TM \odot TM$ the vector bundle over M with the symmetric tensor products $T_xM \odot T_xM$ as fiber x, there is a $C^\infty(M)$ linear mapping

$$\hat{}\ : \Gamma(T^2M) \to \Gamma(TM \odot TM), \quad L \mapsto \hat{L}, \tag{1.9.10}$$

determined by

$$\hat{A} = 0, \quad (A\cdot B)^\wedge = \frac{1}{2}\big(A\otimes B + B\otimes A\big) \equiv A \odot B, \quad A, B \in \Gamma(TM).$$

Writing $L \in \Gamma(T^2M)$ in a chart (h, U) as $L|U = \sum_i b^i\,\partial_i + \sum_{i,j} a^{ij}\,\partial_i\partial_j$ where $\partial_i = \frac{\partial}{\partial h^i}$, then obviously $\hat{L}|U = \sum_{i,j} a^{ij}\,(\partial_i \odot \partial_j)$. Note that the map (1.9.10) is characterized by the property

$$(d\varphi \odot d\psi)\,\hat{L} = (d\varphi\cdot d\psi)L = \Gamma(\varphi, \psi), \quad \varphi, \psi \in C^\infty(M).$$

REMARK 1.9.14. For a manifold M we have the following exact sequence of vector bundles over M

$$0 \longrightarrow TM \hookrightarrow T^2M \xrightarrow{\ \wedge\ } TM \odot TM \longrightarrow 0. \tag{1.9.11}$$

By dualization of (1.9.11) we obtain the exact sequence

$$
\begin{array}{ll}
(1.9.12) \qquad & 0 \longrightarrow T^*M \odot T^*M \xrightarrow{H} T^{*2}M \xrightarrow{R} T^*M \longrightarrow 0\,, \\
& \qquad \alpha_x \odot \beta_x \longmapsto (\alpha\cdot\beta)_x
\end{array}
$$

where R represents the restriction of T^2M to the subbundle TM.

THEOREM 1.9.15. *For a manifold M we have the exact sequence*

$$0 \longrightarrow \Gamma(T^*M \odot T^*M) \xrightarrow{H} \Gamma(T^{*2}M) \xrightarrow{R} \Gamma(T^*M) \longrightarrow 0 \tag{1.9.13}$$

where $H(\alpha \odot \beta) = \alpha\cdot\beta$, as well as $R(d^2\varphi) = d\varphi$ and $R(d\varphi\cdot d\psi) = 0$.
The sequence (1.9.13) *of $C^\infty(M)$-modules possesses an $\mathbb{R}$-linear splitting*

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(T^*M \odot T^*M) \arrow[r, "H"] & \Gamma(T^{*2}M) \arrow[r, "R"] & \Gamma(T^*M) \arrow[l, bend left, dashed, "d"] \arrow[r] & 0\,;
\end{tikzcd} \tag{1.9.14}
$$

More precisely, there exists an $\mathbb{R}$-linear mapping $d\colon \Gamma(T^*M) \to \Gamma(T^{*2}M)$ with the properties $d(d\varphi) = d^2\varphi$ and $d(\varphi\alpha) = d\varphi\cdot\alpha + \varphi\cdot d\alpha$ such that $R \circ d = \mathrm{id}.$.

PROOF. For $\alpha = \sum_\nu \varphi_\nu\, dh^\nu \in \Gamma(T^*M)$, we want to verify that

$$d\alpha := \sum_\nu d\varphi_\nu\cdot dh^\nu + \sum_\nu \varphi_\nu\, d^2h^\nu$$

is well-defined. Assume that for instance $\alpha = \sum_\nu \varphi_\nu\, dh^\nu = 0$. We then have to show that

$$\vartheta := \sum_\nu d\varphi_\nu\cdot dh^\nu + \sum_\nu \varphi_\nu\, d^2h^\nu = 0.$$

To this end, it is sufficient to show that $\vartheta(L) = 0$ for each section $L \in \Gamma(T^2M)$ where we may assume without restrictions that either $L = A$ or $L = A\cdot B$ with $A, B \in \Gamma(TM)$.

(1) If $L = A \in \Gamma(TM)$, then $\vartheta(A) = \sum_\nu (d\varphi_\nu\cdot dh^\nu)\, A + \sum_\nu (\varphi_\nu\, d^2h^\nu)\, A = 0$ where the first term vanishes, since $d\varphi_\nu\cdot dh^\nu|\Gamma(TM) = 0$, while the second term equals $\alpha(A)$ and vanishes since $\alpha = 0$ by assumption.

(2) Let now $L = A\cdot B$ where $A, B \in \Gamma(TM)$: At first we have $\vartheta(AB - BA) = \vartheta\big([A,B]\big) = 0$ by (1), and thus

$$
\begin{aligned}
2\,\vartheta(A\cdot B) &= \vartheta(AB + BA) + \vartheta\big([A,B]\big) \\
&= \sum_\nu (d\varphi_\nu\cdot dh^\nu)\,(A\cdot B + B\cdot A) + \sum_\nu (\varphi_\nu\, d^2h^\nu)\,(A\cdot B + B\cdot A) \\
&= \sum_\nu \Big[(A\varphi_\nu)\,(Bh^\nu) + (B\varphi_\nu)\,(Ah^\nu) + \varphi_\nu\,(A\cdot B)\,(h^\nu) + \varphi_\nu\,(B\cdot A)\,(h^\nu)\Big] \\
&= \sum_\nu \Big[A\big(\varphi_\nu\,(Bh^\nu)\big) + B\big(\varphi_\nu\,(Ah^\nu)\big)\Big] = A\big(\alpha(B)\big) + B\big(\alpha(A)\big) = 0\,,
\end{aligned}
$$

which gives the claim. □

LEMMA 1.9.16. *Let X be a semimartingale taking values in a manifold M.*

(i) *For $\varphi, \psi \in C^\infty(M)$ it holds $\int\langle d\varphi\cdot d\psi, dX\rangle = \frac{1}{2}\,[\varphi(X), \psi(X)]$.*

(ii) *For $\vartheta, \sigma \in \Gamma(T^{*2}M)$ we have $R\vartheta, R\sigma \in \Gamma(T^*M)$, and it holds:*

$$\int\langle R\vartheta\cdot R\sigma, dX\rangle = \frac{1}{2}\,\big[\int\langle\vartheta, dX\rangle, \int\langle\sigma, dX\rangle\big].$$

In particular, if $\vartheta \in \Gamma(T^{*2}M)$ such that $R\vartheta = 0$, then $\int\langle\vartheta, dX\rangle \in \mathscr{A}$.

PROOF. To (i): By $d\varphi\cdot d\psi = \frac{1}{2}\left[d^2(\varphi\psi) - \varphi\, d^2\psi - \psi\, d^2\varphi\right]$ we have

$$\begin{aligned}
&2\textstyle\int\langle d\varphi\cdot d\psi, dX\rangle \\
&\quad = \big[(\varphi\psi)(X) - (\varphi\psi)(X_0)\big] - \textstyle\int \varphi(X)\, d\big(\psi(X)\big) - \int \psi(X)\, d\big(\varphi(X)\big) \\
&\quad = [\varphi(X), \psi(X)].
\end{aligned}$$

To (ii): According to Lemma 1.9.9, ϑ and σ have representations of the form $\vartheta = \sum_\nu \vartheta_\nu\, d^2\varphi^\nu$, respectively $\sigma = \sum_\mu \sigma_\mu\, d^2\psi^\mu$. Hence we have

$$R\vartheta = \sum_\nu \vartheta_\nu\, d\varphi^\nu, \quad R\sigma = \sum_\mu \sigma_\mu\, d\psi^\mu, \quad R\vartheta\cdot R\sigma = \sum_{\nu,\mu} \vartheta_\nu\, \sigma_\mu\, d\varphi^\nu\cdot d\psi^\mu,$$

and by means of (i) we obtain

$$\begin{aligned}
\int\langle R\vartheta\cdot R\sigma, dX\rangle &= \frac{1}{2}\sum_{\nu,\mu}\int \vartheta_\nu(X)\sigma_\mu(X)\, d[\varphi^\nu(X), \psi^\mu(X)] \\
&= \frac{1}{2}\Big[\sum_\nu\int \vartheta_\nu(X)\, d\big(\varphi^\nu(X)\big), \sum_\mu \textstyle\int \sigma_\mu(X)\, d\big(\psi^\mu(X)\big)\Big] \\
&= \frac{1}{2}\Big[\int\langle\vartheta, dX\rangle, \int\langle\sigma, dX\rangle\Big].
\end{aligned}$$

The additional claim follows from part (ii) with $\vartheta = \sigma$. ☐

THEOREM 1.9.17. *Let X be an M-valued semimartingale.*

(i) *For $b \in \Gamma(T^*M \odot T^*M)$ it holds that $\int\langle Hb, dX\rangle = \frac{1}{2}\int b(dX, dX)$.*
(ii) *For $\alpha \in \Gamma(T^*M)$ it holds that $\int\langle d\alpha, dX\rangle = \int_X \alpha$.*

PROOF. It is sufficient to verify the defining properties.

To (i): For $b = d\varphi \odot d\psi$ we have $Hb = d\varphi\cdot d\psi$ and by Lemma 1.9.16 (i) then $\int\langle Hb, dX\rangle = \frac{1}{2}[\varphi(X), \psi(X)]$. On the other hand, we have $H(\varphi\, b) = \varphi\, H(b)$ from where the relation $\int\langle H(\varphi\, b), dX\rangle = \int \varphi(X)\, b(dX, dX)$ follows.

To (ii): If $\alpha = d\varphi$, then $d(d\varphi) = d^2\varphi$ and hence $\int\langle d\alpha, dX\rangle = \varphi(X) - \varphi(X_0)$. On the other hand we have $d(\varphi\alpha) = \varphi\cdot d\alpha + d\varphi\cdot\alpha$, and hence by means of Lemma 1.9.16 (ii), applied to $d\varphi\cdot\alpha = R(d^2\varphi)\cdot R(d\alpha)$,

$$\begin{aligned}
\textstyle\int\langle d(\varphi\alpha), dX\rangle &= \textstyle\int\langle\varphi\cdot d\alpha, dX\rangle + \int\langle d\varphi\cdot\alpha, dX\rangle \\
&= \textstyle\int \varphi(X)\,\langle d\alpha, dX\rangle + \dfrac{1}{2}\left[\textstyle\int\langle d^2\varphi, dX\rangle, \int\langle d\alpha, dX\rangle\right] \\
&= \textstyle\int \varphi(X)\, d\big(\int\langle d\alpha, dX\rangle\big) + \dfrac{1}{2}\left[\varphi(X), \textstyle\int\langle d\alpha, dX\rangle\right] \\
&= \textstyle\int \varphi(X)\circ d\big(\int\langle d\alpha, dX\rangle\big)
\end{aligned}$$

which shows the defining properties and hence the claim. ☐

EXAMPLE 1.9.18. Let X be a M-valued semimartingale of locally bounded variation, in the sense that all compositions $\varphi(X)$ with functions $\varphi \in C^\infty(M)$ lie in $\mathscr{A}$. Then we have

$$\textstyle\int\langle\Theta, dX\rangle = \int\langle dR\Theta, dX\rangle = \int(R\Theta)(\circ dX) \tag{1.9.15}$$

for each continuous adapted $T^{*2}M$-valued process Θ over X. Indeed, both $\int(R\Theta)(\circ dX)$ and $\int\langle dR\Theta, dX\rangle$ have the defining properties for $\int\langle\Theta, dX\rangle$. In particular, we then have

$$\textstyle\int\langle\vartheta, dX\rangle = \int_X R\vartheta, \quad \vartheta \in \Gamma(T^{*2}M),$$

a formula which uncovers why in classical Differential Geometry second order forms do not appear explicitly.

THEOREM 1.9.19. *Let M be a manifold. There is a one-to-one correspondence between the following objects:*

(i) *Torsion-free linear connections ∇ on M.*

(ii) *Bundle homomorphisms $F\colon T^2M \to TM$ with $F \circ \iota = \mathrm{id}$ (where ι denotes the canonical inclusion $TM \hookrightarrow T^2M$), i.e., splittings of the following exact sequence of vector bundles over M*

$$\begin{tikzcd} 0 \arrow[r] & TM \arrow[r, "\iota"] & T^2M \arrow[r] \arrow[l, bend left, dashed, "F"] & TM \odot TM \arrow[r] & 0. \end{tikzcd} \tag{1.9.16}$$

(iii) *Bundle homomorphisms $G\colon T^*M \to T^{*2}M$ with $R \circ G = \mathrm{id}$ (where R denotes the restriction to TM), i.e., splittings of the following exact sequence of vector bundles over M*

$$\begin{tikzcd} 0 \arrow[r] & T^*M \odot T^*M \arrow[r, "H"] & T^{*2}M \arrow[r, "R"] & T^*M \arrow[r] \arrow[l, bend left, dashed, "G"] & 0. \end{tikzcd} \tag{1.9.17}$$

PROOF. Obviously (ii) and (iii) correspond to each other by dualization.

(i) $\to$ (ii): Let ∇ be a torsion-free linear connection on M. Recall that by 1.4.30 torsion-freeness means that for any $\varphi \in C^\infty(M)$, $(A, B) \mapsto \nabla d\varphi(A, B)$ is symmetric. We define $F\colon \Gamma(T^2M) \to \Gamma(TM)$ by

$$(FL)(\varphi) = L\varphi - \langle H\nabla d\varphi, L\rangle, \quad L \in \Gamma(T^2M),\ \varphi \in C^\infty(M), \tag{1.9.18}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing of $T^{*2}M$ and T^2M. Obviously, it holds that

$$FL = \begin{cases} L & \text{for } L = A \in \Gamma(TM),\\ \nabla_A B & \text{for } L = A\cdot B \text{ with } A, B \in \Gamma(TM). \end{cases}$$

Indeed for $L = A \in \Gamma(TM)$ we have $\langle H\nabla d\varphi, A\rangle = 0$; on the other hand for $L = A\cdot B$ we have

$$F(A\cdot B)(\varphi) = (A\cdot B)(\varphi) - \langle H\nabla d\varphi, A\cdot B\rangle = (A\cdot B)(\varphi) - \nabla d\varphi(A, B) = (\nabla_A B)(\varphi),$$

where we used that

$$\langle Hb, A\cdot B\rangle = \frac{1}{2}\big(b(A, B) + b(B, A)\big) = b(A, B)$$

for each $b \in \Gamma(T^*M \odot T^*M)$. This shows in particular that $F\colon \Gamma(T^2M) \to \Gamma(TM)$ is by Eq. (1.9.18) well-defined and that $F|\,\Gamma(TM) = \mathrm{id}$; moreover F is $C^\infty(M)$-linear and hence defines a bundle homomorphism $F\colon T^2M \to TM$ with the wanted properties.

(ii) $\to$ (i): Now let be given a bundle homomorphism $F\colon T^2M \to TM$ such that $F|\,TM = \mathrm{id}$, and let $F\colon \Gamma(T^2M) \to \Gamma(TM)$ be the induced mapping at the level of sections. Inversely to (1.9.18), F induces a linear connection ∇ on M, namely as

$$\nabla_A B := F(A\cdot B), \quad A, B \in \Gamma(TM). \tag{1.9.19}$$

∇ is obviously $C^\infty(M)$-linear in A and derivative in B, since $\nabla_A(\varphi B) = F\big(A\cdot(\varphi B)\big) = F\big(\varphi\, A\cdot B + A(\varphi)\, B\big) = \varphi\, F(A\cdot B) + A(\varphi)\, F(B) = \varphi\, \nabla_A B + A(\varphi)\, B$. Moreover, we observe that

$$\nabla_A B - \nabla_B A = F(AB) - F(BA) = F(AB - BA) = F([A, B]) = [A, B],$$

which shows that ∇ is torsion-free. $\square$

By symmetrization, the $C^\infty(M)$-linear map

$$H\colon \Gamma(T^*M \odot T^*M) \to \Gamma(T^{*2}M), \quad H(\alpha \odot \beta) = \alpha{\cdot}\beta,$$

can be extended to a $C^\infty(M)$-linear mapping

$$H\colon \Gamma(T^*M \otimes T^*M) \to \Gamma(T^{*2}M), \quad H(\alpha \otimes \beta) := H(\alpha \odot \beta) = \alpha{\cdot}\beta.$$

REMARK 1.9.20. Explicitly, the bundle homomorphism $G\colon T^*M \to T^{*2}M$, induced from a torsion-free linear connection ∇ on M by Theorem 1.9.19 (iii), is given by

$$G\alpha = d\alpha - H\nabla\alpha, \quad \alpha \in \Gamma(T^*M). \tag{1.9.20}$$

PROOF. By construction, G is determined by

$$\langle G\alpha, L\rangle = \langle \alpha, FL\rangle := \alpha(FL), \quad \alpha \in \Gamma(T^*M),\ L \in \Gamma(T^2M),$$

with FL being defined by Eq. (1.9.18). Since by $\nabla(\varphi\alpha) = d\varphi \otimes \alpha + \varphi\,\nabla\alpha$ the right-hand side of (1.9.20) is $C^\infty(M)$-linear in α, it is sufficient to show Eq. (1.9.20) for $\alpha = d\varphi$ with $\varphi \in C^\infty(M)$. We have however

$$\langle Gd\varphi, L\rangle = \langle d\varphi, FL\rangle = (FL)(\varphi) = \langle d^2\varphi - H\nabla d\varphi, L\rangle, \quad L \in \Gamma(T^2M),$$

from where the relation $Gd\varphi = d^2\varphi - H\nabla d\varphi$ follows. □

REMARK 1.9.21. Theorem 1.9.19 shows explicitly that torsion-free linear connections on M are exactly the required extra structure to split differential operators $L \in \Gamma(T^2M)$ of second order canonically in a first order part (the *drift* of L), namely FL, and a part of purely second order, namely $L - FL = L - \iota FL$. We call a PDO $L \in \Gamma(T^2M)$ *of purely second order* if $FL \equiv 0$. Writing L in a chart (h,U) as $L|U = \sum_i b^i \frac{\partial}{\partial h^i} + \sum_{i,j} a^{ij} \frac{\partial}{\partial h^i}\frac{\partial}{\partial h^j}$, then

$$(FL)|U = \sum_k \Big(b^k + \sum_{i,j} a^{ij}\,\Gamma^k_{ij}\Big)\,\tfrac{\partial}{\partial h^k}$$

is the corresponding first order part.

THEOREM 1.9.22. *Let M be a manifold, ∇ a torsion-free linear connection on M and γ a differentiable curve taking values in M. The following conditions are equivalent:*

(i) *γ is a geodesic curve.*
(ii) *$\ddot\gamma \equiv \gamma_*\big(\frac{d^2}{dt^2}\big)$ is purely second order, i.e., $F(\ddot\gamma) \equiv 0$.*

PROOF. Letting (h,U) be a chart for M, then we have for $t \in \mathbb{R}$ such that $\gamma(t) \in U$

$$\begin{aligned}
\dot\gamma(t) &= \sum_i \dot\gamma^i(t)\,\big(\tfrac{\partial}{\partial h^i}\big)_{\gamma(t)},\\
\ddot\gamma(t) &= \sum_i \ddot\gamma^i(t)\,\big(\tfrac{\partial}{\partial h^i}\big)_{\gamma(t)} + \sum_{i,j} \dot\gamma^i(t)\,\dot\gamma^j(t)\,\big(\tfrac{\partial}{\partial h^i}\big)_{\gamma(t)}\,\big(\tfrac{\partial}{\partial h^j}\big)_{\gamma(t)},
\end{aligned}$$

and consequently

$$F\big(\ddot\gamma(t)\big) = \sum_k \bigg(\ddot\gamma^k(t) + \sum_{i,j} \Gamma^k_{ij}\big(\gamma(t)\big)\ \dot\gamma^i(t)\,\dot\gamma^j(t)\bigg)\big(\tfrac{\partial}{\partial h^k}\big)_{\gamma(t)},$$

from where we read off the claim with Eq. (1.4.10). □

THEOREM 1.9.23. *Let M be a manifold and let $L\colon C^\infty(M)\to C^\infty(M)$ be a second order* PDO *without constant term, i.e. $L\in\Gamma(T^2M)$ where $(L\varphi)(x)=L_x(\varphi)$. Furthermore, let X be an M-valued semimartingale which solves the martingale problem for L, i.e.,*

$$d\big(\varphi(X)\big)-(L\varphi)(X)\,dt \overset{\mathrm{m}}{=} 0 \quad \textit{for each } \varphi\in C^\infty(M). \tag{1.9.21}$$

For a torsion-free linear connection ∇ on M, the following two conditions are equivalent:

(i) *X is a ∇-martingale on M*
(ii) *L is of purely second order along X, i.e., $(FL)(X)\equiv 0$ $\mathbb{P}$-almost surely.*

PROOF. We want to check first of all that the property (1.9.21) implies

$$\langle\vartheta,dX\rangle \overset{\mathrm{m}}{=} \langle\vartheta(X),L(X)\rangle\,dt,\quad \vartheta\in\Gamma(T^{*2}M). \tag{1.9.22}$$

Indeed, any $\vartheta\in\Gamma(T^{*2}M)$ has by Lemma 1.9.9 a representation of the form $\vartheta=\sum_\nu\vartheta_\nu\,d^2\varphi^\nu$ where $\vartheta_\nu\in C^\infty(M)$, and hence

$$\langle\vartheta,dX\rangle=\sum_\nu(\vartheta_\nu\circ X)\,d\big(\varphi^\nu(X)\big) \overset{\mathrm{m}}{=} \sum_\nu\vartheta_\nu(X)\,L\varphi^\nu(X)\,dt=\langle\vartheta,L\rangle(X)\,dt.$$

Recall that by definition X is a ∇-martingale if

$$d\big(\varphi(X)\big) \overset{\mathrm{m}}{=} \frac{1}{2}\,(\nabla d\varphi)(dX,dX),\quad \varphi\in C^\infty(M).$$

By Theorem 1.9.17 (i) we have $\nabla d\varphi(dX,dX)=2\,\langle H\nabla d\varphi,dX\rangle$, and from relation (1.9.22) we get $\langle H\nabla d\varphi,dX\rangle \overset{\mathrm{m}}{=} \langle H\nabla d\varphi,L(X)\rangle\,dt$, so that X is ∇-martingale if and only if

$$d\big(\varphi(X)\big)-\langle(H\nabla d\varphi)(X),L(X)\rangle\,dt \overset{\mathrm{m}}{=} 0,\quad \varphi\in C^\infty(M).$$

Using $d\big(\varphi(X)\big) \overset{\mathrm{m}}{=} (L\varphi)(X)\,dt$ we conclude that X is a ∇-martingale if and only if for each $\varphi\in C^\infty(M)$:

$$\big(L\varphi-\langle H\nabla d\varphi,L\rangle\big)(X)\,dt=0\,,$$

which is because of $L\varphi-\langle H\nabla d\varphi,L\rangle=FL$ just the claim. □

We finally want use relation (1.6.34) to define the Itô integral of one-forms along semimartingales in a more general context.

DEFINITION 1.9.24 (Itô integral along semimartingales). Let M be a manifold, ∇ a torsion-free linear connection on M and X a semimartingale taking values in M. For a T^*M-valued process J over X, we call

$$\int\langle J,FdX\rangle:=\int\langle GJ,dX\rangle$$

the *Itô integral of J along X*. If in particular $J=\alpha(X)$ where $\alpha\in\Gamma(T^*M)$, then $\int\langle\alpha(X),FdX\rangle$ is also called *Itô integral of α along X* and the following notations are used for it:

$$(\nabla)\!\int_X\alpha=\int\langle\alpha,F\,dX\rangle=\int\langle G\alpha,dX\rangle.$$

REMARK 1.9.25. By (1.9.20) we have $G\alpha=d\alpha-H\nabla\alpha$ and $J\mapsto I_J:=\int\langle GJ,dX\rangle$ is hence determined by the following properties:

(i) $I_{d\varphi(X)}=\varphi(X)-\varphi(X_0)-\frac12\int\nabla d\varphi(dX,dX)$ for $\varphi\in C^\infty(M)$;
(ii) $I_{KJ}=\int K\,dI_J$ for each continuous adapted $\mathbb{R}$-valued process K.

In particular, for each differential form $\alpha \in \Gamma(T^*M)$ the following relation between the Itô Integral and Stratonovich integral of α along X holds:

$$(\nabla)\int_X \alpha = \int_X \alpha - \frac{1}{2}\int \nabla\alpha(dX, dX). \tag{1.9.23}$$

THEOREM 1.9.26. *Let M be a manifold, ∇ a torsion-free linear connection on M and X a M-valued semimartingale. The following statements are equivalent:*

(i) *X is a ∇-martingale;*
(ii) *$(\nabla)\int_X \alpha$ is a local martingale for any differential form $\alpha \in \Gamma(T^*M)$;*
(iii) *$\int\langle J, F\, dX\rangle$ is a local martingale for any continuous adapted T^*M-valued process J above X.*

PROOF. For $\alpha = d\varphi$, respectively $J = \alpha(X)$, the assertions reduce to the definition of ∇-martingales. The general case follows with Lemma 1.9.9. □

The following Remark finally justifies the notion Itô integral, respectively Stratonovich integral of α along X.

THEOREM 1.9.27. *Let M be a manifold, ∇ a torsion-free linear connection on M and X an M-valued semimartingale. Furthermore let U be a horizontal lift of X to $\mathrm{L}(TM)$ and $Z = \int_U \vartheta$ the anti-development of X in $\mathbb{R}^n$. Then for the Itô integral, respectively Stratonovich integral of a differential form $\alpha \in \Gamma(T^*M)$ along X the following formulas hold:*

$$\int_X \alpha = \sum_i \int \alpha(X)\, Ue_i \circ dZ^i, \quad \textit{as well as} \quad (\nabla)\int_X \alpha = \sum_i \int \alpha(X)\, Ue_i\, dZ^i.$$

PROOF. The first formula is already shown in Theorem 1.6.30 (ii); the second one reduces for $\alpha = d\varphi$ with Eq. (1.9.23) to the geometric Itô formula (1.6.32); the general case follows again with Lemma 1.9.9. □

CHAPTER 2

Geometry of Brownian Motion

In this Chapter we focus on stochastic tools in Riemannian Geometry. We start by studying some questions concerning the geometry of Riemannian manifolds in connection with the long-term behaviour of Brownian motion. In particular, using a few selected problems, we want to illustrate the basic idea of stochastic Riemannian geometry, namely to relate differential geometric problems to stochastic questions and to deal with them using stochastic methods (see for instance, [**32**] and [**23, 24**]).

2.1. The Curvature Tensor and Jacobi Fields

The notion of curvature of a Riemannian manifold is one of the key concepts to control the asymptotic behaviour of Brownian motions for large times. Brownian motion is a sensitive instrument to measure curvature. Negative curvature amplifies the tendency of Brownian motion to exit compact sets and to drift off to ∞ if the topology of the manifold permits. Strongly divergent negative curvature, for instance, can have the effect that even on metrically complete manifolds BM(M,g) explodes in finite times.

Brownian motion on Riemannian manifolds can have other asymptotic properties not known from Euclidean Brownian motion. For instance, trajectories of BM(M,g) on certain negatively curved simply connected manifolds M, when considered in polar coordinates from some fixed point, stay with high probability in the entered angular sector and have an asymptotic direction.

Before discussing the concept of curvature we want to consider Riemannian manifolds under the aspect of metric spaces. Recall that the $d(x,y)$ of two points x and y is given by

$$d(x,y) := \inf\bigl\{L(\alpha) \bigm| \alpha\colon [0,1] \to M \text{ piecewise } C^\infty \text{ with } \alpha(0)=x \text{ and } \alpha(1)=y\bigr\}.$$

One of the most elementary questions about asymptotics of Brownian motions is the distance behaviour of BM(M,g) with respect to a given point $x \in M$, i.e. properties of "radial process" $d(x,X_t)$, $t \geq 0$.

EXAMPLE 2.1.1 (Bessel process). Let $M = \mathbb{R}^n$ endowed with the Euclidean metric. The function $r(x) := d(0,x) = |x|$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$ with

$$(\operatorname{grad} r)(x) = \frac{x}{|x|} \quad \text{and} \quad \Delta r = \frac{n-1}{r}.$$

Consequently, if X is a BM$(\mathbb{R}^n)$ such that $X_0 \neq 0$ a.s. ($n \geq 2$), then by Itô's formula

$$d(r(X)) = dN + dA = \sum_{i=1}^n \frac{X^i}{|X|}\, dX^i + \frac{1}{2}\,\frac{n-1}{r(X)}\, dt.$$

As $d[N,N] = \sum_{i,j} X^i X^j \, |X|^{-2} \, dX^i dX^j = \sum_{i,j} X^i X^j \, |X|^{-2} \, \delta_{ij} \, dt = dt$, the process N is a one-dimensional Brownian motion W, and one gets

$$r(X) = r(X_0) + W + \frac{1}{2}\int_0^t \frac{n-1}{r(X)}\, ds \quad \text{with } W \text{ a BM}(\mathbb{R}).$$

In other words, $r(X)$ is solution of an SDE of the type

$$dR = dW + \frac{n-1}{2}\, R^{-1}\, dt. \tag{2.1.1}$$

Any such process is called *n-dimensional Bessel process* or *Bessel process of index $n/2-1$*.

Example 2.1.1 rises the question to what extent in general for Brownian motions X on (M,g) and $x \in M$, radial processes of the form $r(X) := d(x, X)$ are semimartingales which can be described by Itô's formula. To this end, first questions concerning differentiability of the distance function $r(\cdot) := d(x, \cdot)$ on M need to be clarified.

DEFINITION 2.1.2 (Variation of a curve). Let (M,g) be a Riemannian manifold and $\gamma\colon [a,b\,] \to M$ a non-constant differentiable curve parametrized proportionally to arc length, i.e. $0 < \ell := |\dot\gamma| = \text{const}$. A (*free*) *variation* of γ is a differentiable map

$$\alpha\colon [a,b\,] \times \,]{-\varepsilon}, \varepsilon[\, \to M$$

such that $\alpha(\cdot, 0) = \gamma$.

In terms of the canonical vector fields $\frac{\partial}{\partial t} = D_1$ and $\frac{\partial}{\partial s} = D_2$ on $[a,b\,] \times \,]{-\varepsilon}, \varepsilon[$, we consider to a variation α of γ the vector fields along α:

$$T := \alpha_* D_1 \equiv \tfrac{\partial}{\partial t}\alpha \in \Gamma(\alpha^* TM) \quad \text{and} \quad V := \alpha_* D_2 \equiv \tfrac{\partial}{\partial s}\alpha \in \Gamma(\alpha^* TM).$$

Let $\gamma_s := \alpha(\cdot, s)$ for $-\varepsilon < s < \varepsilon$; in particular $\gamma = \gamma_0$. Furthermore denote by $Y := V(\cdot\,, 0) \in \Gamma(\gamma^* TM)$ the "variational field" of α and by $\dot\gamma = T(\cdot\,, 0) \in \Gamma(\gamma^* TM)$ the tangential vector field along γ.

THEOREM 2.1.3 (First variation of arc length). *Let (M,g) be a Riemannian manifold with the Levi-Civita connection and let $\alpha\colon [a,b\,] \times \,]{-\varepsilon}, \varepsilon[\, \to M$ be the differentiable variation of a smooth curve $\gamma\colon [a,b] \to M$ such that $\ell = |\dot\gamma| = \text{const} > 0$. Let $Y = (\alpha_* D_2)(\cdot\,, 0) \in \Gamma(\gamma^* TM)$. Then the lengths $L(\gamma_s) = \int_a^b |\dot\gamma_s(t)|\, dt$ of the curves $\gamma_s = \alpha(\cdot\,, s)\colon [a,b\,] \to M$ satisfy the "variational formula":*

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{1}{\ell}\Big\{ \langle Y, \dot\gamma\rangle \,\big|_{t=a}^{t=b} - \int_a^b \langle Y, \nabla_D \dot\gamma\rangle \, dt \Big\}. \tag{2.1.2}$$

PROOF. By Theorem 1.5.6 (iii) (on the characterization of Riemannian connections) we have $\frac{\partial}{\partial s}\langle T,T\rangle \equiv D_2\langle T,T\rangle = 2\,\langle \nabla_{D_2} T, T\rangle$, and thus

$$\frac{d}{ds} L(\gamma_s) = \frac{d}{ds}\int_a^b \langle T,T\rangle^{1/2}\, dt = \int_a^b \frac{1}{2}\langle T,T\rangle^{-1/2}\, \tfrac{\partial}{\partial s}\langle T,T\rangle\, dt \tag{2.1.3}$$

$$= \int_a^b \tfrac{1}{|T|}\, \langle \nabla_{D_2} T, T\rangle\, dt = \int_a^b \tfrac{1}{|T|}\, \langle \nabla_{D_1} V, T\rangle\, dt \tag{2.1.4}$$

$$= \int_a^b \tfrac{1}{|T|}\, \big[D_1\langle V, T\rangle - \langle V, \nabla_{D_1} T\rangle \big]\, dt. \tag{2.1.5}$$

Here the second to the last equality is a consequence of the first structural equation of Cartan (Theorem 1.4.27): $\nabla_{D_1}(\alpha_* D_2) - \nabla_{D_2}(\alpha_* D_1) = \alpha_*[D_1, D_2] = 0$, whereas the last equality comes from $D_1\langle V,T\rangle = \langle \nabla_{D_1}V,T\rangle + \langle V,\nabla_{D_1}T\rangle$ which is a consequence of Theorem 1.5.6 (iii).

For $s=0$ this gives the wanted equation

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{1}{\ell}\Big\{\langle V,\dot\gamma\rangle\,(t,0)\,\big|_{t=a}^{t=b} - \int_a^b \langle V,\nabla_D\dot\gamma\rangle\,(t,0)\,dt\Big\}. \tag{2.1.6}$$ □

In the particular case of a variation $(\gamma_s)_{-\varepsilon<s<\varepsilon}$ of γ with fixed initial and end point, i.e. $\gamma_s(a)=\gamma(a)$ and $\gamma_s(b)=\gamma(b)$ for all s, we get $V_{(a,\,\cdot\,)}=0$ and $V_{(b,\,\cdot\,)}=0$, which combined with Eq. (2.1.2) gives the following characterization of geodesic curves:

COROLLARY 2.1.4 (Geodesics as critical points of the length functional). *Let (M,g) be a Riemannian manifold and $\gamma\colon [a,b]\to M$ a differentiable curve. Then γ is a geodesic curve, i.e. $\nabla_D\dot\gamma=0$, if and only if $\frac{d}{ds}\big|_{s=0} L(\gamma_s)=0$ for all variations (γ_s) of γ with fixed initial and end point.*

We discuss first some local properties of geodesic curves. On a Riemannian manifold (M,g), to each $x\in M$ and $v\in T_xM$ there exists exactly one geodesic curve γ_v with $\gamma_v(0)=x$ and $\dot\gamma_v(0)=v$; since $|\dot\gamma_v|\equiv|v|$ the geodesic γ_v is a *normal* geodesic in the sense that $|\dot\gamma_v|\equiv 1$, if and only if $|v|=1$.

We consider $\mathscr{O}(M) := \{v\in TM : \gamma_v \text{ is defined for } t=1\}$ and the exponential map of (M,g)

$$\exp\colon \mathscr{O}(M)\to M\times M,\quad v\mapsto \big(\pi(v),\gamma_v(1)\big).$$

As a consequence of the theory of ordinary differential equations, $\mathscr{O}(M)$ is open in TM, and contains obviously the zero section in TM. By Definition 1.8.10 the Riemannian manifold (M,g) is metrically complete if and only if $\mathscr{O}(M)=TM$. As already observed (see the proof to Lemma 1.7.22), the differential of

$$\exp_x = \exp|\big(T_xM\cap\mathscr{O}(M)\big)\colon T_xM\cap\mathscr{O}(M)\to\{x\}\times M\equiv M$$

at the zero element $0_x\in T_xM$ is given by the identity. In general, along the zero section of a vector bundle E over M, the tangent spaces $T_{0_x}E$ decompose canonically as $T_{0_x}E = T_xM\oplus E_x$ with E_x the part in fiber direction, and $(d\exp)_{0_x}$ read as map $(d\exp)_{0_x}\colon T_xM\oplus T_xM\to T_xM\oplus T_xM$ given by the matrix

$$\begin{pmatrix}\mathrm{id} & 0\\ \mathrm{id} & \mathrm{id}\end{pmatrix}.$$

Hence $d\exp$ has full rank at the zero section and by the inverse function theorem $\exp$ maps an open neighbourhood of the zero section in TM locally diffeomorphically to an open neighbourhood of the diagonal in $M\times M$. For $x\in M$ let $V_\varepsilon(0) := \{v\in T_xM : |v|<\varepsilon\}$, then

$$\varrho(x) := \sup\big\{\varepsilon>0 : \exp_x|V_\varepsilon(0) \text{ is an embedding}\big\}\in\,]0,\infty]$$

is called *injectivity radius at* x, and hence for $0<\varepsilon\le\varrho(x)$ there are diffeomorphisms

$$\exp_x|V_\varepsilon(0)\colon V_\varepsilon(0)\xrightarrow{\ \sim\ }\exp_x\big(V_\varepsilon(0)\big) =: B_\varepsilon(x).$$

Note that the map $\varrho\colon M\to\overline{\mathbb{R}}$ is lower semi-continuous: $\{\varrho>c\}$ is open in M for any $c\ge 0$.

For $\varepsilon<\varrho(x)$ we may consider besides the normal coordinates $\big(h,B_\varepsilon(x)\big)$ at x where $h=\big(\exp_x|V_\varepsilon(0)\big)^{-1}$ the so-called "geodesic polar coordinates" with center x, which is the diffeomorphism

$$B_\varepsilon(x)\backslash\{x\} \xrightarrow{\ \sim\ } V_\varepsilon(0)\backslash\{0\} \xrightarrow{\ \sim\ }]0,\varepsilon[\, \times S^{n-1} \qquad x \longmapsto \big(|x|, \tfrac{x}{|x|}\big). \tag{2.1.7}$$

The inverse map to (2.1.7) is given by $\phi\colon (r,v)\mapsto \exp_x(r\,v)=\gamma_v(r)$ Note that through

$$S_r(x) := \big\{\exp_x(v) : v\in T_xM,\ |v|=r\big\} \equiv \phi\big(\{r\}\times S^{n-1}\big), \quad 0<r<\varrho(x),$$

then hypersurfaces in M (one-codimensional submanifolds) are given.

THEOREM 2.1.5 (Gauss Lemma). *Let (M,g) be a Riemannian manifold equipped with the Levi-Civita connection, $x\in M$ and $v\in T_xM\cap\mathscr{O}(M)$ such that $\exp_x v$ is defined. For any $w\in T_xM\cong T_v(T_xM)$ then*

$$\big\langle (d\exp_x)_v v, (d\exp_x)_v w\big\rangle = \langle v,w\rangle. \tag{2.1.8}$$

In particular, the geodesics through the point x are perpendicular on the hypersurfaces $S_r(x)$ for $0<r<\varrho(x)$.

PROOF. Decomposing $w=w'+w^\perp$ such that w' is parallel and $w^\perp$ orthogonal to v, the formula

$$\big\langle (d\exp_x)_v v, (d\exp_x)_v w'\big\rangle = \langle v,w'\rangle$$

is immediate from the Definition of $\exp_x$. By means of the linearity of $d\exp_x$, to verify (2.1.8) it is thus sufficient to consider the case $\langle v,w\rangle=0$.

We show the following: If $c\colon\,]-\varepsilon,\varepsilon[\, \to T_xM$ is a curve $T_xM\cap\mathscr{O}(M)$ such that $|c(s)|=$ const, $c(0)=v$ and $\dot c(0)=w$, then for any $0<t_0\le 1$ it holds that

$$\frac{d}{ds}\Big|_{s=0}\exp_x\big(t_0\,c(s)\big) \perp \frac{d}{dt}\Big|_{t=t_0}\exp_x\big(t\,c(0)\big).$$

For $t_0=1$ this shows $(\exp_x)_*w\perp\dot\gamma_v(1)=(\exp_x)_*v$ as claimed.

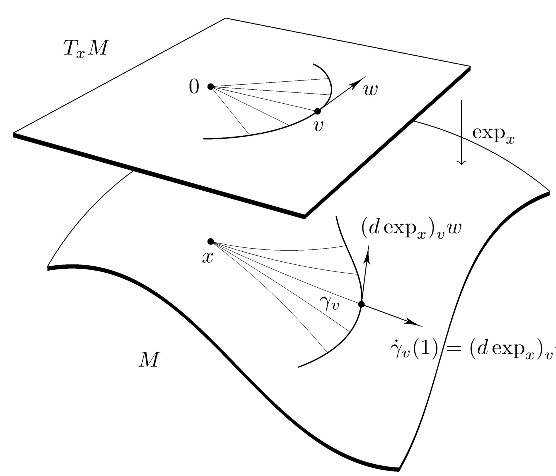


Figure 2.1.1. Exponential function

Denting $\alpha_s(t)=\exp_x\big(t\,c(s)\big)$ for $0\le t\le 1$, then on one hand $L\big(\alpha_s|[0,t_0]\big)$ is independent of s and by means of formula (2.1.2) (first variation of length) we have

$$0=\frac{d}{ds}\Big|_{s=0}L\big(\alpha_s|[0,t_0]\big) = \Big\langle \frac{d}{ds}\Big|_{s=0}\alpha_s(t), \dot\alpha_0(t)\Big\rangle\Big|_{t=0}^{t=t_0}$$

$$= \Big\langle \frac{d}{ds}\Big|_{s=0} \exp_x\big(t_0\, c(s)\big), \dot\gamma_v(t_0)\Big\rangle.$$

The second part of the claim is obvious: If $s \mapsto c(s)$ is a differentiable curve in $S^{n-1} \subset T_xM$ and $\beta(s) := \exp_x\big(r\, c(s)\big)$ die corresponding curve in $S_r(x)$, then as above $\dot\beta(s) \perp \dot\gamma_{c(s)}(r)$. □

THEOREM 2.1.6. *Let (M, g) be a Riemannian manifold and $x \in M$. Furthermore let $V_\varepsilon(0) \subset T_xM$ be an open ε-ball such that $\exp_x |V_\varepsilon(0)$ is an embedding. Then for any $v \in V_\varepsilon(0)$ the geodesic curve*

$$\gamma_v\colon [0,1] \to M, \quad t \mapsto \exp_x(t\, v), \tag{2.1.9}$$

has length $L(\gamma_v) = |v| = d(x, \exp_x v)$, and is modulo parametrization the only curve of length $d(x, y)$ connecting x and $y := \exp_x v$. In addition,

$$\exp_x\big(V_\varepsilon(0)\big) = \big\{p \in M : d(x,p) < \varepsilon\big\}.$$

NOTATION 2.1.7. We call $B_r(x) = \{p \in M : d(x,p) < r\}$ *geodesic ball* about x of radius r. For $\varepsilon < \varrho(x)$ we then have $B_\varepsilon(x) = \exp_x\big(V_\varepsilon(0)\big)$ and the hypersurface

$$S_\varepsilon(x) = \big\{p \in M : d(x,p) = \varepsilon\big\}$$

is called *geodesic sphere* about x of radius ε. The geodesics in $B_\varepsilon(x)$ emanating from the center x are called *radial geodesics*; by the Gauss Lemma they pass orthogonally through geodesic spheres about x.

PROOF. (of Theorem 2.1.6): We use "geodesic polar coordinates" centered at x on $B_\varepsilon(x)\backslash\{x\} = \exp_x\big(V_\varepsilon(0)\backslash\{0\}\big)$ and identify

$$\phi\colon\,]0,\varepsilon[\, \times S^{n-1} \xrightarrow{\ \sim\ } B_\varepsilon(x)\backslash\{x\}, \quad (r,\vartheta) \mapsto \exp_x(r\,\vartheta).$$

By the Gauss Lemma we then have

$$\phi^*\big(g|\,(B_\varepsilon(x)\backslash\{x\})\big) = dr \otimes dr + h_r,$$

where h_r denotes the Riemannian metric on S^{n-1} defined by pullback under ϕ from the Riemannian metric on the geodesic r-sphere $S_r(x)$ induced by g.

(1) We show first that every piecewise differentiable curve $c\colon [0,1] \to M$ starting at x which exits $B_\varepsilon(x) = \exp_x\big(V_\varepsilon(0)\big)$, has length $\geq \varepsilon$ hat.

To this end denote by $t_1 \in\,]0,1]$ the first time such that $c(t_1) \in \partial B_\varepsilon(x) = S_\varepsilon(x)$. Then $c|]0,t_1]$ has a unique representation of the form

$$c(t) = \exp_x\big(r(t)\,\vartheta(t)\big) \equiv \phi\big(r(t),\vartheta(t)\big), \quad 0 < t \leq t_1,$$

with piecewise differentiable curves $t \mapsto \vartheta(t)$ in $S^{n-1} \subset T_xM$ and $t \mapsto r(t)$ in $]0,\infty[$ (without restriction we may assume that $c(t_0) \neq x$ for $t_0 \in\,]0,t_1]$; otherwise we neglect the interval $[0,t_0[$ which only decreases the length of c. Then we have (up to a finite number of points)

$$|\dot c(t)|^2 = |\dot r(t)|^2 + h_{r(t)}\big(\dot\vartheta(t),\dot\vartheta(t)\big),$$

and we may estimate

$$L(c) \geq \int_0^{t_1} |\dot c(t)|\,dt = \int_0^{t_1} \Big[|\dot r(t)|^2 + h_{r(t)}\big(\dot\vartheta(t),\dot\vartheta(t)\big)\Big]^{1/2} dt \geq \int_0^{t_1} |\dot r(t)|\,dt \geq \varepsilon.$$

(2) Let now $\gamma = \gamma_v$ as in 2.1.9 the geodesic from x to $y := \exp_x v$ and let $c\colon [0,1] \to M$ be any piecewise differentiable curve connecting x and y. Then $L(c) \geq L(\gamma)$ with equality if and only if $c = \gamma$ modulo parametrization. Indeed by (1) we assume that c stays

entirely in $B_\varepsilon(x)$ and hence takes the form $c(t) = \exp_x\bigl(r(t)\cdot\vartheta(t)\bigr)$ as in (1). This implies again

$$L(c) = \int_0^1 \Bigl[|\dot r(t)|^2 + h_{r(t)}\bigl(\dot\vartheta(t), \dot\vartheta(t)\bigr)\Bigr]^{1/2} dt \geq \int_0^1 |\dot r(t)|\, dt \geq r(1) - r(0) = L(\gamma),$$

with equality if and only if $\dot\vartheta(t) \equiv 0$ and $t \mapsto r(t)$ isotone. □

Geodesic curves, which realize the distance between two points are called *minimal geodesics*. Theorem 2.1.6 shows in particular that any curve which realizes the distance between its end-points, is (after reparametrization) a minimal geodesic.

COROLLARY 2.1.8 (Geodesics as locally shortest curves). *Let (M, g) be a Riemannian manifold, $I \subset \mathbb{R}$ an open interval, and let $c\colon I \to M$ be a differentiable curve in M parametrized proportional to arc length. The curve c is a geodesic if and only if for each $t \in I$ there exists $\varepsilon > 0$ such that $d\bigl(c(t), c(t+\varepsilon)\bigr) = L\bigl(c|[t, t+\varepsilon]\bigr)$.*

Altogether, we can already give the following partial answer to the mentioned question concerning differentiability of the distance function $d(x, \cdot)$: If $V_\varepsilon(0) \subset T_xM$ is an open ε-ball with $\varepsilon \leq \varrho(x)$, then

$$d(x, \cdot)|B_\varepsilon(x) = |\ \ | \circ \bigl(\exp_x |V_\varepsilon(0)\bigr)^{-1}$$

is differentiable on the punctured geodesic ball $B_\varepsilon(x) \setminus \{x\}$ about x of radius ε. The question, how large ε can be chosen, requires hence information about the injectivity radius $\varrho(x)$ at x.

Before turning to such questions we note some facts about the metric structure of Riemannian manifolds. It is easy to verify that the distance function $d\colon M \times M \to \mathbb{R}_+$ defines indeed a metric on M and that the topology of M coincides with the metric topology of (M, d).

THEOREM 2.1.9 (Hopf-Rinow). *For a connected Riemannian manifold (M, g) the following conditions are equivalent:*

(i) (M, d) *is a complete metric space* (*i.e., every Cauchy sequence in M is convergent*).
(ii) (M, g) *is metrically complete* (*i.e., the domain of any geodesic can be extended to all of* $\mathbb{R}$).
(iii) *The exponential function* $\exp_x$ *is defined on all of* T_xM *for at least one* $x \in M$.

All three conditions imply that any two points in M can be connected by a minimal geodesic.

PROOF. (i) $\Rightarrow$ (ii): Otherwise there is a maximal geodesic $\gamma\colon\,]a, b[\, \to M$ such that $b < \infty$; without restrictions let $|\dot\gamma| \equiv 1$. We choose a monotonic sequence $(t_n)_{n\in\mathbb{N}}$ in $]a, b[$ such that $t_n \to b$. Since $d\bigl(\gamma(t_m), \gamma(t_n)\bigr) \leq |t_m - t_n|$ the sequence $\bigl(\gamma(t_n)\bigr)_{n\in\mathbb{N}}$ is then Cauchy, so that $x := \lim_{n\to\infty} \gamma(t_n)$ exists by assumption.

About the point x we choose a geodesic ball $B_\varepsilon(x)$ of radius $\varepsilon > 0$ such that the injectivity radius ϱ satisfies $\varrho|B_\varepsilon(x) > 2\varepsilon$. For large n we then have $\gamma(t_n) \in B_\varepsilon(x)$, and $\gamma|[t_n, t_{n+k}]$ is the minimal geodesic connecting $\gamma(t_n)$ and $\gamma(t_{n+k})$, in particular then

$$d\bigl(\gamma(t_{n+k}), \gamma(t_n)\bigr) = t_{n+k} - t_n,$$

and with $k \to \infty$ then $d\bigl(x, \gamma(t_n)\bigr) = b - t_n$. Hence $\gamma|[t_n, t_{n+k}]$ is a curve with length $t_{n+k} - t_n$, which starts on the geodesic sphere $S_{b-t_n}(x)$ of radius $b - t_n$ and ends on the geodesic sphere $S_{b-t_{n+k}}(x)$ of Radius $b - t_{n+k}$. By Theorem 2.1.6, $\gamma|[t_n, t_{n+k}]$ lies on a radial geodesic starting at x, and it follows that

$$\gamma(t) = \exp_x\big((b-t)\,v\big), \quad b - \varepsilon < t < b,$$

for some $v \in T_xM$ with $|v| = 1$. This shows that γ can be extended beyond b via

$$\gamma(t) = \exp_x\big((b-t)\,v\big), \quad b \le t < b + \varepsilon,$$

in contradiction to the maximality of γ.

(ii) $\Rightarrow$ claim of the addition: Let $x, y \in M$ such that $d(x,y) = r > 0$; we want to show that x and y can be joined by a geodesic of length r. We fix a geodesic ball $B_\varepsilon(x)$ about x of radius $\varepsilon < \varrho(x)$ and $y \notin B_\varepsilon(x)$. As a consequence of the compactness of $S_\varepsilon(x)$, there exists $x_0 \in S_\varepsilon(x)$ of minimal distance to y; we get $x_0 = \exp_x(\varepsilon\, v)$ for some $v \in T_xM$ with $|v| = 1$.

Consider the geodesic curve $\gamma(t) = \exp_x(t\,v)$, by assumption defined for $t \in \mathbb{R}_+$. We want to show that $y = \exp_x(rv)$. Since $r = d(x,y) \le d\big(x, \gamma(t)\big) + d\big(\gamma(t), y\big) \le t + d\big(\gamma(t), y\big)$ we have $d\big(\gamma(t), y\big) \ge r - t$. We show that for any t with $\varepsilon \le t \le r$ even

$$d\big(\gamma(t), y\big) = r - t, \tag{2.1.10}$$

holds which then gives the claim for $t = r$. First we verify (2.1.10) for $t = \varepsilon$: indeed $d(x_0, y) = r - \varepsilon$ since

$$r = d(x,y) = \min_{x' \in S_\varepsilon(x)} \big(d(x,x') + d(x',y)\big) = \varepsilon + d(x_0, y).$$

If (2.1.10) holds for $t \in [\varepsilon, r]$, then also for all t' with $\varepsilon \le t' \le t$, since

$$d\big(\gamma(t'), y\big) \le d\big(\gamma(t'), \gamma(t)\big) + d\big(\gamma(t), y\big) \le (t - t') + (r - t) = r - t'.$$

Let now $t_0 := \sup\big\{t \in [\varepsilon, r] : d\big(\gamma(t), y\big) = r - t\big\}$; then (2.1.10) holds in particular also for t_0 and it remains to show that $t_0 = r$. Supposing $t_0 < r$, we may choose a sufficiently small geodesic ball $B_{\varepsilon_1}\big(\gamma(t_0)\big)$ about $\gamma(t_0)$ and $x_1 \in S_{\varepsilon_1}\big(\gamma(t_0)\big)$ with minimal distance to y. Since $d(x_1, y) = d\big(\gamma(t_0), y\big) - \varepsilon_1 = (r - t_0) - \varepsilon_1$, we then have

$$d(x, x_1) \ge d(x,y) - d(x_1, y) = r - (r - t_0 - \varepsilon_1) = t_0 + \varepsilon_1.$$

Since the curve $\gamma|[0, t_0]$ from x to $\gamma(t_0)$, prolongated by the radial geodesic from $\gamma(t_0)$ to x_1, has length $t_0 + \varepsilon_1$, the curve realizes the distance $d(x, x_1)$; by the Corollary above it must be geodesic and hence coincide with $\gamma|[0, t_0 + \varepsilon_1]$. In particular, then $x_1 = \gamma(t_0 + \varepsilon_1)$ and

$$d\big(\gamma(t_0 + \varepsilon_1), y\big) = d(x_1, y) = r - (t_0 + \varepsilon_1),$$

in contradiction to the Definition of t_0.

(ii) $\Rightarrow$ (iii) is a weakening.

(iii) $\Rightarrow$ (i): Since Cauchy sequences are bounded, it is sufficient to show that each bounded subset A of M is contained in a compact subset $K \subset M$. Let $x \in M$ be such that $\exp_x : T_xM \to M$ is well-defined on all of T_xM. Assume that $d(a, x) \le r$ for all $a \in A$. By Theorem 2.1.9 then $a = \exp_x v$ for some $v \in T_xM$ with $|v| \le r$, so that $A \subset \exp_x\big(\bar{V}_r(0)\big)$, where $K := \exp_x\big(\bar{V}_r(0)\big)$ is compact as image of a compact set under a continuous map. $\square$

In the sequel we assume that (M, g) is metrically complete and without restriction connected. For any $x \in M$ then $\exp_x$ is defined on all of T_xM and defines for $r \le \varrho(x)$ a diffeomorphism of $V_r(0)$ to $B_r(x)$. A point $x \in M$ is called *pole* for (M, g) if $\varrho(x) = \infty$.

DEFINITION 2.1.10 (Cut locus). Let (M,g) be a metrically complete Riemannian manifold. For $x \in M$ and $v \in T_xM$ with $|v| = 1$ denote by γ_v the geodesic curve starting at x such that $\dot\gamma_v(0) = v$, i.e. $\gamma_v(t) = \exp_x(tv)$, and let

$$\begin{aligned} s(v) :&= \sup\big\{t \geq 0 : d\big(x, \gamma_v(t)\big) = t\big\} \\ &\equiv \sup\big\{t \geq 0 : \gamma_v|[0,t] \text{ is minimal geodesic}\big\} \in\,]0, \infty]. \end{aligned}$$

Then $C_x := \big\{s(v){\cdot}v : v \in T_xM,\ |v| = 1,\ s(v) < \infty\big\}$ is called *cut locus of* $\exp_x$ *in* x and die set

$$\text{cut}(x) := \exp_x(C_x) \subset M$$

cut locus of M with respect to x.

In case $s(v) < \infty$, the curve $\gamma_v|[0,t]$ stops to be the shortest connection between $x = \gamma_v(0)$ and $\gamma_v(t)$ for $t > s(v)$. The point $\gamma_v\big(s(v)\big)$ is then also called *cut point of* x *along* γ_v. By the Theorem of Hopf-Rinow, the curve γ_v is cut at each point $\gamma_v\big(s(v) + \varepsilon\big)$ (for $\varepsilon > 0$) by a shorter geodesic curve emanating from x.

We will show that always $\varrho(x) = d\big(x, \text{cut}(x)\big)$; by Theorem 2.1.6 it obvious that $\exp_x|V_r(0)$ for $r > d\big(x, \text{cut}(x)\big)$ is no longer an embedding, hence either no longer injective or it has critical points. We deal first with critical points of the exponential function.

DEFINITION 2.1.11 (Conjugate locus). Let (M,g) be a metrically complete Riemannian manifold and $x \in M$. Critical points $v \in T_xM$ of $\exp_x\colon T_xM \to M$ are called *vectors conjugate to* x. Then

$$K_x := \{v \in T_xM : v \text{ is a vector conjugate to } x\}$$

is called *conjugate locus of* $\exp_x$ *in* T_xM and the set

$$\text{Conj}(x) := \exp_x\big(K_x\big) \equiv \{y \in M\colon\ y \text{ is critical value of } \exp_x\}$$

the *conjugate locus of* x *in* M. If $y \in \text{Conj}(x)$ such that $y = \exp_x v$ with $v \in K_x$, one says that "y is conjugate to x along the geodesic $\gamma_v(t) = \exp_x(tv)$ $(0 \leq t \leq 1)$".

The next theorem gives a basic characterization of the cut locus.

THEOREM 2.1.12. *Let* (M,g) *be a metrically complete Riemannian manifold,* γ *a normal geodesic curve on* M *and* $\gamma(t_0)$ *a cutting point of* $x = \gamma(0)$ *along* γ. *Then either*

(i) $\gamma(t_0)$ *is conjugate to* $\gamma(0)$ *along* γ, *or*
(ii) *there is a geodesic curve* $\sigma \neq \gamma$ *from* x *to* $\gamma(t_0)$ *such that* $L(\sigma) = L\big(\gamma|[0,t_0]\big)$.

PROOF. Let $\gamma(t_0)$ be as described and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of real numbers such that $0 < \varepsilon_n \to 0$. For any $n \in \mathbb{N}$ let σ_n a normal minimal geodesic connecting x and $\gamma(t_0 + \varepsilon_n)$. Then $\dot\sigma_n(0) \in T_xM$ and $|\dot\sigma_n(0)| = 1$; by compactness of the unit sphere $S^{n-1} \subset T_xM$ we may assume (after eventually passing to a subsequence) that $\dot\sigma_n(0)$ converges in S^{n-1}. Hence there is a geodesic curve σ starting at x such that $\dot\sigma_n(0) \to \dot\sigma(0)$. By the continuity of the exponential function, σ is a minimal geodesic from x to $\gamma(t_0)$ and hence $L\big(\sigma|[0,t_0]\big) = L\big(\gamma|[0,t_0]\big)$. If now $\sigma \neq \gamma$, then part (ii) of the claim is satisfied; it is hence sufficient to verify assertion (i) if $\sigma = \gamma$; thus we have to show that $d\exp_x$ is singular at $t_0\dot\gamma(0)$ if $\sigma = \gamma$ gilt.

Assume that $\dot\sigma(0) = \dot\gamma(0)$ and $d\exp_x$ not singular at $t_0\,\dot\gamma(0)$. Then there is an open neighbourhood V of $t_0\dot\gamma(0)$ on which $\exp_x$ is an embedding. By Definition of σ_n we have $\gamma(t_0 + \varepsilon_n) = \sigma_n(t_0 + \tilde\varepsilon_n)$ with $\tilde\varepsilon_n \leq \varepsilon_n$, since σ_n is a minimal geodesic, and $\tilde\varepsilon_n \to 0$ for

$n \to \infty$. For n sufficiently large then $(t_0 + \tilde{\varepsilon}_n)\dot{\sigma}_n(0)$ und $(t_0 + \varepsilon_n)\dot{\gamma}(0)$ lie in V, and we have

$$\exp_x\big((t_0 + \varepsilon_n)\,\dot{\gamma}(0)\big) = \gamma(t_0 + \varepsilon_n) = \sigma_n(t_0 + \tilde{\varepsilon}_n) = \exp_x\big((t_0 + \tilde{\varepsilon}_n)\,\dot{\sigma}_n(0)\big);$$

hence $(t_0+\varepsilon_n)\dot{\gamma}(0) = (t_0+\tilde{\varepsilon}_n)\dot{\sigma}_n(0)$ and then $\dot{\gamma}(0) = \dot{\sigma}_n(0)$ for large n, in contradiction to the Definition of σ_n. $\square$

If (M, g) is a metrically complete Riemannian manifold, $x \in M$ a given point and $S^{n-1} = \{v \in T_xM : |v| = 1\}$ the unit sphere in T_xM, one can show (e.g. [**27**], p. 98) that the map in Definition 2.1.10

$$s\colon S^{n-1} \to \overline{\mathbb{R}}_+, \quad s(v) = \sup\big\{t \geq 0 : \gamma_v|[0,t] \text{ is a minimal geodesic}\big\},$$

is continuous. Since in addition s is strictly positive, the set

$$U_x := \big\{t\,v \in T_xM : v \in S^{n-1},\ 0 \leq t < s(v)\big\}$$

defines an open star-shaped neighbourhood of 0 in T_xM with $\partial U_x = C_x$; according to Definition 2.1.10 then $\text{cut}(x) = \exp_x(\partial U_x)$.

THEOREM 2.1.13. *If (M, g) is a metrically complete Riemannian manifold and $x \in M$, then*

$$M = \exp_x(U_x) \mathbin{\dot{\cup}} \text{cut}(x).$$

PROOF. Let $y \in M$. By Theorem 2.1.9 (Hopf-Rinow) there is a minimal geodesic

$$\gamma_v\colon\ \gamma_v(t) = \exp_x(tv), \quad |v| = 1, \quad 0 \leq t \leq b,$$

connecting x and y; hence $b \leq s(v)$. It rests to show the disjointness of the union. Suppose that $y \in \exp_x(U_x) \cap \text{cut}(x)$, then $y = \exp_x(t_0v_0) = \exp_x(t_1v_1)$ with $v_0, v_1 \in S^{n-1} \subset T_xM$ such that $t_0 < s(v_0)$ and $t_1 = s(v_1)$. Both $\gamma_{v_0}|[0,t_0]$ and $\gamma_{v_1}|[0,t_1]$ are then minimal geodesics connecting x and y, hence $t_0 = t_1$. In addition, $\gamma_{v_0}|[0, t_0 + \varepsilon]$ is still minimal for $\varepsilon > 0$ sufficiently small. By the following Lemma, γ_{v_0} can however not be minimal beyond the interval $[0, t_0]$. $\square$

LEMMA 2.1.14. *Let $\gamma\colon \mathbb{R} \to M$ be a geodesic on a Riemannian manifold (M, g). If there is a geodesic curve $\sigma \neq \gamma$ connecting $\gamma(0)$ and $\gamma(t_0)$ with the same length as γ, then $\gamma|[0, t_0 + \varepsilon]$ cannot be minimal for $\varepsilon > 0$.*

PROOF. Assume that $\sigma\colon [0, t_0] \to M$ is a further geodesic connecting $\gamma(0)$ and $\gamma(t_0)$ of length $L(\sigma) = L\big(\gamma|[0,t_0]\big)$; furthermore suppose that $\gamma|[0, t_0 + \varepsilon]$ is still minimal for some $\varepsilon > 0$. Then also $c\colon [0, t_0 + \varepsilon] \to M$ defined by $c|[0,t_0] = \sigma|[0,t_0]$ and $c|[t_0, t_0 + \varepsilon] = \gamma|[t_0, t_0 + \varepsilon]$, is a curve connecting x and $\gamma(t_0 + \varepsilon)$ with the length as $\gamma|[0, t_0 + \varepsilon]$ as well. Thus also c is a minimal geodesic curve which must coincide with γ since $c|[t_0, t_0 + \varepsilon] = \gamma|[t_0, t_0 + \varepsilon]$. Consequently also γ coincides with σ on $[0, t_0]$. $\square$

Theorem 2.1.13 combined with Lemma 2.1.14 gives the following result.

COROLLARY 2.1.15. *Let (M, g) be a metrically complete Riemannian manifold and $x \in M$. Then, for each point $y \in M \setminus \text{cut}(x)$ there is exactly one minimal geodesic connecting x and y.*

EXAMPLE 2.1.16. If $M = S^n$ denotes the n-dimensional sphere (considered as part of $\mathbb{R}^{n+1}$ with the induced canonical Riemannian metric), then for each point $x \in S^n$

$$\text{Conj}(x) = \text{cut}(x) = \{-x\}.$$

For $y \in \mathrm{Conj}(x)$ such that $y = \exp_x v$, one says that "v lies in the first conjugate locus in T_xM" if $\exp_x$ is regular at tv for $0 \le t < 1$. We will see that each geodesic curve γ in M emanating from x meets the cut locus cut(x) along γ not later than the first point conjugate to x. This then shows that $\exp_x |U_x$ is not only injective but a local diffeomorphism, and hence defines a diffeomorphism of U_x to $\exp_x(U_x)$. Thus $M \setminus \mathrm{cut}(x)$ is diffeomorphic to an open ball in $\mathbb{R}^n$, and cut(x) itself is a strong deformation retract of $M \setminus \{x\}$. In this sense the cut locus cut(x) contains the topology of M and will hence in general have a complicated structure which indicates that Example 2.1.16 is not typical.

REMARK 2.1.17. $M \setminus \mathrm{cut}(x)$ can be characterized as the maximal open subset of M with the property that each of its points can be uniquely joined with x by a minimal geodesic curve.

Before turning to one of the fundamental questions of the theory, i.e. the problem when along a radial ray $t \mapsto tv$ in T_xM the first conjugate vector in in T_xM shows up, we insert a discussion of the general notion of curvature of a Riemannian manifold. The basic answer to the question above is then given by the so-called "comparison principle" which roughly speaking says that the first conjugate vector comes later the smaller curvature is.

REMARK 2.1.18. Let (M, g) be a Riemannian manifold of of dimension at least 2 and ∇ the Levi-Civita connection on M. By Definition 1.4.24, the *Riemann curvature tensor* $R \in \Gamma(T^*M^{\otimes 3} \otimes TM)$ is given by

$$R(X,Y,Z) \equiv R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \Gamma(TM)$. One may read R equally either as $C^\infty(M)$-trilinear map

$$\Gamma(TM)^3 \to \Gamma(TM)$$

or as

$$\Gamma(TM \otimes TM) \longrightarrow \mathrm{Hom}_{C^\infty(M)}\big(\Gamma(TM), \Gamma(TM)\big).$$

LEMMA 2.1.19 (curvature identities). *For $X, Y, Z, U \in \Gamma(TM)$ one has:*

(i) $\langle R(X,Y)Z, U\rangle = -\langle R(Y,X)Z, U\rangle = -\langle R(X,Y)U, Z\rangle$

(ii) $\langle R(X,Y)Z, U\rangle = \langle R(Z,U)X, Y\rangle$

(iii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(Bianchi identity).*

PROOF. (i): Anti-symmetry in the first and second argument is trivial, in the third and fourth argument it holds because of

$$\langle \nabla_X \nabla_Y Z, Z\rangle = \frac{1}{2} X\big(Y\langle Z, Z\rangle\big) - \langle \nabla_Y Z, \nabla_X Z\rangle$$
$$\langle \nabla_{[X,Y]} Z, Z\rangle = \frac{1}{2}[X,Y]\langle Z, Z\rangle.$$

(iii): The Bianchi identity follows from the fact that

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0$$

for $X, Y, Z \in \Gamma(TM)$ (Jacobi identity for the Lie product of vector fields); on the other hand by torsion-freeness of ∇ one has

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0.$$

(ii): According to the Bianchi identity one has

$$\langle R(X,Y)Z, U\rangle + \langle R(Y,Z)X, U\rangle + \langle R(Z,X)Y, U\rangle = 0$$
$$-\langle R(X,Y)U, Z\rangle + \langle R(Y,U)X, Z\rangle + \langle R(U,X)Y, Z\rangle = 0$$

$$-\langle R(Z,U)X,Y\rangle + \langle R(U,X)Z,Y\rangle + \langle R(X,Z)U,Y\rangle = 0$$
$$\langle R(Z,U)Y,X\rangle + \langle R(U,Y)Z,X\rangle + \langle R(Y,Z)U,X\rangle = 0.$$

Addition of the four equations and taking into account i) leads to

$$2\langle R(X,Y)Z,U\rangle - 2\langle R(Z,U)X,Y\rangle = 0$$

which gives the claim. □

Further curvature identities are obtained from the Riemannian curvature tensor by contraction.

DEFINITION 2.1.20 (Ricci curvature, scalar curvature). For a Riemannian manifold (M,g) the tensor $\mathrm{Ric}^M \in \Gamma(T^*M \otimes T^*M)$, defined by

$$\mathrm{Ric}^M_x(u,v) := \mathrm{trace}\big(T_xM \to T_xM\,,\ w \mapsto R(w,u,v)\big) \equiv \sum_{i=1}^n \big\langle R(e_i,u)v, e_i\big\rangle,$$

where $(e_1,\dots,e_d)$ denotes an orthonormal basis of T_xM, is called *Ricci tensor* of (M,g); the symmetric bilinear form

$$\mathrm{Ric}^M_x\colon T_xM \times T_xM \to \mathbb{R}$$

is called *Ricci curvature* at x. The real-valued function k^M finally,

$$k^M(x) := \mathrm{trace}\ \mathrm{Ric}^M_x = \sum_{j=1}^n \mathrm{Ric}^M_x(e_j,e_j) \equiv \sum_{i=1}^n\sum_{j=1}^n \big\langle R(e_i,e_j)e_j, e_i\big\rangle,$$

where $(e_1,\dots,e_d)$ denotes again an orthonormal basis of T_xM, is called *scalar curvature* of (M,g).

DEFINITION 2.1.21 (Sectional curvature). Let (M,g) be a Riemannian manifold with $\dim M \geq 2$. Furthermore let $G_2TM \to M$ be the Grassmann 2-bundle, defined as set by $G_2TM := \bigcup_{x\in M} G_2T_xM$ where

$$G_2T_xM := \{E \subset T_xM\colon\ E \text{ two-dimensional real subspace}\}.$$

Then the map $\mathrm{Riem}^M\colon G_2TM \to \mathbb{R}$,

$$\mathrm{Riem}^M\big|G_2T_xM \equiv \mathrm{Riem}^M_x\colon\ E = \mathrm{span}\{u,v\} \mapsto \frac{\langle R(u,v)v,u\rangle}{|u|^2|v|^2 - \langle u,v\rangle^2},$$

is well-defined (i.e. independent of the choice of u and v) and is called *Riemannian sectional curvature of* M. Here $|u|^2|v|^2 - \langle u,v\rangle^2 = |u\wedge v|^2$ is the squared area of the parallelogram spanned by u and v.

REMARK 2.1.22. The sectional curvature determines the Riemann curvature tensor.

PROOF. Indeed, for $X,Y \in \Gamma(TM)$ at first $k(X,Y) := \langle R(X,Y)Y,X\rangle$ is uniquely determined by Riem^M. By means of the curvature identities it holds however

$$\begin{aligned}6\,\langle R(X,Y)Z,U\rangle = k(X+U,Y+Z) &- k(X+U,Y) - k(X+U,Z)\\ &- k(X,Y+Z) - k(U,Y+Z) + k(X,Z) + k(U,Y)\\ &- k(Y+U,X+Z) + k(Y+U,X) + k(Y+U,Z)\\ &+ k(Y,X+Z) + k(U,X+Z) - k(Y,Z) - k(U,X),\end{aligned}$$

so that R is determined by k. □

DEFINITION 2.1.23. A Riemannian manifold (M, g) is said to have *constant* (resp., *positive*, *negative*) *curvature*, if the sectional curvature Riem^M is constant (resp., positive, negative). The Riemannian manifold (M, g) is said to be *flat* if $\mathrm{Riem}^M \equiv 0$ (equivalently, $R \equiv 0$). A Riemannian manifold (M, g) is called *Einstein manifold* if $\mathrm{Ric}^M = cg$ for some real constant c.

We now turn again to the conjugacy behaviour of the exponential map. Let γ be a geodesic curve on M and $\gamma(t_0)$ a cut point of $x = \gamma(0)$ along γ. Then $\gamma|[0, t_0 + \varepsilon]$ is for $\varepsilon > 0$ no longer the shortest connection of $x = \gamma(0)$ and $\gamma(t_0 + \varepsilon)$, which can mean either that a deformation of γ provides shorter curves, or that there exist non-neighbouring curves of shorter length connecting $\gamma(0)$ and $\gamma(t_0 + \varepsilon)$ for $\varepsilon > 0$. We shall see that in the first case $\gamma(t_0)$ will be conjugates to x along γ, i.e., $\gamma(t_0) \in \mathrm{Conj}(x)$. In general it will turn out that a geodesic curve γ emanating from x, which does not hit the conjugate locus $\mathrm{Conj}(x)$ up to time t_0, is the shortest connection of x and $\gamma(t_0)$, compared to all (piecewise differentiable) curves from x to $\gamma(t_0)$ which are sufficiently close to $\gamma|[0, t_0]$.

In Corollary 2.1.8 we characterized geodesics as critical points of the length functional under smooth variation of curves. This point of view motivates to consider in addition to the first derivative (first variation) also the second derivative of the length functional.

THEOREM 2.1.24 (Second variation of arc length). *Let (M, g) be a Riemannian manifold with the Levi-Civita connection, $\gamma\colon [a, b\,] \to M$ a normal geodesic and let*

$$\alpha\colon [a, b\,] \times \left]-\varepsilon, \varepsilon\right[\to M$$

be a differentiable variation of γ. In terms of $\gamma_s = \alpha(\cdot, s)$, along with $T = \frac{\partial}{\partial t}\alpha \equiv \alpha_* D_1 \in \Gamma(\alpha^* TM)$ and $V = \frac{\partial}{\partial s}\alpha \equiv \alpha_* D_2 \in \Gamma(\alpha^* TM)$, for the second derivative of the length functional $L(s) := L(\gamma_s) \equiv \int_a^b |\dot\gamma_s(t)|\, dt$ at $s = 0$ the so-called Synge formula holds:

$$L''(0) = \big\langle \nabla_{D_2} V, T \big\rangle(t, 0)\Big|_{t=a}^{t=b} + \int_a^b \Big\{ |\nabla_{D_1} V|^2 - \big\langle R(V, T)T\,, V \big\rangle - \big(D_1 \langle V, T\rangle\big)^2 \Big\}(t, 0)\, dt.$$

PROOF. Recall that by (2.1.3) we have $L'(s) = \frac{d}{ds} L(\gamma_s) = \int_a^b \frac{1}{|T|} \left\langle \nabla_{D_1} V, T \right\rangle dt$. By means of Cartan's structural equations (Theorem 1.4.27) and the characterization of Riemannian connections in Theorem 1.5.6 (iii), this gives

$$\begin{aligned}
L''(s) &= \int_a^b \left\{ \frac{D_2 \langle \nabla_{D_1} V, T\rangle}{|T|} - \frac{1}{2} \frac{\langle \nabla_{D_1} V, T\rangle\, D_2 \langle T, T\rangle}{|T|^3} \right\} dt \\
&= \int_a^b \left\{ \frac{\langle \nabla_{D_2} \nabla_{D_1} V, T\rangle + \langle \nabla_{D_1} V, \nabla_{D_2} T\rangle}{|T|} - \frac{\langle \nabla_{D_1} V, T\rangle\, \langle \nabla_{D_2} T, T\rangle}{|T|^3} \right\} dt \\
&= \int_a^b \left\{ \frac{\big\langle R(V, T)V, T\big\rangle + \langle \nabla_{D_1} \nabla_{D_2} V, T\rangle + \langle \nabla_{D_1} V, \nabla_{D_2} T\rangle}{|T|} - \frac{\langle \nabla_{D_1} V, T\rangle^2}{|T|^3} \right\} dt.
\end{aligned}$$

Now since $\nabla_{D_1} T = 0$ and $|T| = 1$ along γ, we obtain for $s = 0$ the formula

$$\begin{aligned}
L''(0) &= \int_a^b \Big\{ \langle \nabla_{D_1} V, \nabla_{D_2} T\rangle - \big\langle R(V, T)T, V\big\rangle + D_1 \langle \nabla_{D_2} V, T\rangle - \big(D_1 \langle V, T\rangle\big)^2 \Big\}(t, 0)\, dt \\
&= \big\langle \nabla_{D_2} V, T\big\rangle(t, 0)\Big|_{t=a}^{t=b} + \int_a^b \Big\{ |\nabla_{D_1} V|^2 - \big\langle R(V, T)T\,, V\big\rangle - \big(D_1 \langle V, T\rangle\big)^2 \Big\}(t, 0)\, dt
\end{aligned}$$

which proves the claim. $\square$

NOTATION 2.1.25. Let $\gamma\colon [a,b] \to M$ be a geodesic curve and $\alpha\colon [a,b] \times]-\varepsilon,\varepsilon[\to M$ a differentiable variation of γ. The variation of γ is called *Jacobi variation* if all neighbouring curves $\gamma_s = \alpha(\cdot, s)$ to γ are geodesics. For $t \in [a,b]$ we say that α *varies geodesically at* t if the induced curve $\alpha(t,\cdot) :]-\varepsilon,\varepsilon[\to M$ is a geodesic.

If in the situation of Theorem 2.1.24 $\alpha\colon [a,b] \times]-\varepsilon,\varepsilon[\to M$ is a variation of $\gamma\colon [a,b] \to M$ with fixed initial and end point (i.e., $\alpha(a,s) \equiv \alpha(a,0)$ and $\alpha(b,s) \equiv \alpha(b,0)$ for $-\varepsilon < s < \varepsilon$), or more generally, if α varies geodesically at the end points $t = a$ and $t = b$, then the term $\langle \nabla_{D_2} V, T\rangle(t,0)\big|_{t=a}^{t=b}$ vanishes in the Synge formula for the second variation of the length, and Theorem 2.1.24 gives the following Corollary.

COROLLARY 2.1.26. *Let (M,g) be a Riemannian manifold, $\gamma\colon [a,b] \to M$ a normal geodesic curve and $\alpha\colon [a,b] \times]-\varepsilon,\varepsilon[\to M$ a variation of γ which varies geodesically at a and b. Denoting by $Y = \alpha_* D_2(\cdot, 0) \in \Gamma(\gamma^*TM)$ the corresponding variational field along γ and $Y^\perp := Y - \langle Y, \dot\gamma\rangle\,\dot\gamma \in \Gamma(\gamma^*TM)$ its orthogonal part, then for the second variation of the length the following formula holds:*

$$L''(0) = \int_a^b \Big\{ |\nabla_D Y^\perp|^2 - \big\langle R(Y^\perp, \dot\gamma)\,\dot\gamma, Y^\perp \big\rangle \Big\}\, dt. \tag{2.1.11}$$

PROOF. Indeed we have $\nabla_D Y^\perp = \nabla_D Y - \langle \nabla_D Y, \dot\gamma\rangle\,\dot\gamma = (\nabla_D Y)^\perp$ and hence

$$|\nabla_D Y^\perp|^2 = |\nabla_D Y|^2 - \langle \nabla_D Y, \dot\gamma\rangle^2,$$

from where the claim follows since $\big\langle R(Y,\dot\gamma)\,\dot\gamma, Y\big\rangle = \big\langle R(Y^\perp,\dot\gamma)\,\dot\gamma, Y^\perp\big\rangle$. ☐

REMARK 2.1.27. Let $\alpha\colon [a,b] \times]-\varepsilon,\varepsilon[\to M$ be a Jacobi variation of the geodesic $\gamma\colon [a,b] \to M$ and let $Y = (\alpha_* D_2)(\cdot,\,0) \in \Gamma(\gamma^*TM)$. Then $D\langle Y, \dot\gamma\rangle$ is constant along $\gamma = \alpha(\cdot, 0)$. Hence, if $\langle Y, \dot\gamma\rangle$ vanishes at the end points a and b, then $\langle Y, \dot\gamma\rangle$ vanishes already identically on $[a,b]$.

PROOF. Let again $T = \alpha_* D_1 \in \Gamma(\alpha^*TM)$ and $V = \alpha_* D_2 \in \Gamma(\alpha^*TM)$. From $\nabla_{D_1} V - \nabla_{D_2} T = \alpha_*[D_1, D_2] = 0$ and $\nabla_{D_1} T = 0$ it follows first that

$$\nabla_{D_1}\nabla_{D_1} V = \nabla_{D_1}\nabla_{D_2} T = \nabla_{D_1}\nabla_{D_2} T - \nabla_{D_2}\nabla_{D_1} T = R(T,V)T \tag{2.1.12}$$

and then

$$D_1 D_1 \langle V, T\rangle = D_1 \langle \nabla_{D_1} V, T\rangle = \langle \nabla_{D_1}\nabla_{D_1} V, T\rangle = \big\langle R(T,V)T, T\big\rangle = 0,$$

where the last equality comes from Lemma 2.1.19 (i). ☐

Hence if $\gamma\colon [a,b] \to M$ is a normal geodesic curve and α a Jacobi variation of γ varying geodesically at the end points such that $\langle Y, \dot\gamma\rangle(a) = \langle Y, \dot\gamma\rangle(b) = 0$ holds for the variational vector field $Y = (\alpha_* D_2)(\cdot,\,0) \in \Gamma(\gamma^*TM)$, than by Remark 2.1.27 the Synge formula simplifies to

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) = \int_a^b \Big\{ |\nabla_D Y|^2 - \big\langle R(Y,T)T\,, Y\big\rangle \Big\}\, dt. \tag{2.1.13}$$

The idea is now to bilinearize the Synge formula (2.1.13) for the second variation of the arc length which leads to the notion of the index form of γ.

DEFINITION 2.1.28 (Index form). Let (M,g) be a Riemannian manifold, $\gamma\colon [a,b] \to M$ a normal geodesic and $\Gamma^\perp(\gamma^*TM)$ the real vector space of piecewise differentiable vector fields X along γ such that $\langle X, \dot\gamma\rangle \equiv 0$. Then

$$I(X,Y) := \int_a^b \Big\{ \langle \nabla_D X, \nabla_D Y\rangle - \big\langle R(X,\dot\gamma)\dot\gamma\,, Y\big\rangle \Big\}\, dt \tag{2.1.14}$$

defines a symmetric bilinear form on $\Gamma^{\perp}(\gamma^*TM)$, the so-called *index form* of γ. The *nullspace* of I is the linear subspace of $X \in \Gamma^{\perp}(\gamma^*TM)$ with the property that $I(X,Y) = 0$ for all $Y \in \Gamma^{\perp}(\gamma^*TM)$.

The index form I of a normal geodesic curve $\gamma\colon [a,b\,] \to M$ thus assigns to each differentiable vector field $Y \in \Gamma^{\perp}(\gamma^*TM)$ the second variation $L''(0)$ of the length L with respect to the following variation of γ (induced by Y),

$$\alpha\colon [a,b\,] \times \left]-\varepsilon,\varepsilon\right[\to M, \quad \alpha(t,s) = \exp_{\gamma(t)}(s\,Y_t), \tag{2.1.15}$$

that is $L''(0) = I(Y,Y)$ and $(\alpha_* D_2)(\,\cdot\,,0) = Y$. Note that (2.1.15) is well-defined for $\varepsilon > 0$ sufficiently small by the compactness of the interval $[a,b]$.

REMARK 2.1.29. If $I(Y,Y) < 0$ for a differentiable vector field $Y \in \Gamma^{\perp}(\gamma^*TM)$ with $Y_a = 0$ and $Y_b = 0$, then there are curves arbitrarily close to γ connecting $\gamma(a)$ and $\gamma(b)$ with a shorter length than γ. If however I is positively definite on the subspace of differentiable vector fields $Y \in \Gamma^{\perp}(\gamma^*TM)$ vanishing at the end points, then the length of γ is minimal compared to all variational curves sufficiently close to γ with the same end points.

DEFINITION 2.1.30 (Jacobi field). Let $\gamma\colon [a,b] \to M$ be a geodesic on a Riemannian manifold (M,g). A vector field $J \in \Gamma(\gamma^*TM)$ along γ is said to be a *Jacobi field* along γ if it satisfies the "Jacobi equation"

$$\nabla_D\nabla_D J + R(J,\dot\gamma)\,\dot\gamma = 0. \tag{2.1.16}$$

A Jacobi field J along γ is called *proper*, if in addition $\langle J,\dot\gamma\rangle = 0$ holds.

It is easy to see that the Jacobi equation (2.1.16) is equivalent to a second order system of linear differential equations. Fixing a parallel section e along γ in $\mathrm{O}(TM)$, then $\big(e_1(t),\dots,e_d(t)\big)$ is an orthonormal basis for $T_{\gamma(t)}M$ and J writes as $J = \sum_i \langle J,e_i\rangle e_i$.

For the scalar functions $\langle J,e_i\rangle$ we have then $\langle J,e_i\rangle' \equiv D\langle J,e_i\rangle = \langle\nabla_D J,e_i\rangle$ and $\langle J,e_i\rangle'' \equiv DD\langle J,e_i\rangle = \langle\nabla_D\nabla_D J,e_i\rangle$, and the Jacobi equation (2.1.16) is equivalent to the system of linear differential equations

$$\langle J,e_j\rangle'' = \sum_{i=1}^{n}\langle R(\dot\gamma,e_i)\dot\gamma,e_j\rangle\,\langle J,e_i\rangle, \quad j=1,\dots,d. \tag{2.1.17}$$

By the theory of ordinary linear differential equations the system (2.1.17) has a $2n$-dimensional space of solutions, and to each initial value and first derivative, corresponding to the data of $J|_{t=t_0}$ and $J'|_{t=t_0} := (\nabla_D J)(t_0)$ for some t_0, there is exactly one solution.

Since $\nabla_D\dot\gamma = 0$ we observe in addition $\langle J,\dot\gamma\rangle'' = \langle\nabla_D\nabla_D J,\dot\gamma\rangle = \langle R(\dot\gamma,J)\dot\gamma,\dot\gamma\rangle = 0$. Each Jacobi field J along γ has hence a unique representation as

$$J = J^{\perp} + (c_1 + t\,c_2)\,\dot\gamma \tag{2.1.18}$$

with $J^{\perp}$ a proper Jacobi field (i.e. $\langle J^{\perp},\dot\gamma\rangle = 0$) and real constants c_1, c_2.

REMARK 2.1.31. Let (M,g) be a Riemannian manifold, $\gamma\colon [0,b\,] \to M$ a geodesic and $\alpha\colon [0,b\,] \times \left]-\varepsilon,\varepsilon\right[\to M$ a Jacobi variation of γ. Then the "variational field"

$$J := (\alpha_* D_2)(\,\cdot\,,0) \equiv \tfrac{\partial}{\partial s}\Big|_{s=0}\alpha(\,\cdot\,,s) \in \Gamma(\gamma^*TM) \tag{2.1.19}$$

is a Jacobi field along γ, and all Jacobi fields along γ are obtained in this way.

PROOF. That (2.1.19) defines a Jacobi field along γ is a consequence of formula (2.1.12) for Jacobi variations. Conversely, let J be an arbitrary Jacobi field along γ. We want to show that J results from a variation of geodesic curves (Jacobi variation). To this end, we fix a curve $c\colon\,]-\varepsilon,\varepsilon[\,\to M$ with $c(0)=\gamma(0)$ and $\dot c(0)=J_0$. Along c we choose a vector field W such that $W(0)=\dot\gamma(0)$ and $(\nabla_D W)_0=(\nabla_D J)_0$ (for instance, $W_s := /\!/_{0,s}\dot\gamma(0)+s\,/\!/_{0,s}(\nabla_D J)_0\in\Gamma(c^*TM)$ with $/\!/_{0,s}$ the parallel transport along c from $T_{c(0)}M$ to $T_{c(s)}M$). Then $\alpha(t,s):=\exp_{c(s)}(t\,W_s)$ defines a Jacobi variation of γ and hence $\bar J=(\alpha_*D_2)(\cdot,0)$ a Jacobi field along γ. But we have $\bar J_0=J_0$ and $(\nabla_D\bar J)_0=(\nabla_D J)_0$ (this follows with $T=\alpha_*D_1$ and $V=\alpha_*D_2$ according to $(\nabla_D\bar J)_0=(\nabla_{D_1}V)_{(0,0)}=(\nabla_{D_2}T)_{(0,0)}=(\nabla_D W)_0=(\nabla_D J)_0$); hence necessarily $\bar J=J$ holds. □

The proof of Remark 2.1.31 provides in particular a method to construct Jacobi fields. The special case described in the following example is of particular importance.

EXAMPLE 2.1.32. Let (M,g) be a Riemannian manifold, $\gamma\colon[0,b]\to M$ a geodesic and $J\in\Gamma(\gamma^*TM)$ a Jacobi field along γ with $J(0)=0$. Then J is the variational field to the variation

$$\alpha\colon[0,b]\times\,]-\varepsilon,\varepsilon[\,\to M,\quad \alpha(t,s)=\exp_x\bigl[\,t\,\bigl(\dot\gamma(0)+s\,J'(0)\bigr)\bigr],$$

where $\gamma(0)=x$ and $J'(0):=(\nabla_D J)_0$; in other words::

$$J(t)=\tfrac{\partial}{\partial s}\big|_{s=0}\,\alpha(t\,,s)=(d\exp_x)_{t\dot\gamma(0)}\bigl(t\,J'(0)\bigr)\in T_{\gamma(t)}M.$$

Consider now the index form I defined in (2.1.14) on the vector space $\Gamma_0^\perp(\gamma^*TM)$ of piecewise differentiable vector fields X along a normal geodesic $\gamma\colon[a,b]\to M$ with $\langle X,\dot\gamma\rangle\equiv 0$ satisfying in addition $X_a=0$ and $X_b=0$. For $X,Y\in\Gamma_0^\perp(\gamma^*TM)$ and $a=t_0<t_1<\ldots<t_n=b$ a subdivision of the interval $[a,b]$ such that X and Y are differentiable on the subintervals $[t_i,t_{i+1}]$, one has $\langle\nabla_D X,\nabla_D Y\rangle=D\langle\nabla_D X,Y\rangle-\langle\nabla_D\nabla_D X,Y\rangle$ on $[t_i,t_{i+1}]$ and hence for the index form:

$$\begin{aligned}I(X,Y)&=\int_a^b\Bigl\{\langle\nabla_D X,\nabla_D Y\rangle-\bigl\langle R(X,\dot\gamma)\dot\gamma\,,Y\bigr\rangle\Bigr\}\,dt\\&=\sum_{i=1}^{n-1}\langle\nabla_D X,Y\rangle\Big|_{t_i+}^{t_i-}-\int_a^b\Bigl\{\langle\nabla_D\nabla_D X,Y\rangle+\bigl\langle R(X,\dot\gamma)\dot\gamma\,,Y\bigr\rangle\Bigr\}\,dt.\end{aligned}$$

If $X|[t_i,t_{i+1}]$ is a Jacobi field, it follows that

$$I(X,Y)=\textstyle\sum_i\langle\Delta_{t_i}(\nabla_D X),Y_{t_i}\rangle:=\sum_i\langle\nabla_D X,Y\rangle\big|_{t_i+}^{t_i-}\,.$$

THEOREM 2.1.33 (Jacobi fields as nullspace of the index form). *Let (M,g) be a Riemannian manifold, $\gamma\colon[a,b]\to M$ a normal geodesic and I the index form on $\Gamma_0^\perp(\gamma^*TM)$. Then the nullspace of I contains exactly the Jacobi fields J along γ vanishing at the end points: $J(a)=0$ and $J(b)=0$.*

PROOF. It is sufficient to show: From $I(X,Y)=0$ for all $Y\in\Gamma_0^\perp(\gamma^*TM)$ follows that X is a Jacobi field. Let $a=t_0<t_1<\ldots<t_n=b$ be a subdivision of $[a,b]$ such that X is differentiable on $[t_i,t_{i+1}]$, and $\varphi\colon[a,b]\to\mathbb{R}$ a differentiable function vanishing exactly at the places t_i for $i=0,\ldots,n$. With

$$Y:=\varphi\cdot\bigl(\nabla_D\nabla_D X+R(X,\dot\gamma)\dot\gamma\bigr)\in\Gamma_0^\perp(\gamma^*TM)$$

one obtains that each $X|[t_i,t_{i+1}]$ is a Jacobi field. Considering then $I(X,Y^0)$ for an arbitrary vector field $Y^0\in\Gamma_0^\perp(\gamma^*TM)$ with $Y^0(t_i)=\Delta_{t_i}(\nabla_D X)$, gives the claim. □

The next Theorem finally connects Jacobi fields to the conjugacy behaviour of the exponential function.

THEOREM 2.1.34. *Let (M,g) be a Riemannian manifold and $x,y \in M$; furthermore let $v \in T_xM$ with $y = \exp_x v$ and $\gamma\colon [0,1] \to M$, $\gamma(t) := \exp_x(t\,v)$ the connecting geodesic segment. The following statements are equivalent:*

(i) *y is conjugate to x along γ (i.e., v is a critical point of $\exp_x$).*
(ii) *There exists a non-identically vanishing Jacobi field J along γ with $J(0) = 0$ and $J(1) = 0$.*

In particular, $x \in \mathrm{Conj}(y)$ if and only if $y \in \mathrm{Conj}(x)$.

PROOF. (i) $\Rightarrow$ (ii): Let $y = \exp_x v$ and v a critical point of $\exp_x$; then there exists $w \in T_v(T_xM) \cong T_xM$ such that $(d\exp_x)_v w = 0$. The variation

$$\alpha\colon [a,b] \times]{-\varepsilon},\varepsilon[\to M \qquad \alpha(t,s) := \exp_x\bigl(t\,(v + s\,w)\bigr),$$

of γ is a Jacobi variation, and hence $J := \alpha_* D_2(\cdot\,,0)$ defines a Jacobi field according to Remark 2.1.31 which satisfies $J(0) = 0$ and

$$J(1) = \tfrac{\partial}{\partial s}\big|_{s=0} \exp_x(v + s\,w) = (d\exp_x)_v w = 0.$$

(i) $\Rightarrow$ (i): Conversely, let now $0 \neq J \in \Gamma(\gamma^*TM)$ be a Jacobi field along γ with $J(0) = 0$ and $J(1) = 0$. Then, by Example 2.1.32,

$$J(t) = (d\exp_x)_{t\dot\gamma(0)}(t\,w), \quad w = J'(0) \neq 0.$$

Since $J(1) = (d\exp_x)_v w = 0$ then v is a critical point of $\exp_x$, and thus $y = \exp_x v$ is conjugate to x. □

COROLLARY 2.1.35. *Let (M,g) be a Riemannian manifold and $\gamma\colon [a,b] \to M$ a geodesic with the property that $\gamma(a)$ and $\gamma(b)$ are conjugate to each other along γ. Then each Jacobi field J along γ is uniquely determined by the values $J(a)$ and $J(b)$.*

PROOF. The difference of two Jacobi fields along γ with identical boundary values defines a Jacobi field which vanishes at a and b, and hence vanishes identically by Theorem 2.1.34. □

The next Theorem, the so-called Index Lemma, will serve as a crucial tool. I shows that Jacobi fields minimize the index form in a certain sense.

In the proof we use the following elementary observation.

LEMMA 2.1.36. *Let $I \subset \mathbb{R}$ be an open real interval containing 0 and $h\colon I \to \mathbb{R}$ a differentiable function. Then there exists a differentiable function $\phi\colon I \to \mathbb{R}$ such that $h(t) = h(0) + t\phi(t)$ for $t \in I$.*

PROOF. Indeed, the function $\phi(t) = \int_0^1 h'(st)\,ds$ satisfies the claim. □

We assume the following situation: (M,g) is a Riemannian manifold, $\gamma\colon [0,b] \to M$ a normal geodesic curve and $\Gamma^\perp(\gamma^*TM)$ the real vector space of piecewise differentiable vector fields X along γ such that $\langle X, \dot\gamma\rangle \equiv 0$. By Definition 2.1.28, on $\Gamma^\perp(\gamma^*TM)$ the index form of γ is given:

$$I(X,Y) = \int_0^b \Bigl\{\langle \nabla_D X, \nabla_D Y\rangle - \bigl\langle R(X,\dot\gamma)\dot\gamma\,,Y\bigr\rangle\Bigr\}\,dt, \quad X,Y \in \Gamma^\perp(\gamma^*TM).$$

THEOREM 2.1.37 (Index Lemma). *Let (M, g) be a Riemannian manifold and suppose that $\gamma\colon [0, b] \to M$ is a normal geodesic with no points conjugate to $\gamma(0)$ along γ. Let J be a Jacobi field along γ with $\langle J, \dot\gamma\rangle = 0$ and X a vector field in $\Gamma^\perp(\gamma^* TM)$. Suppose that $J(0) = X(0) = 0$ and $J(b) = X(b)$. Then*

$$I(J, J) \le I(X, X),$$

with equality if and only if $J = X$.

PROOF. (1) The real vector space $\mathcal{J}$ of Jacobi fields J along γ with $J(0) = 0$ and $\langle J, \dot\gamma\rangle = 0$ is of dimension $n - 1$ where $n = \dim M$. Let $(J_1, \dots, J_{n-1})$ be a basis of $\mathcal{J}$ so that $J = \sum_i \alpha_i J_i$ with real constants $\alpha_1, \dots, \alpha_n$. Since there is no t such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ, according to Theorem 2.1.34, $\big(J_1(t), \dots, J_{n-1}(t)\big)$ forms a basis of the orthogonal complement $\big\{\dot\gamma(t)\big\}^\perp$ of $\dot\gamma(t)$ in $T_{\gamma(t)}M$ for each $t \in\ ]0, b]$. Consequently, for any $t \in\ ]0, b]$, the vector field X has a representation as

$$X(t) = \sum_{i=1}^{n-1} f_i(t)\, J_i(t) \tag{2.1.20}$$

with f_i piecewise differentiable functions on $]0, b]$. We want to check first that each f_i can be differentiably extended to $t = 0$, and hence to a piecewise differentiable function on $[0, b]$. Lemma 2.1.36, applied to the components $\langle J_i\,, e_k\rangle$ with respect to a parallel orthonormal basis $e = (e_1 \dots, e_n) \in \Gamma\big(\gamma^* \mathrm{O}(TM)\big)$ along γ, gives $J_i(t) = t\, A_i(t)$ with vector fields $A_i \in \Gamma(\gamma^* TM)$. In particular, then $(\nabla_D J_i)(0) = A_i(0)$ which shows the linear independence of $\big(A_1(0), \dots, A_{n-1}(0)\big)$. For any $t \in [0, b]$ hence $\big(A_1(t), \dots, A_{n-1}(t)\big)$ is a basis for $\big\{\dot\gamma(t)\big\}^\perp$ in $T_{\gamma(t)}M$, and one has $X(t) = \sum_i g_i(t)\, A_i(t)$ for $t \in [0, b]$, where g_i are piecewise differentiable functions on $[0, b]$ with $g_i(0) = 0$. Applying Lemma 2.1.36 one more time gives $g_i(t) = t\, h_i(t)$ with h_i piecewise differentiable functions on $[0, b]$. Since $f_i(t) = h_i(t)$ for $t \ne 0$, this shows the wanted continuability.

(2) Next we show that on the interior of each subinterval, on which the f_i are differentiable, the following formula holds:

$$\begin{aligned} &\langle \nabla_D X, \nabla_D X\rangle - \big\langle R(X, \dot\gamma)\, \dot\gamma\,, X\big\rangle \\ &\qquad = \big\langle \textstyle\sum_i f_i'\, J_i, \sum_i f_i'\, J_i\big\rangle + D\big\langle \sum_i f_i\, J_i, \sum_i f_i\, \nabla_D J_i\big\rangle. \end{aligned} \tag{2.1.21}$$

To shorting the notation we write $\langle A, A\rangle + D\langle X, B\rangle$ for the right-hand side of (2.1.21) where $A := \sum_i f_i'\, J_i$ and $B := \sum_i f_i\, \nabla_D J_i$. Firstly we have

$$R(X, \dot\gamma)\, \dot\gamma = \textstyle\sum_i f_i\, R(J_i, \dot\gamma)\, \dot\gamma = -\sum_i f_i\, \nabla_D \nabla_D J_i = -C$$

with $C = \sum_i f_i\, \nabla_D \nabla_D J_i$, and hence for the left-hand side of (2.1.21):

$$\begin{aligned} \langle \nabla_D X, \nabla_D X\rangle - \big\langle R(X, \dot\gamma)\, \dot\gamma\,, X\big\rangle &= \langle A + B, A + B\rangle - \big\langle R(X, \dot\gamma)\, \dot\gamma\,, X\big\rangle \\ &= \langle A, A\rangle + \langle A, B\rangle + \langle B, A\rangle + \langle B, B\rangle + \langle C, X\rangle. \end{aligned}$$

On the other hand, letting $Q := \sum_i f_i'\, \nabla_D J_i$, we get for the right-hand side of (2.1.21):

$$\begin{aligned} \langle A, A\rangle + D\langle X, B\rangle &= \langle A, A\rangle + \langle A + B, B\rangle + \langle X, Q + C\rangle \\ &= \langle A, A\rangle + \langle A, B\rangle + \langle B, B\rangle + \langle X, Q\rangle + \langle X, C\rangle. \end{aligned}$$

To verify (2.1.21) it is hence sufficient to show $\langle B, A\rangle = \langle X, Q\rangle$, or equivalently:

$$\big\langle \textstyle\sum_i f_i\, \nabla_D J_i, \sum_i f_i'\, J_i\big\rangle = \big\langle \sum_i f_i\, J_i, \sum_i f_i'\, \nabla_D J_i\big\rangle. \tag{2.1.22}$$

For the verification of (2.1.22) we consider for fixed indices i, j the function

$$h\colon [0, b] \to \mathbb{R}\,, \quad h := \langle \nabla_D J_i\,, J_j\rangle - \langle J_i\,, \nabla_D J_j\rangle\,;$$

since $h(0) = 0$ and

$$\begin{aligned} h' &= \langle \nabla_D \nabla_D J_i \,, J_j \rangle + \langle \nabla_D J_i \,, \nabla_D J_j \rangle - \langle \nabla_D J_i \,, \nabla_D J_j \rangle - \langle J_i \,, \nabla_D \nabla_D J_j \rangle \\ &= \big\langle R(\dot\gamma, J_i)\,\dot\gamma \,, J_j \big\rangle - \big\langle J_i \,, R(\dot\gamma, J_j)\,\dot\gamma \big\rangle = 0 \end{aligned}$$

we have $h \equiv 0$ on $[0, b]$. This shows Eq. (2.1.22), Using

$$\sum_{i,j} f_i\, f_j' \,\langle \nabla_D J_i \,, J_j \rangle = \sum_{i,j} f_i\, f_j' \,\langle J_i \,, \nabla_D J_j \rangle,$$

this shows Eq. (2.1.22), and completes the proof of formula (2.1.21).

(3) Integration of (2.1.21) gives

$$I(X, X) = \big\langle \textstyle\sum_i f_i\, J_i, \sum_j f_j\, \nabla_D J_j \big\rangle(b) + \displaystyle\int_0^b \big\langle \textstyle\sum_i f_i'\, J_i, \sum_j f_j'\, J_j \big\rangle\, dt;$$

analogously one obtains for the Jacobi field J the equation

$$I(J, J) = \big\langle \textstyle\sum_i \alpha_i\, J_i, \sum_j \alpha_j\, \nabla_D J_j \big\rangle(b).$$

By assumption, we have $J(b) = X(b)$, and hence $\alpha_i = f_i(b)$, which implies

$$I(X, X) = I(J, J) + \int_0^b \big|\textstyle\sum_i f_i'\, J_i\big|^2\, dt \geq I(J, J). \tag{2.1.23}$$

This completes the proof of the first part of the Index Lemma.

(4) If now $I(X, X) = I(J, J)$, then $\sum_i f_i'\, J_i = 0$ by (2.1.23).

By part (1) $\big(J_1(t), \ldots, J_{n-1}(t)\big)$ is linearly independent for each $t \in\,]0, b]$ which gives first $f_i' = 0$ on $]0, b]$ and by continuity then also on $[0, b]$. This shows $f_i =$ const for each i, since $f_i(b) = \alpha_i$ hence $f_i \equiv \alpha_i$ for each i, and hence $J = X$. □

A first consequence from the Index Lemma is that geodesics γ minimize the length up to the first conjugate point compared to sufficiently close neighbouring curves of γ with the same end points. Indeed, considering the case $J(b) = 0$ in the Index Lemma (without restrictions assume that γ is normal), we read off the following: If $\gamma(t_0)$ is the first point conjugate to $\gamma(0)$ along γ, then $I(X, X) > 0$ for any vector field $X \neq 0$ along $\gamma|[0, t]$ with $\langle X, \dot\gamma \rangle = 0$, provided $t < t_0$ and X vanishes at the end points, i.e., $X(0) = 0$ and $X(t) = 0$. This shows $L''(0) > 0$ for all variations of $\gamma|[0, t]$ with fixed end points.

In addition the following conversion holds:

COROLLARY 2.1.38. *Let (M, g) be a Riemannian manifold, $\gamma\colon [0, \infty[\, \to M$ a geodesic curve such that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. Then $\gamma|[0, t]$ is not minimal for $t > t_0$.*

PROOF. Without restriction let γ be normal; denote by $\gamma(t_0)$ the first point conjugate to $\gamma(0)$ on γ. By Theorem 2.1.34, there is a Jacobi field $J \neq 0$ along $\gamma|[0, t_0]$ with $J(0) = 0$ and $J(t_0) = 0$. We choose $\varepsilon > 0$ sufficiently small so that no pair of conjugate points exists on $\gamma|[t_0 - \varepsilon, t_0 + \varepsilon]$, and extend J to a piecewise differentiable vector field X along $\gamma|[0, t_0 + \varepsilon]$ via

$$X|[0, t_0] = J \quad \text{and} \quad X|[t_0, t_0 + \varepsilon] = 0.$$

Besides X we consider another piecewise differentiable vector field Y along $\gamma|[0, t_0 + \varepsilon]$ given by

$$Y|[0, t_0 - \varepsilon] = J \quad \text{and} \quad Y|[t_0 - \varepsilon, t_0 + \varepsilon] = \tilde J,$$

where $\tilde J$ is the unique Jacobi field along $\gamma|[t_0-\varepsilon, t_0+\varepsilon]$ such that $\tilde J(t_0-\varepsilon) = J(t_0-\varepsilon)$ and $\tilde J(t_0+\varepsilon) = 0$. Note that $\langle X, \dot\gamma\rangle = \langle Y, \dot\gamma\rangle = 0$. Since X and Y agree on $[0, t_0-\varepsilon]$, but $X|[t_0-\varepsilon, t_0+\varepsilon]$ is no Jacobi field, we obtain

$$I(Y,Y) < I(X,X) = 0$$

by the Index Lemma 2.1.37. Since Y induces a variation of $\gamma|[0, t_0+\varepsilon]$ with fixed end points according to (2.1.15) so that $L''(0) = I(Y,Y)$ for the corresponding second variation of the length, there exists a variation of $\gamma|[0, t_0+\varepsilon]$ which keeps the end point fixed and shortens the length of $\gamma|[0, t_0+\varepsilon]$. ☐

Absolute values of Jacobi fields can be compared by means of curvature relations. This is the content of the Comparison Theorem of Rauch.

THEOREM 2.1.39 (Rauch Comparison Theorem). *Let (M,g) and $(\tilde M, \tilde g)$ be Riemannian manifolds with $2 \le \dim M \le \dim \tilde M$ and $\gamma\colon [0,b] \to M$, respectively $\tilde\gamma\colon [0,b]\to \tilde M$ normal geodesic curves. Furthermore let J and $\tilde J$ be Jacobi fields along γ, resp. $\tilde\gamma$ with $J(0), \tilde J(0)$ parallel to $\gamma(0)$, resp. $\tilde\gamma(0)$ such that:*

$$\big|J(0)\big| = \big|\tilde J(0)\big|, \quad \big\langle \nabla_D J(0), \dot\gamma(0)\big\rangle = \big\langle \nabla_D \tilde J(0), \dot{\tilde\gamma}(0)\big\rangle, \quad \big|\nabla_D J(0)\big| = \big|\nabla_D \tilde J(0)\big|.$$

Suppose that there are no points along $\tilde\gamma$ conjugate to $\tilde\gamma(0)$ and that the curvature of M along γ does not exceed the curvature of $\tilde M$ along $\tilde\gamma$, i.e., for any $t \in [0,b]$ and for all planes $E \subset T_{\gamma(t)}M$ with $\dot\gamma(t) \in E$, resp. $\tilde E \subset T_{\tilde\gamma(t)}\tilde M$ with $\dot{\tilde\gamma}(t) \in \tilde E$ the sectional curvatures of the planes E, $\tilde E$ satisfy the inequality $\mathrm{Riem}^M(E) \le \mathrm{Riem}^{\tilde M}(\tilde E)$. *Then for all $t\in[0,b]$,*

$$\big|J(t)\big| \ge \big|\tilde J(t)\big|.$$

PROOF. (1) It is sufficient to prove the statements for Jacobi fields $J, \tilde J$ such that $J(0) = 0$, $\tilde J(0) = 0$ and $\langle J, \dot\gamma\rangle = \langle \tilde J, \dot{\tilde\gamma}\rangle \equiv 0$, since by (2.1.18) one has

$$J = J^\perp + (c_1 + t\,c_2)\,\dot\gamma \quad \text{and} \quad \tilde J = \tilde J^\perp + (\tilde c_1 + t\,\tilde c_2)\,\dot{\tilde\gamma};$$

but by assumption $J_0^\perp = 0$, $\tilde J_0^\perp = 0$, as well as $c_1 = |J(0)| = |\tilde J(0)| = \tilde c_1$ and

$$c_2 = D\langle J, \dot\gamma\rangle = \langle \nabla_D J, \dot\gamma\rangle = \langle \nabla_D J(0), \dot\gamma(0)\rangle = \langle \nabla_D \tilde J(0), \dot{\tilde\gamma}(0)\rangle = \langle \nabla_D \tilde J, \dot{\tilde\gamma}\rangle = \tilde c_2.$$

Hence if $|J^\perp(t)| \ge |\tilde J^\perp(t)|$ is shown, we have because of $\langle J, \dot\gamma\rangle(t) = \langle \tilde J, \dot{\tilde\gamma}\rangle(t)$ also $|J(t)| \ge |\tilde J(t)|$ for $t \in [0,b]$. On the other hand, since $\nabla_D(J^\perp) = (\nabla_D J)^\perp$, resp. $\nabla_D(\tilde J^\perp) = (\nabla_D \tilde J)^\perp$, it is easy to see that with J and $\tilde J$ also $J^\perp$ and $\tilde J^\perp$ satisfy the assumptions of the theorem.

In addition, we may assume that $|\nabla_D J(0)| = |\nabla_D \tilde J(0)| > 0$, since in the case $|\nabla_D J(0)| = |\nabla_D \tilde J(0)| = 0$ we have $|J| = |\tilde J| = 0$, and the claim trivially holds true.

(2) Letting $h(t) := |J(t)|^2$ and $\tilde h(t) := |\tilde J(t)|^2$, then $h(t)/\tilde h(t)$ for $t \in \,]0,b]$ is well-defined, since along $\tilde\gamma$ there are conjugate points to $\tilde\gamma(0)$. An application of l'Hospital's rule then gives

$$\lim_{t\to 0}\frac{h(t)}{\tilde h(t)} = \lim_{t\to 0}\frac{h''(t)}{\tilde h''(t)} = \lim_{t\to 0}\frac{\langle \nabla_D\nabla_D J, J\rangle(t) + \langle \nabla_D J, \nabla_D J\rangle(t)}{\langle \nabla_D\nabla_D \tilde J, \tilde J\rangle(t) + \langle \nabla_D \tilde J, \nabla_D \tilde J\rangle(t)} = \frac{|\nabla_D J(0)|^2}{|\nabla_D \tilde J(0)|^2} = 1,$$

and for the verification of $|\tilde J| \le |J|$ it is sufficient to check $\frac{d}{dt}\big(h(t)/\tilde h(t)\big) \ge 0$ on $]0,b]$, or equivalently: $h'\tilde h \ge h\,\tilde h'$ on $]0,b]$.

To this end, we fix $t_0 \in \,]0,b]$ for the rest of the proof and show that

$$h'(t_0)\,\tilde h(t_0) \geq h(t_0)\,\tilde h'(t_0). \tag{2.1.24}$$

Without loss of generality, we may assume $h(t_0) > 0$ and $\tilde h(t_0) > 0$: For instance, if $h(t_0) = 0$, then $h'(t_0) = 2\big\langle \nabla_D J(t_0), J(t_0)\big\rangle = 0$ and (2.1.24) holds trivially; analogously for $\tilde h(t_0) = 0$.

(3) Considering the vector fields $X := \frac{J}{|J(t_0)|}$ and $\tilde X := \frac{\tilde J}{|\tilde J(t_0)|}$ along γ, resp. along $\tilde\gamma$, we have:

$$\begin{aligned}\frac{h'(t_0)}{h(t_0)} &= \langle X, X\rangle'(t_0) = \int_0^{t_0} \langle X, X\rangle''\,dt \\ &= 2\int_0^{t_0} \Big\{\langle \nabla_D X, \nabla_D X\rangle - \big\langle R(X,\dot\gamma)\dot\gamma\,, X\big\rangle\Big\}\,dt = 2\,I_{t_0}(X,X)\end{aligned}$$

where $I_{t_0}(X,X) = I\big(X|[0,t_0], X|[0,t_0]\big)$; analogously it holds $\tilde h'(t_0)/\tilde h(t_0) = 2\,I_{t_0}(\tilde X, \tilde X)$.

To verify (2.1.24) it is hence sufficient to show $I_{t_0}(\tilde X, \tilde X) \leq I_{t_0}(X,X)$.

(4) We choose parallel orthonormal bases $e = (e_1, \ldots, e_n)$ and $\tilde e = (\tilde e_1, \ldots, \tilde e_{n+k})$ (where $n + k = \dim \tilde M$) along γ, resp. $\tilde\gamma$, such that

$$e_1 = \dot\gamma, \quad e_2(t_0) = X(t_0) \quad \text{and} \quad \tilde e_1 = \dot{\tilde\gamma}, \quad \tilde e_2(t_0) = \tilde X(t_0).$$

To each vector field $A \in \Gamma(\gamma^* TM)$ we associate a vector field $\iota A \in \Gamma(\tilde\gamma^* T\tilde M)$ via

$$A = \sum_{i=1}^n a^i\, e_i \;\mapsto\; \iota A = \sum_{i=1}^n a^i\, \tilde e_i.$$

Denoting by $\iota_0 \colon T_{\gamma(0)}M \to T_{\tilde\gamma(0)}\tilde M$ the isometric embedding defined by $e_i(0) \mapsto \tilde e_i(0)$, we have $(\iota A)(t) = (\widetilde{/\!/}_{0,t} \circ \iota_0 \circ /\!/_{t,0})A(t) =: \iota_t\, A(t)$ with $/\!/_{t,0}$ and $\widetilde{/\!/}_{t,0}$ the corresponding parallel transports along γ, resp. along $\tilde\gamma$. In particular for $A, B \in \Gamma(\gamma^* TM)$ it holds

$$\langle \iota A, \iota B\rangle = \langle A, B\rangle \quad \text{and} \quad \nabla_D(\iota A) = \iota\nabla_D A.$$

By the curvature assumption and the fact that both geodesics are normal, we hence conclude $I_{t_0}(\iota X, \iota X) \leq I_{t_0}(X,X)$.

On the other hand, $\tilde X, \iota X$ are both vector fields along $\tilde\gamma$, and $\tilde X$ a Jacobi field, hence the assumptions of the Index Lemma (Theorem 2.1.37) are satisfied. In this situation the Index Lemma then gives

$$I_{t_0}(\tilde X, \tilde X) \leq I_{t_0}(\iota X, \iota X) \leq I_{t_0}(X,X)$$

which completes the proof of the Theorem. □

COROLLARY 2.1.40 (Comparison Principle). *Let (M,g), $(\tilde M, \tilde g)$ be Riemannian manifolds such that $2 \leq \dim M \leq \dim \tilde M$ and let $\gamma\colon [0,b] \to M$, resp. $\tilde\gamma\colon [0,b] \to \tilde M$ be normal geodesic curves. If*

$$\mathrm{Riem}^M(E) \leq \mathrm{Riem}^{\tilde M}(\tilde E)$$

for all planes $E \subset T_{\gamma(t)}M$ with $\dot\gamma(t) \in E$, resp. $\tilde E \subset T_{\tilde\gamma(t)}\tilde M$ with $\dot{\tilde\gamma}(t) \in \tilde E$ and all $t \in [0,b]$, then along γ the first conjugate point to $\gamma(0)$ does not appear before the first conjugate point to $\tilde\gamma(0)$ along $\tilde\gamma$.

PROOF. We assume that $\tilde{\gamma}$ has no conjugate points $\tilde{\gamma}(0)$ along $\tilde{\gamma}$ on $[0, t_0]$. Let J be a Jacobi field along γ with $J(0) = 0$, but $J \neq 0$. Then $\nabla_D J(0) \neq 0$, and we choose a Jacobi field $\tilde{J}$ along $\tilde{\gamma}$ with $\tilde{J}(0) = 0$ such that

$$\big\langle \nabla_D J(0), \dot{\gamma}(0) \big\rangle = \big\langle \nabla_D \tilde{J}(0), \dot{\tilde{\gamma}}(0) \big\rangle, \quad \big|\nabla_D J(0)\big| = \big|\nabla_D \tilde{J}(0)\big|.$$

Then $|J(t)| \geq |\tilde{J}(t)| > 0$ for $t \in \,]0, t_0]$ where the fist inequality comes from the Comparison Theorem of Rauch, the second inequality holds according to Theorem 2.1.34. Applying Theorem 2.1.34 one more time then shows that also $\gamma|[0, t_0]$ has no points conjugate to $\gamma(0)$ along γ. $\square$

For a given manifold in general there there are topological obstructions for the existence of a Riemannian metric satisfying certain curvature conditions. For instance, negatively curved metrically complete Riemannian manifolds, which in addition are simply connected, are necessarily topologically trivial, as is shown in the next Theorem. We always assume metrically complete Riemannian manifolds to be connected.

THEOREM 2.1.41 (Theorem of Hadamard-Cartan). *Any simply connected, metrically complete Riemannian manifold* (M, g) *of curvature* $\mathrm{Riem}^M \leq 0$ *is diffeomorphic to* $\mathbb{R}^n$. *More precisely: If* (M, g) *is a metrically complete Riemannian manifold with* $\mathrm{Riem}^M \leq 0$*, then* $\exp_x \colon T_x M \to M$ *is a covering for each* $x \in M$*, and hence a diffeomorphism if* M *is in addition simply connected.*

A differentiable map $f \colon \tilde{M} \to M$ between manifolds is said to be a *covering*, if to each point $x \in M$ there exists an open neighbourhood U such that $f^{-1}U = \dot{\bigcup}_{i \in I} \tilde{U}_i$ for some disjoint family $(\tilde{U}_i)_{i \in I}$ of open sets $\tilde{U}_i$ in $\tilde{M}$ with the property that $f|\tilde{U}_i : \tilde{U}_i \xrightarrow{\sim} U$ is a diffeomorphism for each $i \in I$.

PROOF OF THEOREM 2.1.41. (1) Let (M, g) be metrically complete and $x \in M$. According to the Theorem of Hopf-Rinow, $\exp_x$ is defined on all of $T_x M$ and surjective. If in addition $\mathrm{Riem}^M \leq 0$, then $\mathrm{Conj}(x) = \varnothing$ by the Comparison Principle with $(\mathbb{R}^n, \mathrm{eucl})$ as comparison manifold. Hence $\exp_x \colon T_x M \to M$ is a local diffeomorphism und $(T_x M, \exp_x^* g)$ a metrically complete Riemannian manifold: Metric completeness follows from the Theorem of Hopf-Rinow; geodesic curves emanating from $0 \in T_x M$ correspond to the half-rays starting at the origin.

(2) It is hence sufficient to show: Each local isometry $f \colon (\tilde{M}, \tilde{g}) \to (M, g)$ between Riemannian manifolds of the same dimension is already a covering in case $(\tilde{M}, \tilde{g})$ is metrically complete. Let now $x \in M$; we show that there exists a connected open neighbourhood U of x in M such that f maps each connected component $\tilde{U}_i$ of $f^{-1}U$ diffeomorphically onto U. To this end, we choose $r > 0$ sufficiently small such that $\exp_x$ maps the r-ball $V_r(0_x)$ about $0_x \in T_x M$ diffeomorphically onto the geodesic r-ball $B_r(x) =: U \subset M$ about x. If $f^{-1}\{x\} = \{\tilde{x}_i : i \in I\}$ let $\tilde{U}_i := B_r(\tilde{x}_i) \subset \tilde{M}$; we claim

$$\dot{\bigcup_{i \in I}} \tilde{U}_i = f^{-1}U \quad \text{and} \quad f|\tilde{U}_i : \tilde{U}_i \xrightarrow{\sim} U.$$

Firstly, for fixed $i \in I$, we have $\exp_x \circ df_{\tilde{x}_i} = f \circ \exp_{\tilde{x}_i}$: indeed if $v \in T_{\tilde{x}_i}\tilde{M}$ and if γ is the geodesic with $\dot{\gamma}(0) = v$, then $f \circ \gamma$ is a geodesic on M, since f maps as local isometry geodesics to geodesics; thus $(f \circ \gamma)(t) = \exp_x(tw)$ with $w := df_{\tilde{x}_i} v \in T_x M$ and hence

$(f \circ \exp_{\tilde{x}_i})(v) = (f \circ \gamma)(1) = \exp_x(df_{\tilde{x}_i} v)$. The diagram

$$\begin{tikzcd} T_{\tilde{x}_i}(\tilde{M}) \arrow[r, "\exp_{\tilde{x}_i}"] \arrow[d, "df_{\tilde{x}_i}"', "\wr"] & \tilde{M} \arrow[d, "f"'] \\ T_x M \arrow[r, "\exp_x"] & M \end{tikzcd} \tag{2.1.25}$$

consequently commutes and in particular also (M, g) is metrically complete. Restriction of the maps in (2.1.25) gives

$$\begin{tikzcd} V_r(0_{\tilde{x}_i}) \arrow[r, "\exp_{\tilde{x}_i}"] \arrow[d, "df_{\tilde{x}_i}"', "\wr"] & B_r(\tilde{x}_i) = \tilde{U}_i \arrow[d, "f"'] \\ V_r(0_x) \arrow[r, "\exp_x", "\sim"'] & B_r(x) = U, \end{tikzcd} \tag{2.1.26}$$

since $\exp_x \circ df_{\tilde{x}_i}$ maps $V_r(0_{\tilde{x}_i})$ diffeomorphically onto $B_r(x) = U$, hence $\exp_{\tilde{x}_i}$ maps $V_r(0_{\tilde{x}_i})$ diffeomorphically to $B_r(\tilde{x}_i)$ and consequently $f|\tilde{U}_i : \tilde{U}_i \xrightarrow{\sim} U$ is a diffeomorphism.

Trivially $\bigcup_{i\in I} \tilde{U}_i \subset f^{-1}U$; we want to verify $f^{-1}U \subset \bigcup_{i\in I} \tilde{U}_i$. To this end, let $\tilde{y} \in f^{-1}U$ and $y := f(\tilde{y})$. We consider the minimal normal geodesic $c\colon [0, t_0] \to M$ connecting y and x; it holds $t_0 = d(x, y) < r$. To $w = \dot{c}(0) \in T_y M$ there is a unique tangent vector $v \in T_{\tilde{y}}\tilde{M}$ with $df_{\tilde{y}}\, v = w$. By the metric completeness, the geodesic $\tilde{c}(t) := \exp_{\tilde{y}}(tv)$ on $\tilde{M}$ is defined on all of $\mathbb{R}$; by construction $f \circ \tilde{c} = c$. Hence $(f \circ \tilde{c})(t_0) = c(t_0) = x$, and thus $\tilde{c}(t_0) = \tilde{x}_i$ for some $i \in I$. But since $d(\tilde{x}_i, \tilde{y}) \le t_0 < r$, we have $\tilde{y} \in B_r(\tilde{x}_i) = \tilde{U}_i$.

It remains to show that $\tilde{U}_i \cap \tilde{U}_j = \varnothing$ for $i \neq j$. Let $\gamma\colon [0, t_1] \to \tilde{M}$ the minimal normal geodesic that connects $\tilde{x}_i$ and $\tilde{x}_j$. Then $f \circ \gamma$ is a closed geodesic curve on M with x as initial and end point. Hence $f \circ \gamma$ does not lie in the geodesic ball $U = B_r(x)$ and must hence have length $> 2r$. But this shows $t_1 = d(\tilde{x}_i, \tilde{x}_j) > 2r$. ☐

Let (M, g) be a metrically complete Riemannian manifold, $x_0 \in M$ and $\gamma\colon [0, b] \to M$ a normal geodesic curve such that $\gamma(0) = x_0$. If for $t_0 \in \,]0, b]$ the point $x := \gamma(t_0)$ is not conjugate to $x_0 = \gamma(0)$ along γ, then $\exp_{x_0}\colon T_{x_0}M \to M$ maps an open neighbourhood of $t_0\dot{\gamma}(0)$ diffeomorphically to an open neighbourhood of x in M. Hence, if there are no points conjugate to $\gamma(0)$ along γ, then $\exp_{x_0}$ maps an open neighbourhood V of the ray $\{t\dot{\gamma}(0)\colon 0 \le t \le b\}$ locally diffeomorphically to an open neighbourhood U of $\gamma\big([0, b]\big)$ in M.

For instance, if $\gamma\colon [0, b] \to M$ is a normal geodesic starting at x_0 which does not hit the cut locus $C(x_0)$, then by Corollary 2.1.38 there are no conjugate points to $\gamma(0) = x_0$ along γ; in addition we may choose V and U such that $\exp_{x_0}$ maps V diffeomorphically to U. In particular, $r = |\cdot| \circ (\exp_{x_0}|V)^{-1}$ is well-defined on U, and coincides there with the distance function $d(x_0, \cdot)$, i.e.

$$r = d(x_0, \cdot) = |\cdot| \circ (\exp_{x_0}|V)^{-1}; \tag{2.1.27}$$

consequently r is differentiable on $U \backslash \{x_0\}$.

Assuming that $\gamma\colon [0, b] \to M$ does not hit the cut locus of $\gamma(0)$, we fix $x = \gamma(t_0)$ with $t_0 \in \,]0, b]$ and consider for $u \in T_x M$ the geodesic curve $c\colon \,]{-\varepsilon}, \varepsilon[\, \to U \subset M$ satisfying $c(0) = x$ and $\dot{c}(0) = u$. Through the induced curve $\beta := \frac{1}{t_0}(\exp_{x_0}|V)^{-1} \circ c$ in $T_x M$ we obtain a Jacobi variation of $\gamma|[0,t_0]$, namely

$$\alpha\colon [0,t_0]\times\,]{-\varepsilon},\varepsilon[\,\to M,\quad \alpha(t,s):=\exp_{x_0}\big(t\,\beta(s)\big).$$

It holds $\alpha(0,\cdot)=x_0$ and $\alpha(t_0,\cdot)=c$; hence $X:=\alpha_*D_2(\cdot,0)\in\Gamma(\gamma^*TM)$ is the unique Jacobi field with $X(0)=0$ and $X(t_0)=u$. Furthermore, we have

$$L\big(\alpha(\cdot,s)\big)=t_0\,|\beta(s)|=|\cdot|\circ(\exp_{x_0}|V)^{-1}\big(c(s)\big)=(r\circ c)(s).$$

By the formulas (2.1.2) and (2.1.11) for the first and second variation of length, we then have:

$$\begin{aligned}(dr)_x(u)&=(r\circ c)'(0)=\langle X,\dot\gamma\rangle\big|_0^{t_0}\\(\nabla dr)_x(u,u)&=(r\circ c)''(0)=I_{t_0}(X^\perp,X^\perp)\end{aligned}\tag{2.1.28}$$

where $X^\perp$ denotes the orthogonal component of the Jacobi fields X along $\gamma|[0,t_0]$.

We want to use (2.1.28) to derive comparison theorems for the Hessian ∇dr depending on curvature relations.

To this end, let $(\tilde M,\tilde g)$ be a further metrically complete Riemannian manifold with $\dim M\le\dim\tilde M$ and $\tilde\gamma\colon[0,b]\to\tilde M$ an additional normal geodesic curve. To put M and $\tilde M$ in relation, as in part (4) of the proof to Theorem 2.1.39, we choose an isometric embedding $\iota_0\colon T_{\gamma(0)}M\to T_{\tilde\gamma(0)}\tilde M$ with $\iota_0\big(\dot\gamma(0)\big)=\dot{\tilde\gamma}(0)$ and extend it via parallel transport to isometric embeddings $\iota_t\colon T_{\gamma(t)}M\to T_{\tilde\gamma(t)}\tilde M$:

$$\begin{array}{ccc}T_{\gamma(0)}M & \xhookrightarrow{\iota_0} & T_{\tilde\gamma(0)}\tilde M\\ {\scriptstyle //_{0,t}}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle \widetilde{//}_{0,t}}\\ T_{\gamma(t)}M & \overset{\iota_t}{\dashrightarrow} & T_{\tilde\gamma(t)}\tilde M\end{array}\tag{2.1.29}$$

In this way an isometric bundle embedding $\iota\colon\gamma^*TM\to\tilde\gamma^*T\tilde M$ over $\mathbb R$ with the properties is obtained:

$$\iota\,\dot\gamma=\dot{\tilde\gamma}\quad\text{and}\quad\nabla_D(\iota A)=\iota\nabla_D A\quad\text{for }A\in\Gamma(\gamma^*TM).$$

We assume that there is no cut point of $x_0=\gamma(0)$ along γ and no cut point of $\tilde x_0=\tilde\gamma(0)$ along $\tilde\gamma$, and fix for some $t_0\in\,]0,b]$ the points $x=\gamma(t_0)\in M$, resp. $\tilde x=\tilde\gamma(t_0)\in\tilde M$. The functions

$$r=|\cdot|\circ(\exp_{x_0}|V)^{-1}\quad\text{and}\quad\tilde r=|\cdot|\circ(\exp_{\tilde x_0}|\tilde V)^{-1}\tag{2.1.30}$$

are defined according to (2.1.27). By definition, then $r(x)=\tilde r(\tilde x)=t_0$, and for the differentials of r and $\tilde r$ at x, resp. $\tilde x$ the following result holds:

LEMMA 2.1.42. *Keeping the notions from above, it holds*

$$d(f\circ r)_x=\iota_{t_0}^*\,d(f\circ\tilde r)_{\tilde x}\equiv d(f\circ\tilde r)_{\tilde x}\circ\iota_{t_0}$$

for each C^1-function $f\colon[0,\infty[\,\to\mathbb R$.

PROOF. By the chain rule the claim is reduced to the case $f(t)=t$. Let now $u\in T_xM$. Consider along $\gamma|[0,t_0]$ the Jacobi field X with $X(0)=0$ and $X(t_0)=u$, and along $\tilde\gamma|[0,t_0]$ the Jacobi field $\tilde X$ with $\tilde X(0)=0$ and $\tilde X(t_0)=\iota_{t_0}u$. Since $\langle X,\dot\gamma\rangle=\langle\iota X,\iota\dot\gamma\rangle=\langle\tilde X,\dot{\tilde\gamma}\rangle$, we obtain with the first part of (2.1.28) the claim:

$$(dr)_x(u)=\langle X,\dot\gamma\rangle\big|_0^{t_0}=\langle\tilde X,\dot{\tilde\gamma}\rangle\big|_0^{t_0}=(d\tilde r)_{\tilde x}(\iota_{t_0}u).$$

□

The Hessians are however no longer equal in the sense above, but they can be estimated against each other by means of curvature relations. To this end, we use the following notation:

NOTATION 2.1.43. For symmetric bilinear forms

$$b \in \Gamma\big(\gamma^*(T^*M \otimes T^*M)\big), \quad \tilde{b} \in \Gamma\big(\tilde{\gamma}^*(T^*\tilde{M} \otimes T^*\tilde{M})\big)$$

along γ, resp. along $\tilde{\gamma}$, we write

$$b \succcurlyeq \tilde{b},$$

if for each $t \in [0,b]$ and each isometric bundle embedding $\iota\colon \gamma^*TM \to \tilde{\gamma}^*T\tilde{M}$ over $\mathbb{R}$, which as in (2.1.29) is induced by parallel transport from an isometric embedding

$$\iota_0\colon T_{\gamma(0)}M \to T_{\tilde{\gamma}(0)}\tilde{M}$$

with $\iota_0\big(\dot{\gamma}(0)\big) = \dot{\tilde{\gamma}}(0)$, the symmetric bilinear form $b_{\gamma(t)} - \iota_t^*\tilde{b}_{\tilde{\gamma}(t)}$ on $T_{\gamma(t)}M$ is positive semidefinite. In other words:

$$b \succcurlyeq \tilde{b} \iff b_{\gamma(t)}(u,u) \geq \tilde{b}_{\tilde{\gamma}(t)}(\tilde{u},\tilde{u}) \quad \text{for } t \in [0,b]\,,\ u \in T_{\gamma(t)}M,\ \tilde{u} \in T_{\tilde{\gamma}(t)}\tilde{M}: \\ |u| = |\tilde{u}|, \quad \langle u, \dot{\gamma}(t)\rangle = \langle \tilde{u}, \dot{\tilde{\gamma}}(t)\rangle.$$

THEOREM 2.1.44 (Hessian Comparison Theorem). *Let (M,g) and $(\tilde{M},\tilde{g})$ be Riemannian manifolds with $2 \leq \dim M \leq \dim \tilde{M}$, and let $\gamma\colon [0,b] \to M$, resp., $\tilde{\gamma}\colon [0,b] \to \tilde{M}$ be minimal normal geodesic curves. If then the curvature of M along γ does not exceed the curvature of $\tilde{M}$ along $\tilde{\gamma}$, in the sense that always*

$$\mathrm{Riem}^M(E) \leq \mathrm{Riem}^{\tilde{M}}(\tilde{E})$$

for $t \in [0,b[$ and all planes $E \subset T_{\gamma(t)}M$ with $\dot{\gamma}(t) \in E$, resp. $\tilde{E} \subset T_{\tilde{\gamma}(t)}\tilde{M}$ with $\dot{\tilde{\gamma}}(t) \in \tilde{E}$, then for any isotone C^2-function $f\colon [0,\infty[\to \mathbb{R}$:

$$\nabla d(f \circ r)_{\gamma(t)} \succcurlyeq \nabla d(f \circ \tilde{r})_{\tilde{\gamma}(t)}, \quad t \in \,]0,b[, \tag{2.1.31}$$

where $r = d(x_0,\cdot)$ and $\tilde{r} = \tilde{d}(\tilde{x}_0,\cdot)$ denote the distance functions from $x_0 = \gamma(0)$ in M, resp. from $\tilde{x}_0 = \tilde{\gamma}(0)$ in $\tilde{M}$.

REMARK 2.1.45. (1) The assumed minimality of $\gamma\colon [0,b] \to \tilde{M}$ has as consequence that for $t < b$ no $\gamma(t)$ is a cut point of $\gamma(0)$; analogously for $\tilde{\gamma}$. Obviously (2.1.31) also holds for $t = b$, if r and $\tilde{r}$ are differentiable at $\gamma(b)$, resp. at $\tilde{\gamma}(b)$.

(2) It would be sufficient to assume that there are no conjugate points to $\tilde{\gamma}(0)$ along $\tilde{\gamma}$. By the Comparison Theorem of Rauch, along with Theorem 2.1.34, then also no $\gamma(t)$ is conjugate to $\gamma(0)$ along γ. However $\exp_{x_0}|V$ and $\exp_{\tilde{x}_0}|\tilde{V}$ may then be no longer invertible; for fixed t then $\nabla d(f \circ r)_{\gamma(t)}$ needs to be replaced by $\rho = (\exp_{x_0}|V_{\mathrm{loc}})^{-1}$ with V_{loc} a sufficiently small neighbourhood of $t\dot{\gamma}(0)$, which is mapped by $\exp_{x_0}$ diffeomorphically to an open neighbourhood of $\gamma(t)$; the right-hand side of (2.1.31) should then be interpreted correspondingly.

PROOF OF THEOREM 2.1.44. Let $t_0 \in \,]0,b]$, to this $x = \gamma(t_0)$ and $\tilde{x} = \tilde{\gamma}(t_0)$. Furthermore, let $u \in T_xM$ and $\iota\colon \gamma^*TM \to \tilde{\gamma}^*T\tilde{M}$ an isometric bundle embedding over $\mathbb{R}$ constructed according to (2.1.29). We then have to show that

$$\nabla d(f \circ r)_x(u,u) \geq \nabla d(f \circ \tilde{r})_{\tilde{x}}(\iota_{t_0}u, \iota_{t_0}u).$$

By formula (1.7.2) one obtains at first

$$\nabla d(f \circ r)_x(u,u) = f''(t_0)\,(dr)_x(u) + f'(t_0)\,(\nabla dr)_x(u,u)$$

$$\nabla d(f \circ \tilde{r})_{\tilde{x}}(\iota_{t_0} u, \iota_{t_0} u) = f''(t_0)\,(d\tilde{r})_{\tilde{x}}(\iota_{t_0} u) + f'(t_0)\,(\nabla d\tilde{r})_{\tilde{x}}(\iota_{t_0} u, \iota_{t_0} u).$$

The first two summands are equal by Lemma 2.1.42; since by assumption $f'(t_0) \geq 0$, it remains to show that $(\nabla dr)_x(u,u) \geq (\nabla d\tilde{r})_{\tilde{x}}(\iota_{t_0} u, \iota_{t_0} u)$, by (2.1.28) hence to verify that

$$I_{t_0}(X^\perp, X^\perp) \geq I_{t_0}(\tilde{X}^\perp, \tilde{X}^\perp);$$

here $X \in \Gamma(\gamma^* TM)$ is the Jacobi field with $X(0) = 0$, $X(t_0) = u$ and $\tilde{X} \in \Gamma(\tilde{\gamma}^* T\tilde{M})$ the Jacobi field with $\tilde{X}(0) = 0$, $\tilde{X}(t_0) = \iota_{t_0} u$. In terms of the vector field $Y := \iota X^\perp \in \Gamma(\tilde{\gamma}^* T\tilde{M})$ however, it holds

$$I_{t_0}(X^\perp, X^\perp) \geq I_{t_0}(Y, Y) \geq I_{t_0}(\tilde{X}^\perp, \tilde{X}^\perp)$$

and thus the claim: The first inequality follows directly from the Definition of the index form, combined with the observations that $|\nabla_D X^\perp| = |\nabla_D Y|$ and that $\big\langle R(X^\perp, \dot{\gamma})\,\dot{\gamma}, X^\perp \big\rangle \leq \big\langle R(Y, \dot{\tilde{\gamma}})\,\dot{\tilde{\gamma}}, Y \big\rangle$ by the assumptions on the sectional curvatures; the second inequality is a consequence of the Index Lemma (Theorem 2.1.37), since Y equals $\tilde{X}^\perp$ at 0 and t_0. □

COROLLARY 2.1.46 (Comparison Theorem for the Laplacian: basic version). *Let (M, g), $(\tilde{M}, \tilde{g})$ be Riemannian manifolds with $2 \leq \dim M \leq \dim \tilde{M}$ and let $\gamma\colon [0, b] \to M$, resp., $\tilde{\gamma}\colon [0, b] \to \tilde{M}$ be minimal normal geodesic curves. If then*

$$\mathrm{Riem}^M(E) \leq \mathrm{Riem}^{\tilde{M}}(\tilde{E})$$

for all planes $E \subset T_{\gamma(t)} M$ with $\dot{\gamma}(t) \in E$, resp.. $\tilde{E} \subset T_{\tilde{\gamma}(t)} \tilde{M}$ with $\dot{\tilde{\gamma}}(t) \in \tilde{E}$ and all $t \in [0, b]$, then for each isotone C^2-function $f\colon [0, \infty[\, \to \mathbb{R}$ the inequality

$$\Delta(f \circ r)\big(\gamma(t)\big) \geq \Delta(f \circ \tilde{r})\big(\tilde{\gamma}(t)\big), \quad t \in\,]0, b[,$$

holds, where $r = d(x_0, \cdot)$ and $\tilde{r} = \tilde{d}(\tilde{x}_0, \cdot)$ denote the distance functions from $x_0 = \gamma(0)$ in M, resp. from $\tilde{x}_0 = \tilde{\gamma}(0)$ in $\tilde{M}$.

PROOF. The claim follows from Theorem 2.1.44 by taking trace. □

Comparison theorems are typically applied by comparing a given Riemannian manifold to simply structured standard manifolds. This procedure obviously depends on the explicit knowledge of suitable comparison manifolds. An important type of model manifolds are covered by the following definition (see [**14**]).

DEFINITION 2.1.47 (Model, rotationally symmetric manifold). Let $(\mathbb{M}, g)$ be an n-dimensional ($n \geq 2$) Riemannian manifold and $0 \in \mathbb{M}$ be a distinguished point. Then $(\mathbb{M}, g)$ is called a *model* with center 0 if 0 is a pole for $(\mathbb{M}, g)$ with $\mathbb{M}$ being rotationally symmetric about 0 in the sense that each linear isometry $\varphi\colon T_0\mathbb{M} \to T_0\mathbb{M}$ is he differential of an isometry $\phi\colon \mathbb{M} \to \mathbb{M}$, i.e., such that $\phi(0) = 0$ and $(d\phi)_0 = \varphi$.

Before entering the discussion on properties of models, we want to collect some facts about isometries. By Definition 1.5.4, isometries are local isometries with the additional property that they are diffeomorphisms.

REMARK 2.1.48 (on local isometries). Let (M, g) be a metrically complete Riemannian manifold and $\phi\colon M \to M$ a local isometry, i.e., $\phi^* g = g$. Then:

(i) ϕ preserves the length of curves, and hence, if ϕ is even an isometry, then also distances, i.e., then it holds: $d\big(\phi(x), \phi(y)\big) = d(x, y)$ for $x, y \in M$.
(ii) ϕ transfers geodesics in geodesics; hence in particular:

$$\phi \circ \exp_x(t\,v) = \exp_{\phi(x)}(t\,\phi_* v), \quad x \in M,\ v \in T_x M.$$

(iii) ϕ preserves the Levi-Civita connection, i.e.,

$$d\phi\, \nabla_A B = \nabla_A (d\phi B), \quad A, B \in \Gamma(TM).$$

In particular, for vector fields X along a curve c, it holds $d\phi\, \nabla_D X = \nabla_D (d\phi\, X)$, and X is hence parallel along c if and only if $d\phi\, X$ is parallel along $\phi \circ c$.

(iv) ϕ preserves the Riemannian sectional curvature, i.e., if $E = \operatorname{span}\{v, w\} \subset T_x M$ and $E' = \operatorname{span}\{\phi_* v, \phi_* w\} \subset T_{\phi(x)} M$, then $\mathrm{Riem}^M_x(E) = \mathrm{Riem}^M_{\phi(x)}(E')$.

PROOF. (i) is a direct consequence of the definition of the length functional. (ii) follows from (i) since ϕ is a local diffeomorphism. (iii) follows from formula (1.7.1) and the observation that ϕ is affine, since ϕ maps geodesics to geodesics. (iv) finally is a consequence of (iii) and the second Cartan structural equation (see Theorem 1.4.27). $\square$

THEOREM 2.1.49. *Let $(\mathbb{M}, g)$ be a model and $\gamma\colon [0, b] \to \mathbb{M}$ a geodesic curve emanating from the distinguished point $0 \in \mathbb{M}$. Then each each proper Jacobi field along γ, which vanishes at 0, is up to a scalar function a parallel vector field along γ. In particular, two Jacobi fields along γ, vanishing at 0, are already orthogonal along γ if they are orthogonal at one place.*

PROOF. Let J be a Jacobi field along γ such that $\langle J, \dot\gamma\rangle = 0$ and $J(0) = 0$; denote $v = \dot\gamma(0) \in T_0\mathbb{M}$ and $w = J'(0)$. Since J is a proper Jacobi field, we have $v \perp w$ in $T_0\mathbb{M}$. With the identifications $\mathbb{M} \cong T_0\mathbb{M}$ via $\exp_0$, and correspondingly $T_{\gamma(t)}\mathbb{M} \cong T_{tv}T_0\mathbb{M} \cong T_0\mathbb{M}$, it holds that $\gamma(t) = \exp_0(tv) \equiv tv$ and $J(t) = (d\exp_0)_{tv}(tw) \equiv tw \in T_{tv}\mathbb{M}$. We have to show that the vector field W along γ given by $W(t) := w \in T_{tv}\mathbb{M}$, coincides up to multiplication by a scalar function with the parallel transport of $w \in T_0\mathbb{M}$ along γ. To this end, we consider the two-dimensional submanifold

$$\mathbb{M}_0 := \exp_0(\mathbb{R}v + \mathbb{R}w) \cong \mathbb{R}v + \mathbb{R}w \subset \mathbb{M}$$

with the induced Riemannian metric. Now γ is also a geodesic in $\mathbb{M}_0$ and it holds $W(t) \perp \dot\gamma(t)$ in $T_0\mathbb{M}_0$. By the isometry of the parallel transport with respect to the Levi-Civita connection, then $\bar W := W/|W|$ must be parallel along γ in $\mathbb{M}_0$. It remains to show that $\bar W$ is also parallel along γ in $\mathbb{M}$. To this end, it is sufficient to show that $\mathbb{M}_0$ as submanifold of $\mathbb{M}$ is totally geodesic, since the inclusion $\iota\colon \mathbb{M}_0 \hookrightarrow \mathbb{M}$ is affine and then

$$\nabla_D \iota_* \bar W = \iota_* \nabla_D \bar W = 0.$$

By definition, we have $\mathbb{M}_0 = \exp_0(\mathbb{R}v + \mathbb{R}w)$ with $v \perp w$ in $T_0\mathbb{M}$. We choose a linear isometry $\varphi\colon T_0\mathbb{M} \to T_0\mathbb{M}$ with $\varphi(v) = v$ and $\varphi(w) = w$, but $\varphi(u) \neq u$ for any $u \in \{\mathbb{R}v + \mathbb{R}w\}^\perp$. Then there is an isometry $\phi\colon \mathbb{M} \to \mathbb{M}$ with $\phi(0) = 0$ and $d\phi_0 = \varphi$. According to Remark 2.1.48 (ii), $\mathbb{M}_0$ is the fixed point set of ϕ, i.e., $\mathbb{M}_0 = \{x \in \mathbb{M} : \phi(x) = x\}$. This already shows the claim since if c is a geodesic in $\mathbb{M}$ with $c(0) \in \mathbb{M}_0$, i.e. $c(t) = \exp_{c(0)}\big(t\,\dot c(0)\big)$, then c lies totally in $\mathbb{M}_0$ if and only if $\phi \circ c = c$; because of $(\phi \circ c)(t) = \exp_{c(0)}\big(t\,\phi_*\dot c(0)\big)$ this is however the case exactly if $\phi_*\dot c(0) = \dot c(0)$, or in other words, if $\dot c(0) \in T_{c(0)}\mathbb{M}_0$.

In general, we have for $X \in T_x\mathbb{M}$ with $x \in \mathbb{M}_0$ that $X \in T_x\mathbb{M}_0$ if and only if $X = \dot\beta(0)$ for a curve β in M_0 with $\beta(0) = x$; indeed, by $\phi \circ \beta = \beta$ this condition implies $\phi_* X = X$, conversely from $\phi_* X = X$ the existence of an M_0-valued curve β follows with $X = \dot\beta(0)$, e.g. $\beta(t) = \exp_x(tX)$ according to Remark (2.1.48) (ii). $\square$

LEMMA 2.1.50. *Let $(\mathbb{M}, g)$ be a model and $0 \in \mathbb{M}$ its center. Fix $x, \tilde x \in \mathbb{M}$ such that $r = d(x, 0) = d(\tilde x, 0)$, and let γ, resp. $\tilde\gamma$, be the normal geodesic curves emanating from 0 with the property that $\gamma(r) = x$ and $\tilde\gamma(r) = \tilde x$. If then $(u_1, \ldots, u_d)$ is an orthonormal* *basis for $T_x\mathbb{M}$ with $u_1 = \dot\gamma(r)$, and analogously $(\tilde u_1, \ldots, \tilde u_d)$ an orthonormal basis for $T_{\tilde x}\mathbb{M}$ with $\tilde u_1 = \dot{\tilde\gamma}(r)$, then there is an isometry $\phi\colon \mathbb{M} \to \mathbb{M}$ with $\phi(x) = \tilde x$, such that*

$$d\phi_x\, u_i = \tilde u_i, \quad i = 1, \ldots, d.$$

PROOF. Let $v := u_1 = \dot\gamma(r)$ and $\tilde v := \tilde u_1 = \dot{\tilde\gamma}(r)$. We identify $\mathbb{M} \cong T_0\mathbb{M}$, so that $x = \exp_0\big(r\,\dot\gamma(0)\big) \equiv r\,\dot\gamma(0)$ und

$$T_x\mathbb{M} \cong T_{r\dot\gamma(0)}T_0\mathbb{M} \cong T_0\mathbb{M}.$$

In this sense we understand $u_1, \ldots, u_n$ as elements of $T_0\mathbb{M}$; in particular then $v \equiv \dot\gamma(0)$. By the Gauss Lemma, we have $v \perp u_i$ for $i = 2, \ldots, n$ in $T_0\mathbb{M}$, and correspondingly $\tilde v \perp \tilde u_i$ for $i = 2, \ldots, n$ in $T_0\mathbb{M}$. On the other hand, the Jacobi fields $J_2, \ldots, J_n$ with

$$J_i(t) = (d\exp_0)_{tv}(tu_i) \equiv t\,u_i \in T_{tv}\mathbb{M}$$

are pairwise orthogonal along γ by Theorem 2.1.49; hence also $(u_2, \ldots, u_n)$ is orthogonal in $T_0\mathbb{M}$. With the same argument one obtains the orthogonality of $(\tilde u_2, \ldots, \tilde u_n)$ in $T_0\mathbb{M}$. Thus we can find a linear isometry $\varphi\colon T_0\mathbb{M} \to T_0\mathbb{M}$ such that $\varphi(v) = \tilde v$ and $\varphi(u_i) = \lambda_i\,\tilde u_i$ for $i = 2, \ldots, n$ where $\lambda_i > 0$. Since $\mathbb{M}$ is a model, there is an isometry $\phi\colon \mathbb{M} \to \mathbb{M}$ such that $\phi(0) = 0$ and $d\phi_0 = \varphi$. Because of

$$\phi \circ \exp_0(t\,v) = \exp_{\phi(0)}(t\,\phi_* v) = \exp_{\phi(0)}(t\,\varphi v) = \exp_{\phi(0)}(t\,\tilde v),$$

we have $\phi \circ \gamma = \tilde\gamma$, in particular then $\phi(x) = \tilde x$ and $d\phi_x\, v = \tilde v$.

It remains to verify that $d\phi_x\, u_i = \tilde u_i$ for $i = 2, \ldots, n$. To this end, we consider for fixed i the Jacobi field J along γ with $J(0) = 0$, $J'(0) = u_i$, and analogously $\tilde J$ the Jacobi field along $\tilde\gamma$ with $\tilde J(0) = 0$, $\tilde J'(0) = \tilde u_i$. According to Example 2.1.32, we have

$$J(t) = \tfrac{\partial}{\partial s}\big|_{s=0} \exp_0\big(t\,(v + su_i)\big) = (d\exp_0)_{tv}(tu_i) \equiv tu_i \in T_{tv}\mathbb{M},$$

and $\tilde J(t) \equiv t\,\tilde u_i \in T_{t\tilde v}\mathbb{M}$. But we have

$$\phi \circ \exp_0\big(t\,(v + su_i)\big) = \exp_0\big(t\,\varphi(v + su_i)\big) = \exp_0\big(t\,(\tilde v + s\,\lambda_i \tilde u_i)\big),$$

from where by differentiating with respect to s at $s = 0$ the relation

$$(d\phi)_{tv}\, J(t) = \lambda_i\, \tilde J(t)$$

is derived, thus $(d\phi)_{tv}\, tu_i = \lambda_i\, t\tilde u_i$. This shows in particular that $(d\phi)_{rv}\, u_i \equiv d\phi_x\, u_i = \lambda_i\, \tilde u_i$. By the isometry of $d\phi_x\colon T_x\mathbb{M} \to T_{\tilde x}\mathbb{M}$, then necessarily $\lambda_i = 1$. ☐

Let (M, g) be a metrically complete Riemannian manifold and $x_0 \in M$. The *radial vector field* $\frac{\partial}{\partial r}$ defined on $M\backslash\big(C(x_0) \cup \{x_0\}\big)$ is given by $\big(\frac{\partial}{\partial r}\big)_x = \dot\gamma(t_0)$ where γ denotes the unique minimal normal geodesic such that $\gamma(0) = x_0$ and $\gamma(t_0) = x$. Obviously, it holds $\frac{\partial}{\partial r} = \operatorname{grad} r$ on $M\backslash\big(C(x_0) \cup \{x_0\}\big)$ with $r := d(x_0, \cdot)$, since $\operatorname{grad} r$ is determined by $\langle \operatorname{grad} r, Y\rangle = Y(r)$ for each vector field Y on $M\backslash\big(C(x_0) \cup \{x_0\}\big)$; on the other hand, one has $Y = Y(r)\,\frac{\partial}{\partial r} + Y^\perp$ with $\langle \frac{\partial}{\partial r}, Y^\perp\rangle = 0$ by the Gauss Lemma, so that also $\langle \frac{\partial}{\partial r}, Y\rangle = Y(r)$ holds.

REMARK 2.1.51. Note that for an arbitrary vector field X on $M\backslash\big(C(x_0) \cup \{x_0\}\big)$ it always holds that

$$(\nabla dr)\big(X, \tfrac{\partial}{\partial r}\big) = 0. \tag{2.1.32}$$

Indeed one has $(\nabla dr)\big(X, \frac{\partial}{\partial r}\big) = X\big(\frac{\partial}{\partial r} r\big) - \big(\nabla_X \frac{\partial}{\partial r}\big)\, r$, and because of $\frac{\partial}{\partial r} r = 1$, then

$$(\nabla dr)\big(X, \tfrac{\partial}{\partial r}\big) = -\big\langle \nabla_X \tfrac{\partial}{\partial r}, \operatorname{grad} r\big\rangle = -\big\langle \nabla_X \tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}\big\rangle = -\tfrac12 X\langle \tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}\rangle = 0.$$

Instead of $\frac{\partial}{\partial r}$ we also write occasionally also ∂ or ∂^M. In the particular case that x_0 is a pole for (M, g), the corresponding radial vector field ∂^M is defined on $M \backslash \{x_0\}$.

DEFINITION 2.1.52. Under *radial curvature* of (M, g) with respect to x_0 we understand the restriction of the sectional curvature Riem^M to radial planes, i.e. planes $E \subset T_x M$ such that $\partial^M_x = \left(\frac{\partial}{\partial r}\right)_x \in E$. Planes in $T_{x_0}M$ are considered as radial by convention.

REMARK 2.1.53. In a model $(\mathbb{M}, g)$ with with center 0, the radial curvature at some point x depends only on $r = r_M(x)$ where $r_M(x) = d(0, x)$.

PROOF. This is a consequence of Lemma 2.1.50 and Remark (2.1.48) (iv) which says that isometries preserve the Riemannian sectional curvature. □

DEFINITION 2.1.54 (Radial curvature function). Let $(\mathbb{M}, g)$ be a model. The function

$$k_{\mathbb{M}} \colon \mathbb{R}_+ \to \mathbb{R}, \quad k_{\mathbb{M}}(t) := \text{radial curvature at } x \in \mathbb{M} \text{ with } r_M(x) = t,$$

is well-defined by Remark 2.1.53 and called *radial curvature function* of the model $(\mathbb{M}, g)$.

The Comparison Theorem for the Laplacian (Theorem 2.1.46) takes a simpler form in the case of a model as comparison manifold: it is then sufficient to compare the Ricci curvature along normal geodesics.

THEOREM 2.1.55 (Laplacian Comparison Theorem: special version). *Let (M, g) be a metrically complete Riemannian manifold with $n = \dim M \geq 2$, and $x_0 \in M$, as well as $\mathbb{M}$ a model of the same dimension with center $0 \in \mathbb{M}$. Let $r_M = d(x_0, \cdot)$ and $r_{\mathbb{M}} = d(0, \cdot)$ be the distance functions to x_0 in M, resp. to 0 in $\mathbb{M}$, and ∂^M resp. $\partial^{\mathbb{M}}$ the corresponding radial vector fields. Suppose that for some $R > 0$,*

$$\mathrm{Ric}^M(\partial^M, \partial^M)(x) \geq \mathrm{Ric}^{\mathbb{M}}(\partial^{\mathbb{M}}, \partial^{\mathbb{M}})(\tilde{x}) \ \big(\equiv (n-1) k_{\mathbb{M}}(r)\big)$$

for all $x \in M \backslash \big(C(x_0) \cup \{x_0\}\big)$ and $\tilde{x} \in \mathbb{M} \backslash \{0\}$ with $r = r_M(x) = r_{\mathbb{M}}(\tilde{x}) < R$. Then for each isotone C^2-function $f \colon [0, R[\to \mathbb{R}$ and all x, $\tilde{x}$ as above, it holds

$$\Delta(f \circ r_M)(x) \leq \Delta(f \circ r_{\mathbb{M}})(\tilde{x}).$$

PROOF. We follow the proof of Theorem 2.1.44. At first we remark that it is again sufficient to consider the case $f(r) = r$, since from

$$\nabla d(f \circ r_M)_x(u, u) = (f'' \circ r_M)\,(d\,r_M)_x(u) + (f' \circ r_M)\,(\nabla d\,r_M)_x(u, u)$$

for $u \in T_x M$ one concludes immediately $\Delta(f \circ r_M) = f'' \circ r_M + (f' \circ r_M)\,\Delta r_M$, and analogously $\Delta(f \circ r_{\mathbb{M}}) = f'' \circ r_{\mathbb{M}} + (f' \circ r_{\mathbb{M}})\,\Delta r_{\mathbb{M}}$.

Let now $x \in M \backslash \big(C(x_0) \cup \{x_0\}\big)$ and $\tilde{x} \in \mathbb{M} \backslash \{0\}$ such that $r = r_M(x) = r_{\mathbb{M}}(\tilde{x}) < R$; to this let $\gamma \colon [0, r] \to M$ be the geodesic emanating from x_0 with $\gamma(r) = x$, and correspondingly $\tilde{\gamma} \colon [0, r] \to \mathbb{M}$ the geodesic emanating from 0 with $\tilde{\gamma}(r) = \tilde{x}$. We fix orthonormal bases $(u_1, \ldots, u_n)$ for $T_x M$ where $u_1 = \partial^M_x$ and $(\tilde{u}_1, \ldots, \tilde{u}_n)$ for $T_{\tilde{x}}\mathbb{M}$ where $\tilde{u}_1 = \partial^{\mathbb{M}}_{\tilde{x}}$. For $i = 2, \ldots, n$ let X_i be unique (proper) Jacobi field along γ such that $X_i(0) = 0$, $X_i(r) = u_i$, and analogously $\tilde{X}_i$ the corresponding Jacobi field along $\tilde{\gamma}$, such that $\tilde{X}_i(0) = 0$, as well as $\tilde{X}_i(r) = \tilde{u}_i$. Taking (2.1.32) into account, one obtains

$$\Delta r_M(x) = \sum_{i=2}^n (\nabla d\,r_M)(u_i, u_i) = \sum_{i=2}^n I(X_i, X_i),$$

respectively,

$$\Delta r_{\mathbb{M}}(\tilde{x}) = \sum_{i=2}^{n} (\nabla d\, r_{\mathbb{M}})(\tilde{u}_i, \tilde{u}_i) = \sum_{i=2}^{n} I(\tilde{X}_i, \tilde{X}_i).$$

But $\mathbb{M}$ is a model so that for any $0 < t \leq r$ the vectors $\tilde{X}_i(t)$ pairwise orthogonal and in addition $|\tilde{X}_i(t)| = |\tilde{X}_j(t)|$: indeed, the first by Theorem 2.1.49 and the second claim, since each $\tilde{X}_i(t)$ can be transfered to $\tilde{X}_j(t)$ by the differential of an isometry which lets $\tilde{\gamma}$ invariant. Hence we have

$$\begin{aligned}\Delta r_{\mathbb{M}}(\tilde{x}) = \sum_{i=2}^{n} I(\tilde{X}_i, \tilde{X}_i) &= \sum_{i=2}^{n} \int_0^r \Big\{ |\nabla_D \tilde{X}_i|^2 - \big\langle R(\tilde{X}_i, \partial^{\mathbb{M}})\, \partial^{\mathbb{M}}, \tilde{X}_i \big\rangle \Big\}\, dt \\ &= \int_0^r \Big\{ \sum_{i=2}^{n} |\nabla_D \tilde{X}_i|^2 - |\tilde{X}_2|^2 \ \mathrm{Ric}^{\mathbb{M}}(\partial^{\mathbb{M}}, \partial^{\mathbb{M}}) \Big\}\, dt.\end{aligned}$$

If now $\iota_r \colon T_{\tilde{x}}\mathbb{M} \to T_x M$ is the linear isometry such that $\iota_r(\tilde{u}_i) = u_i$ for $i = 1, \ldots, n$ and if one extends it canonically via parallel transport according to

$$\begin{array}{ccc} T_{\tilde{\gamma}(r)}\mathbb{M} & \xhookrightarrow{\ \iota_r\ } & T_{\gamma(r)}M \\ {\scriptstyle //_{t,r}}\uparrow\wr & & \wr\uparrow{\scriptstyle //_{t,r}} \\ T_{\tilde{\gamma}(t)}\mathbb{M} & \xhookrightarrow{\ \iota_t\ } & T_{\gamma(t)}M\,, \end{array}$$

then one obtains an isometric bundle embedding $\iota\colon \tilde{\gamma}^* T\mathbb{M} \to \gamma^* TM$ over $\mathbb{R}$, which commutes with the covariant derivative ∇_D of vector fields and transfers the radial vector field along $\tilde{\gamma}$ into the radial vector field along γ. One applies now again the index lemma (Theorem 2.1.37):

$$\begin{aligned}\Delta r_M(x) &= \sum_{i=2}^{n} I(X_i, X_i) \leq \sum_{i=2}^{n} I(\iota\tilde{X}_i, \iota\tilde{X}_i) \\ &= \sum_{i=2}^{n} \int_0^r \Big\{ |\nabla_D \iota\tilde{X}_i|^2 - \big\langle R(\iota\tilde{X}_i, \partial^M)\, \partial^M, \iota\tilde{X}_i \big\rangle \Big\}\, dt \\ &= \int_0^r \Big\{ \sum_{i=2}^{n} |\iota\nabla_D \tilde{X}_i|^2 - \sum_{i=2}^{n} |\iota\tilde{X}_i|^2 \ \mathrm{Ric}^M(\partial^M, \partial^M) \Big\}\, dt \\ &= \int_0^r \Big\{ \sum_{i=2}^{n} |\nabla_D \tilde{X}_i|^2 - |\tilde{X}_2|^2 \ \mathrm{Ric}^M(\partial^M, \partial^M) \Big\}\, dt \\ &\leq \int_0^r \Big\{ \sum_{i=2}^{n} |\nabla_D \tilde{X}_i|^2 - |\tilde{X}_2|^2 \ \mathrm{Ric}^{\mathbb{M}}(\partial^{\mathbb{M}}, \partial^{\mathbb{M}}) \Big\}\, dt = \Delta r_{\mathbb{M}}(\tilde{x}),\end{aligned}$$

where the last inequality comes from the assumption on the Ricci curvature. □

An important tool for the explicit description and construction of models are Euclidean spheres, even if themselves they are not covered by the class of models. For $a > 0$ let

$$\mathbb{S}_a^n := \{x \in \mathbb{R}^{n+1} : |x| = a\} \xhookrightarrow{\ \iota\ } \mathbb{R}^{n+1}$$

be the sphere of radius a, equipped with the Riemannian metric $g = \iota^*\,\mathrm{eucl}$ induced from $(\mathbb{R}^{n+1}, \mathrm{eucl})$ (see Example 1.5.12), where occasionally for historical reasons the metric is written $g = d\vartheta^2$.

For $x \in \mathbb{S}^n_a$ we identify canonically

$$T_x\mathbb{S}^n_a \cong \{x\}^\perp \subset \mathbb{R}^{n+1},$$

where $\{x\}^\perp$ is the orthogonal complement of $\mathbb{R}x$ in $\mathbb{R}^{n+1}$. Each orthogonal transformation $A \in \mathrm{O}(n+1)$ defines by restriction

$$A|\mathbb{S}^n_a \colon \mathbb{S}^n_a \to \mathbb{S}^n_a$$

an isometry of $\mathbb{S}^n_a$. In particular, $(\mathbb{S}^n_a, d\vartheta^2)$ has constant sectional curvature, since for given $x, y \in \mathbb{S}^n_a$ and orthonormal vectors $u_1, u_2 \in T_x\mathbb{S}^n_a$, resp. $v_1, v_2 \in T_y\mathbb{S}^n_a$, there is an orthogonal transformation $A \in \mathrm{O}(n+1)$ such that $Ax = y$ and $A_* u_i = v_i$ for $i = 1, 2$. Modulo multiplication by a constant, $d\vartheta^2$ is however the only Riemannian metric on $\mathbb{S}^n_a$ invariant under the full orthogonal group $\mathrm{O}(n+1)$. For $a = 1$ we write simply $\mathbb{S}^n$ instead of $\mathbb{S}^n_1$.

REMARK 2.1.56. As a compact manifold $(\mathbb{S}^n_a, d\vartheta^2)$ is metrically complete and the maximal geodesics coincide with the great circles on $\mathbb{S}^n_a$: For instance, fix $x, y \in \mathbb{S}^n_a$, $x \neq y$ and $r = d(x, y)$ sufficiently small such that there is exactly one geodesic $\gamma\colon [0, r] \to \mathbb{S}^n_a$ with $\gamma(0) = x$ and $\gamma(r) = y$. To the plane $E = \mathbb{R}x + \mathbb{R}y$ consider now an orthogonal transformation $A \in \mathrm{O}(n+1)$ which has E as fixed point set, e.g. the mirror map at E. Then also $A \circ \gamma$ is a minimal geodesic connecting x and y, hence $A \circ \gamma = \gamma$ and γ lies on the great circle $E \cap \mathbb{S}^n_a$.

Let now $(\mathbb{M}, g)$ again be a model and $0 \in \mathbb{M}$ its center. Then

$$\exp_0 \colon (T_0\mathbb{M}, \exp_0^* g) \to (\mathbb{M}, g)$$

defines an isometry of Riemannian manifolds. Without restrictions, we may identify $\mathbb{M} \cong \mathbb{R}^n$, where the center $0 \in \mathbb{M}$ corresponds to the origin in $\mathbb{R}^n$ and where we identify $T_0\mathbb{M}$ and $\mathbb{R}^n$ isometically as Euclidean $\mathbb{R}$-vector spaces. The metric $\exp_0^* g$ restricted to $\mathbb{R}^n \backslash \{0\}$ takes under pull-back with $]0, \infty[\, \times \mathbb{S}^{n-1} \xrightarrow{\sim} \mathbb{R}^n \backslash \{0\}$, $(r, v) \mapsto rv$ by the Gauss Lemma the form $dr \otimes dr + h_r$ where h_r denotes the metric on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ induced by $\exp_0^* g$ on $\mathbb{S}^{n-1}_r$.

Recall that $\mathbb{M}$ is a model and that the metric h_r is hence invariant under the full n-dimensional orthogonal group: thus h_r coincides up to a positive constant (depending on r) with the standard metric $d\vartheta^2$ on $\mathbb{S}^{n-1}$. We write $h_r = f(r)^2\, d\vartheta^2$. With the positive function $f\colon\,]0, \infty[\, \to \mathbb{R}$ defined in this way, each n-dimensional model $(\mathbb{M}, g)$ takes the form

$$\big(\mathbb{R}^n, dr \otimes dr + f(r)^2\, d\vartheta^2\big).$$

THEOREM 2.1.57 (Elementary properties of models). *Let $(\mathbb{M}, g)$ be a model with $\mathbb{M} \cong \mathbb{R}^n$ and $g = dr \otimes dr + f(r)^2\, d\vartheta^2$ on $\mathbb{R}^n \backslash \{0\}$, as well as $k = k_{\mathbb{M}}$ the radial curvature function of $\mathbb{M}$. Then the following items hold:*

(i) (Jacobi equation) $f''(t) + k(t)\, f(t) \equiv 0$ *with* $f(0) = 0$ *and* $f'(0) = 1$.

(ii) $\nabla dr = \big((f'/f) \circ r\big)\, (g - dr \otimes dr)$ *with $r = d(0, \cdot)$ the radial function of the model. In particular, it holds that*

$$\Delta r = (n-1)\, (f'/f) \circ r.$$

The statement $f(0) = 0$ *and* $f'(0) = 1$ *are to read as* $f(0+) = 0$ *and* $f'(0+) = 1$.

PROOF. (i) At first let $v, w \in T_0\mathbb{M}$ such that $|v| = |w| = 1$ and $v \perp w$. We consider the Jacobi field J along the geodesic γ where $\gamma(t) = tv$ such that $J(0) = 0$ and $J'(0) = w$. By Theorem 2.1.49 J is up to a scalar function a parallel vector field along γ, hence $J(t) = c(t)W(t)$ with $\nabla_D W = 0$ and without restriction $|W| \equiv 1$ as well as $c(t) > 0$

for $t > 0$; on the other hand $J(t) = (d\exp_0)_{tv}(tw) \equiv tw \in T_{tv}\mathbb{M}$. Thus $c(t)^2 = |c(t)\,W(t)|^2 = |tw|^2 = f(t)^2$, hence $J(t) = f(t)\,W(t)$ and in particular $f(0) = 0$. In addition, it holds $(\nabla_D J)(t) = f'(t)\,W(t)$; because of $(\nabla_D J)(0) = w = W(0)$ (see the proof of Theorem 2.1.49), hence $f'(0) = 1$.

Now $J = f\,W$ is a Jacobi field along γ, and by (2.1.16) hence

$$\big\langle \nabla_D \nabla_D J, J\big\rangle = -\big\langle R(J,\dot\gamma)\dot\gamma, J\big\rangle.$$

This means $\big\langle f''W, fW\big\rangle = -\big\langle R(fW,\dot\gamma)\,\dot\gamma, fW\big\rangle = -f^2\,\big\langle R(W,\dot\gamma)\,\dot\gamma, W\big\rangle$, from where by

$$\big\langle R(W,\dot\gamma)\dot\gamma, W\big\rangle(t) = k(t)\,|W(t)|^2 = k(t)$$

the relation $f''(t)\,f(t) = -f^2(t)\,k(t)$ (or equivalently $f''(t) + k(t)\,f(t) = 0$) follows.

(ii) According to (2.1.32) we have $\nabla dr(\partial^{\mathbb{M}}, X) = 0$ for each vector field X on $\mathbb{M}\backslash\{0\}$, so that in particular $\nabla dr(\partial^{\mathbb{M}}, \partial^{\mathbb{M}}) = 0$. It is hence sufficient to show that

$$(\nabla dr)_x(u,u) = (f'/f)\,\big(r(x)\big) \quad \text{for } u \in T_x\mathbb{M}\,,\ x \neq 0\,,\ |u| = 1 \text{ and } u \perp \partial^{\mathbb{M}}_x.$$

Let $\gamma\colon [0,b] \to \mathbb{M}$ be the normal geodesic with $\gamma(0) = 0$ and $\gamma(b) = x$, and J the unique Jacobi field along γ with $J(0) = 0$ and $J(b) = u$. Then, according to (2.1.28) along with the Jacobi equation for J, it holds that

$$\begin{aligned}(\nabla dr)_x(u,u) &= \int_0^b \Big\{|\nabla_D J|^2 - \big\langle R(J,\dot\gamma)\dot\gamma\,, J\big\rangle\Big\}\,dt\\ &= \int_0^b \Big\{|\nabla_D J|^2 + \langle \nabla_D\nabla_D J, J\rangle\Big\}\,dt = \langle \nabla_D J, J\rangle(b) = \big\langle (\nabla_D J)(b), u\big\rangle.\end{aligned}$$

On the other hand, by Theorem 2.1.49 and part (i), we have $J = fW$ with W a parallel vector field along γ; in particular then $\nabla_D J = f'\,W$ and $f(b)\,W(b) = u$. This shows

$$(\nabla dr)_x(u,u) = \big\langle (\nabla_D J)(b), u\big\rangle = \big\langle f'(b)\,W(b), u\big\rangle = \big\langle f'(b)/f(b)\,u, u\big\rangle = f'(b)/f(b)$$

and hence the claim. □

On the other hand, Theorem 2.1.57 (i) opens a simple strategy for the construction of models: Starting with a differentiable function $k\colon [0,\infty[\ \to \mathbb{R}$, one determines f as solution to the equation

$$f''(t) = -k(t)f(t), \quad f(0) = 0,\ f'(0) = 1. \tag{2.1.33}$$

If then $f > 0$ on $]0,\infty[$, then $dr\otimes dr + f(r)^2\,d\vartheta^2$ defines a Riemannian metric on $\mathbb{R}^n\backslash\{0\}$, and one shows that because of $f(0) = 0$ and $f'(0) = 1$ this metric allows a differentiable continuation to $\mathbb{R}^n$, in other words, there exists a Riemannian metric g on $\mathbb{R}^n$ which restricted to $\mathbb{R}^n\backslash\{0\}$ coincides with $dr\otimes dr + f(r)^2\,d\vartheta^2$ (see [**14**] p. 60). Obviously $(\mathbb{R}^n, g)$ is then a model with $k_{\mathbb{M}} = k$ as radial curvature function. The problem which functions $k\colon \mathbb{R}_+ \to \mathbb{R}$ can serve as radial curvature function of a model thus reduces to the question whether the corresponding solution f to (2.1.33) stays positive on all of $]0,\infty[$.

LEMMA 2.1.58. *Let $n \geq 2$ and $k\colon [0,\infty[\ \to \mathbb{R}$ be a C^∞-function such that either*

(a) $k \leq 0$*, or*
(b) $k \geq 0$ *and* $\int_0^\infty sk(s)\,ds \leq 1$.

Then, up to isometry, there exists a unique model $(\mathbb{R}^n, g)$ with radial curvature function k.

PROOF. We show that in both cases $f' > 0$ on $[0,\infty[$ must hold; since $f(0)=0$ then also $f>0$ on $]0,\infty[$ holds. Assume that $r := \inf\{t>0 : f'(t)=0\}$ is finite. Because of $f'(0)=1$ one has $r>0$, and then $f>0$ and $f'>0$ on $]0,r[$.

(a) Assume that $k \leq 0$. Then $f'' \geq 0$ on $]0,r[$ and then $f'(r)-f'(0) = \int_0^r f''(s)\,ds \geq 0$, in contradiction to the definition of r.

(b) Assume now that $k \geq 0$. Then $f'' \leq 0$ on $]0,r[$ and hence $f' \leq 1$ on $]0,r[$. This implies $f(s) \leq s$ for $s \in [0,r]$ and hence $\int_0^r f(s)k(s)\,ds \leq \int_0^r sk(s)\,ds$ with equality if and only if $f(s)=s$ for each $s \in [0,r]$, which however would imply $f'(r)=1$ and is excluded by the definition of r. But then we have

$$-1 = f'(r) - f'(0) = \int_0^r f''(s)\,ds = -\int_0^r f(s)k(s)\,ds > -\int_0^r sk(s)\,ds \geq -1$$

which is a contradiction. □

The case of constant radial curvature is of particular interest. Let $c>0$ be a constant and suppose that $f''(t) = -k(t)\,f(t)$ with $f(0)=0$ and $f'(0)=1$. Then:

(i) $f(t)=t$ for $t \in [0,\infty[$ if $k \equiv 0$ on $[0,\infty[$.
(ii) $f(t) = (1/c)\,\sin ct$ for $t \in [0,r]$ if $k \equiv c^2$ on $[0,r]$ with $r < \pi/c$.
(iii) $f(t) = (1/c)\,\sinh ct$ for $t \in [0,\infty[$ if $k \equiv -c^2$ on $[0,\infty[$.

We want to investigate the different cases and to give descriptions of the spaces of constant curvature.

A. (Euclidean space) Let $(\mathbb{M},g) = (\mathbb{R}^n, \mathrm{eucl})$ be the Euclidean space $\mathbb{R}^n$ with the standard metric. Obviously $(\mathbb{R}^n,\mathrm{eucl})$ is a model: it holds

$$\exp_0|\big(\mathbb{R}^n\backslash\{0\}\big) : \,]0,\infty[\,\times\,\mathbb{S}^{n-1} \to \mathbb{R}^n, \quad (t,v) \mapsto tv$$

and hence $\exp_0^* g = dr \otimes dr + r^2\,d\vartheta^2$. This corresponds to case (i) and gives up to isometry the unique model with radial curvature $k \equiv 0$; in addition the sectional curvature of $(\mathbb{R}^n,\mathrm{eucl})$ vanishes as well.

B. (Sphere) Let $\mathbb{S}^n_a = \{x \in \mathbb{R}^{n+1} : |x| = a\} \subset \mathbb{R}^{n+1}$ be the sphere of radius $a>0$, equipped with Riemannian metric g induced from $\mathbb{R}^{n+1}$. Geodesics stay on great circles; fixing an arbitrary point on $\mathbb{S}^n_a$, for simplicity the north pole n, and $v \in T_n\mathbb{S}^n_a \cong \{n\}^\perp$ with $|v|=1$, we have

$$\exp_n(tv) = \cos(t/a)\,n + a\,\sin(t/a)\,v.$$

Therefore it holds $\exp_n^* g = dr \otimes dr + a^2\,\sin^2(r/a)\,d\vartheta^2$ on $]0,a\pi\,[\,\times\,\mathbb{S}^{n-1}$. Recall that $(\mathbb{S}^n_a, g)$ has constant sectional curvature, as already deduced from symmetry arguments. The representation of the metric in polar coordinates locally about the north pole as $dr \otimes dr + f(r)^2\,d\vartheta^2$ with $f(r) = a\,\sin(r/a)$ gives for radial planes $E \subset T_x\mathbb{S}^n_a$ as value of the sectional curvature

$$k(r) = -f''(r)/f(r) = 1/a^2$$

where $r = d(n,x) < a\pi$; hence the sectional curvature of $(\mathbb{S}^n_a, g)$ is constant and equal to $1/a^2$. However there is no model with positive radial curvature.

C. (Hyperbolic space) For $a>0$ let $M = \mathbb{B}^n_a = \big\{x \in \mathbb{R}^n : |x| < a\big\}$ be the open unit ball endowed with the Riemannian metric g given by

$$g_x(u,v) := \frac{4\,\langle u,v\rangle}{\big(1-|x|^2/a^2\big)^2}, \quad u,v \in T_xM \cong \mathbb{R}^n.$$

The normal geodesics γ emanating from 0 with $\dot\gamma(0) = v \in T_0M$ obviously take the form $\gamma(t) = \kappa(t)v$ with κ a scalar function such that $\kappa(0)=0$ and $\dot\kappa(0)=1$; from $|\dot\gamma(t)|=1$

we conclude that $\dot\kappa(t) = 1 - \kappa(t)^2/(4a^2)$ and hence $\kappa(t) = 2a\,\tanh(t/2a)$. This gives

$$\begin{array}{rcccl}
 & \mathbb{R}^n\backslash\{0\} & & T_0M & \\
 & \| & & \| & \\
\exp_0\big|\big(\mathbb{R}^n\backslash\{0\}\big)\colon\ ]0,\infty[\,\times\,\mathbb{S}^{n-1} & \xrightarrow{\sim} &]0,\infty[\,\times S_1(0) & \longrightarrow & M \\
(t,v) & \longmapsto & (2t, v/2) & \longmapsto & \kappa(2t)\,v/2 = a\tanh(t/a)v
\end{array}$$

and $\exp_0^* g = dr \otimes dr + f(r)^2\, d\vartheta^2$ with $f(r) = a\,\sinh(r/a)$. Hence $(\mathbb{B}_a^n, g)$ with the origin as distinguished point is the up to isometry unique model with radial curvature $k \equiv -1/a^2$. From invariance properties of the metric g one deduces that $(\mathbb{B}_a^n, g)$ has constant sectional curvature $-1/a^2$. One calls $(\mathbb{B}_a^n, g)$ the *n-dimensional hyperbolic space with curvature* $-1/a^2$; in the case of constant negative curvature -1 one calls it simply the (n-dimensional) *hyperbolic space* and writes $\mathbb{B}^n$ instead of $\mathbb{B}_1^n$.

There are other classical realizations of the hyperbolic space; $(\mathbb{B}_a^n, g)$ is usually called the *ball model* of hyperbolic geometry. We sketch two equivalent models, where we restrict ourselves to the case of curvature -1:

(a) Let $(\mathbb{H}^n, h)$ be the upper half space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x^n > 0\}$ with the metric

$$h_x(u,v) = \langle u, v\rangle/(x^n)^2, \quad u, v \in T_x\mathbb{H}^n \cong \mathbb{R}^n.$$

$(\mathbb{H}^n, h)$ is called *Poincaré model* of the n-dimensional hyperbolic space.

(b) Let $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ equipped with the "Lorentz metric" $\langle x|y\rangle = -x^0y^0 + \sum_{i=1}^n x^iy^i$ and

$$\mathbb{L}^n := \{x \in \mathbb{R}^{n+1} : \langle x|x\rangle = -1,\ x^0 > 0\}.$$

As inverse image of a regular value under a differentiable function, $\mathbb{L}^n$ is an n-dimensional submanifold of $\mathbb{R}^{n+1}$: the sheet determined by the positive sign of x^0 of the two-sheet hyperboloid $\{x \in \mathbb{R}^{n+1} : \langle x|x\rangle = -1\}$. For $x \in \mathbb{L}^n$ one identifies

$$T_x\mathbb{L}^n \cong \{y \in \mathbb{R}^{n+1} : \langle x|y\rangle = 0\} =: \{x\}^\perp.$$

Because of $\langle x|x\rangle = -1$, for $x \in \mathbb{L}^n$ the restriction of $\langle\cdot|\cdot\rangle$ to $\{x\}^\perp$ is positive definite, namely

$$\langle y|y\rangle = \sum_{i=1}^n (y^i)^2 - \Big(\frac{1}{x^0}\sum_{i=1}^n x^iy^i\Big)^2 \ge \sum_{i=1}^n (y^i)^2 - \frac{1}{(x^0)^2}\sum_{i=1}^n (x^i)^2 \sum_{i=1}^n (y^i)^2 = \sum_{i=1}^n \Big(\frac{y^i}{x^0}\Big)^2 \ge 0,$$

and $\langle\cdot|\cdot\rangle$ defines canonically a Riemannian metric k on $\mathbb{L}^n$. We call $(\mathbb{L}^n, k)$ *hyperboloid model* of the n-dimensional hyperbolic space.

THEOREM 2.1.59. *$(\mathbb{B}^n, g)$, $(\mathbb{H}^n, h)$ and $(\mathbb{L}^n, k)$ are isometric models of hyperbolic geometry.*

PROOF. The map $f(x) := \dfrac{(x^1, \ldots, x^n)}{1 + x^0}$ defines an isometry $f\colon \mathbb{L}^n \to \mathbb{B}^n$, i.e., f is a diffeomorphism and it holds:

$$g_{f(x)}(df_xu, df_xv) = k_x(u,v), \quad u, v \in T_x\mathbb{L}^n,\ x \in \mathbb{L}^n.$$

Likewise, an isometry $\mathbb{H}^n \to \mathbb{B}^n$ is given by $\phi \circ \sigma\colon \mathbb{H}^n \to \mathbb{B}^n$ where σ denotes the reflection at the plane $x^n = 0$ and

$$\phi(x) := e_d + \frac{2(x - e_d)}{|x - e_d|^2}$$

where $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^n$. The verification of these properties is left to the reader. $\square$

2.2. Brownian Motion and Curvature

After these differential geometric preparations we continue again with probabilistic questions and start with the description of the distance process $d_M(o, X)$ of an M-valued Brownian motion X to a given point $o \in M$. To this end, we refer to some elementary facts about one-dimensional diffusion processes, which are put together in Appendix A.1.

THEOREM 2.2.1. *Let (M, g) be a metrically complete Riemannian manifold with $n = \dim M \geq 2$ and let $o \in M$ be a fixed point. Let $r(\cdot) = d_M(o, \cdot)$ denote the distance function to o and X be a Brownian motion on (M, g) such that $X_0 = x_0 \neq o$ a.s. If*

$$\tau = \inf\bigl\{t \geq 0 : X_t \in \mathrm{cut}(o)\bigr\}$$

denotes the first hitting time of X of the cut locus $\mathrm{cut}(o)$ *of M with respect to o, then there is a one-dimensional Brownian motion $\hat{W}$ (on some possibly enlarged filtered probability space) such that on $[0, \tau[$ it holds:*

$$d(r(X)) = d\hat{W} + \frac{1}{2}(\Delta r \circ X)\, dt. \tag{2.2.1}$$

PROOF. Consider first the stopping time $\tau' = \inf\bigl\{t \geq 0 : X_t \in \mathrm{cut}(o) \cup \{o\}\bigr\}$. Since $r = d_M(o, \cdot)$ is differentiable on $M \backslash \bigl(\mathrm{cut}(o) \cup \{o\}\bigr)$, the Geometric Itô formula can be applied and one obtains on $[0, \tau'[$

$$d(r(X)) = (dr)(X)\,(U dW) + \frac{1}{2}(\Delta r \circ X)\, dt.$$

Letting $d\hat{W} := (dr)(X)\,(U dW) \equiv \sum_i (dr)(X)\,(Ue_i)\, dW^i$ with $\hat{W}_0 = 0$, we obtain

$$d[\hat{W}, \hat{W}] = \sum_i \bigl((dr)(X)\, Ue_i\bigr)^2\, dt = \sum_i \bigl\langle (\mathrm{grad}\, r) \circ X, Ue_i \bigr\rangle^2\, dt = \bigl|(\mathrm{grad}\, r)(X)\bigr|^2 = dt.$$

Hence $\hat{W}$ defines a (stopped) Brownian motion which can be extended to all of $\mathbb{R}_+$ with the usual methods.

It remains to show that $\tau' = \tau$ a.s. To this end, we have to verify that X does not hit the point o a.s. We fix $\varepsilon > 0$ with $B_{2\varepsilon}(o) \cap \mathrm{cut}(o) = \varnothing$ such that $d_M(o, x_0) > \varepsilon$, and consider for $R := d_M(o, X)$ inductively the following stopping times

$$\begin{aligned} &\sigma_0 = \tau_0 = 0, \quad \text{and} \\ &\sigma_n = \inf\{t \geq \tau_{n-1} : R_t = \varepsilon\} \wedge \tau', \quad \tau_n = \inf\{t \geq \sigma_n : R_t = 2\varepsilon\} \wedge \tau', \quad n \geq 1. \end{aligned}$$

It is obviously sufficient to show that the process $R|[\sigma_n, \tau_n[$ does not hit 0 a.s. for any n. Without restrictions let $\sigma_n < \infty$ a.s. Now the Riemannian sectional curvature on $B_{2\varepsilon}(o)$ is bounded, i.e. $\mathrm{Riem}^M\,|B_{2\varepsilon}(o) \leq c^2$ for some $c > 0$. After possibly diminishing ε we may assume that $\varepsilon < \pi/2c$. Comparison with the sphere $S^n_{1/c}$ combined with Theorems 2.1.55 and 2.1.57 (ii) gives

$$(\Delta r)(x) \geq (n-1)\, c \cot ct, \quad t = d(x, o) < 2\varepsilon, \quad x \in B_{2\varepsilon}(o) \subset M.$$

Using the abbreviations $\tilde{X}_t = X_{(\sigma_n + t) \wedge \tau_n}$ and $\tilde{R}_t = R_{(\sigma_n + t) \wedge \tau_n}$, we get in terms of the Brownian motion $\tilde{W}_t = \hat{W}_{\sigma_n + t} - \hat{W}_{\sigma_n}$ starting anew at σ_n and the stopping time $\tilde{\tau} = \inf\{t \geq 0 : \tilde{R}_t = 2\varepsilon \text{ or } \tilde{R}_0 = 0\}$ (both with respect to the transformed filtration) on the interval interval $[0, \tilde{\tau}[$ the equation

$$d\tilde{R} = d\tilde{W} + \frac{1}{2}\, \Delta r(\tilde{X})\, dt, \quad \tilde{R}_0 = \varepsilon. \tag{2.2.2}$$

We compare $\tilde{R}$ with the solution to the SDE

$$dY = d\tilde{W} + \frac{n-1}{2}\, c\, \cot(cY)\, dt, \quad Y_0 = \varepsilon, \tag{2.2.3}$$

on the real interval $]0, 2\varepsilon[$. At first we conclude from Theorem A.1.9 (ii) for the SDE (2.2.3) that 0 is a non-reachable boundary point of Y. It is hence sufficient to show that that $\tilde{R} \geq Y$ on $[0, \tilde{\tau}[$ which implies that also $\tilde{R}$ does not hit the point 0 a.s. To this end we can conclude as in Comparison Theorem A.1.8: If $[a_n, b_n] \uparrow]0, 2\varepsilon[$ denotes a compact exhaustion with $a_n < \varepsilon < b_n$, then one has first for

$$t \leq \inf\bigl\{s \geq 0 : \tilde{R}_s \notin [a_n, b_n] \text{ or } Y_s \notin [a_n, b_n]\bigr\}$$

the pathwise inequalities

$$\begin{aligned}(Y_t - \tilde{R}_t)_+ &= \int_0^t 1_{\{Y_s > \tilde{R}_s\}}\, d(Y - \tilde{R})_s \\ &= \int_0^t 1_{\{Y_s > \tilde{R}_s\}}\, \frac{1}{2}\, \bigl[(n-1)\, c\, \cot(cY_s) - (\Delta r \circ \tilde{X}_s)\bigr]\, ds \\ &\leq \int_0^t 1_{\{Y_s > \tilde{R}_s\}}\, \frac{1}{2}\, (n-1)\, c\, \bigl[\cot(cY_s) - \cot(c\tilde{R}_s)\bigr]\, ds \\ &\leq C_n \int_0^t 1_{\{Y_s > \tilde{R}_s\}}\, |Y_s - \tilde{R}_s|\, ds = C_n \int_0^t (Y_s - \tilde{R}_s)_+\, ds\end{aligned}$$

with a real constant C_n. By the Gronwall lemma it follows $(Y_t - \tilde{R}_t)_+ = 0$, hence $Y_t \leq \tilde{R}_t$, and then the claim as $n \to \infty$. $\square$

Theorem 2.2.1 indicates the general procedure: the distance process $r(X) = d_M(o, X)$ of an M-valued Brownian motion X to a fixed reference point o is (at least up to the first entrance in the cut locus $\mathrm{cut}(o)$ of M with respect to o) of the form

$$r(X_t) = r(X_0) + \hat{W}_t + \frac{1}{2} \int_0^t (\Delta r \circ X_s)\, ds \tag{2.2.4}$$

with a one-dimensional Brownian motion $\hat{W}$, where the drift part in (2.2.4) is controlled by curvature bounds according to Theorem 2.1.55.

THEOREM 2.2.2 (Comparison Theorem for Brownian motion). *Let (M, g) be a metrically complete Riemannian manifold of dimension $n = \dim M \geq 2$ and let $B_\rho(o)$ be an open geodesic ball of radius $\rho > 0$ about a fixed point $o \in M$ which does not intersect the cut locus* $\mathrm{cut}(o)$ *of M with respect to o. To this, suppose that there is a model $\mathbb{M}$ of the same dimension with center 0 and radial curvature function $k_{\mathbb{M}}$ such that for any $x \in M\backslash\{o\}$ with $0 < d_M(o, x) = r < \rho$ it holds:*

$$\mathrm{Ric}_x^M(\partial^M, \partial^M) \geq (n-1)\, k_{\mathbb{M}}(r),$$

respectively,

$$\mathrm{Riem}_x^M(E) \leq k_{\mathbb{M}}(r) \quad \textit{for any radial plane } E \textit{ in } T_xM]$$

Let X be a Brownian motion on (M, g), starting from a point $x_0 \in B_\rho(o)$, and τ_ρ its exit time from $B_\rho(o)$. Correspondingly let $\tilde{X}$ be a Brownian motion on $\mathbb{M}$, starting from

$\tilde{x}_0 \in \mathbb{M}$ *with* $d_{\mathbb{M}}(0, \tilde{x}_0) = d_M(o, x_0)$*, and* $\tilde{\tau}_\rho$ *the exit time of* $\tilde{X}$ *from the open geodesic* ρ*-ball about* 0*. Then for any antitone function* $\varphi\colon [0, \rho[\to \mathbb{R}$,

$$E\big[\big(\varphi \circ d_M(o, X_t)\big)\, 1_{\{t<\tau_\rho\}}\big] \underset{[\leq]}{\geq} E\big[\big(\varphi \circ d_{\mathbb{M}}(0, \tilde{X}_t)\big)\, 1_{\{t<\tilde{\tau}_\rho\}}\big]. \tag{2.2.5}$$

In particular, for $0 < \rho' < \rho$ *the following inequalities hold:*

$$\mathbb{P}\big\{d_M(o, X_t) < \rho' \text{ und } t < \tau_\rho\big\} \underset{[\leq]}{\geq} \mathbb{P}\big\{d_{\mathbb{M}}(0, \tilde{X}_t) < \rho' \text{ und } t < \tilde{\tau}_\rho\big\}.$$

PROOF. Denote by $r_M(\cdot) = d_M(o, \cdot)$ and $r_{\mathbb{M}}(\cdot) = d_{\mathbb{M}}(0, \cdot)$ the distance processes to the distinguished points $o \in M$, $0 \in \mathbb{M}$ and let $r_0 := r_M(x_0) = r_{\mathbb{M}}(\tilde{x}_0)$. Then, for $t < \tau_\rho$, respectively $t < \tilde{\tau}_\rho$,

$$r_M(X_t) = r_0 + \hat{W}_t + \frac{1}{2}\int_0^t \Delta r_M(X_s)\, ds. \tag{2.2.6}$$

$$r_{\mathbb{M}}(\tilde{X}_t) = r_0 + \tilde{W}_t + \frac{1}{2}\int_0^t \Delta r_{\mathbb{M}}(\tilde{X}_s)\, ds \tag{2.2.7}$$

Since $\mathbb{M}$ is a model, we have $\Delta r_{\mathbb{M}} = (n-1)\,(f'/f) \circ r_{\mathbb{M}} =: a \circ r_{\mathbb{M}}$ where f denotes the radial function of the model. If the curvature of M can be estimated from below in the way indicated, then as in the proof of Theorem 2.2.1, by Theorem 2.1.55 (Laplacian Comparison Theorem) the radial process $r_M(X)$ may be compared to the solution of the SDE

$$dY = d\hat{W} + \frac{1}{2}\, a(Y)\, dt, \quad Y_0 = r_0 \tag{2.2.8}$$

and one obtains $r_M(X_t) \leq Y_t$ for $t < \tau_\rho$ a.s. Hence if φ is antitone, i.e. monotonically decreasing, then $\varphi \circ r_M \circ X_t \geq \varphi \circ Y_t$ for $t < \tau_\rho$ a.s. By the uniqueness in law for solutions of (2.2.8) we then have, as claimed,

$$\mathbb{E}\big[\big(\varphi \circ d_M(o, X_t)\big)\, 1_{\{t<\tau_\rho\}}\big] \geq \mathbb{E}\big[\big(\varphi \circ d_{\mathbb{M}}(0, \tilde{X}_t)\big)\, 1_{\{t<\tilde{\tau}_\rho\}}\big].$$

The case of upper curvature bounds for M can be treated completely analogously by means of Corollary 2.1.46. □

COROLLARY 2.2.3. *Keeping the assumptions and notation of Theorem 2.2.2, we have in addition*

$$\mathbb{P}\{\tau_\rho \leq t\} \underset{[\geq]}{\leq} \mathbb{P}\{\tilde{\tau}_\rho \leq t\}$$

for any $t \geq 0$*, and since* $\mathbb{E}[\tau_\rho] = \int_0^\infty \mathbb{P}\{\tau_\rho > t\}\, dt$*, then in particular*

$$\mathbb{E}[\tau_\rho] \underset{[\leq]}{\geq} \mathbb{E}[\tilde{\tau}_\rho].$$

EXAMPLE 2.2.4. Let (M, g) be a simply connected, metrically complete Riemannian manifold with $n = \dim M \geq 2$ and $o \in M$ be a fixed point. Suppose that $\mathrm{Riem}^M \leq 0$. If then X is a Brownian motion on (M, g) with $X_0 = x_0 \in M$, we have

$$\mathbb{P}\big\{d_M(o, X_t) < \rho\big\} \leq \mathbb{P}\big\{R_t < \rho\big\},$$

for any $\rho > 0$ and $t > 0$ where R denotes a weak solution of the SDE

$$dR = dW + (n-1)/(2R)\, dt, \quad R_0 = d_M(o, x_0),$$

with W representing a one-dimensional Brownian motion.

PROOF. By Theorem 2.1.41 (Cartan-Hadamard), the cut locus of M with respect to any point is empty, and the claim follows from Theorem 2.2.2 by comparison with $(\mathbb{R}^n, \text{eucl})$. $\square$

Before continuing the discussion of the radial part of M-valued Brownian motions, we want to note some general properties of Brownian motions on Riemannian manifolds.

REMARK 2.2.5 (Strong Markov property of Brownian motion). Let (M, g) be a Riemannian manifold. For $x \in M$, let X^x denote a Brownian motion on (M, g), starting at x, which we extend to a continuous process defined on $\mathbb{R}_+$ and taking values in the one-point-compactification $\hat{M}$ of M. If then $H\colon C(\mathbb{R}_+; \hat{M}) \to \mathbb{R}_+$ is a bounded measurable function, then for any Brownian motion X on (M, g) and each stopping time τ, it holds

$$\mathbb{E}^{\mathscr{F}_\tau}[H(X_{\tau+\bullet})] = \mathbb{E}[H(X^y_\bullet)]\big|_{y=X_\tau} \quad \text{a.s. on } \{\tau < \infty\}. \tag{2.2.9}$$

PROOF. Taking into account the specific construction of Brownian motions as solutions of SDEs on the orthonormal frame bundle, the claim reduces to the strong Markov property of maximal solutions of SDEs with locally Lipschitz-continuous coefficients. $\square$

In general, the cut locus $\text{cut}(x)$ on a metrically complete Riemannian manifold (M, g) with respect to a point x is not a polar set; Brownian motions may hit the cut locus with positive probability, as can be seen from simple examples. However, for almost all paths of an M-valued Brownian motion, the occupation time on the cut locus equals zero, which comes from the fact that the cut locus is a nullset of the canonical Riemannian volume measure. We want briefly discuss this point.

DEFINITION 2.2.6 (Riemannian volume measure). On a Riemannian manifold (M, g) there is exactly one measure vol on the Borel σ-algebra $\mathscr{B}(M)$ with the property that for each measurable function $f\colon M \to \mathbb{R}_+$ with $\text{supp}(f)$ in the domain of a chart (φ, U) for M, it holds that

$$\int f \, d\,\text{vol} = \int_{\varphi(U)} \left(f \sqrt{g^{(\varphi)}}\right) \circ \varphi^{-1} \, dx \tag{2.2.10}$$

where $g^{(\varphi)} = \det G^{(\varphi)} > 0$ with $G^{(\varphi)}_{ij} = g(\partial_i, \partial_j) \in C^\infty(U)$ and $\partial_i = \partial/\partial\varphi^i$. The measure vol is called *Riemannian volume measure* on (M, g).

REMARK 2.2.7. Note that if (ψ, V) is another chart, then

$$\sqrt{g^{(\varphi)}} = \sqrt{g^{(\psi)}}\, |\det J(\psi \circ \varphi^{-1})| \circ \varphi \quad \text{on } U \cap V.$$

On the other hand, if $\phi\colon D_1 \to D_2$ is a diffeomorphism between two domains of $\mathbb{R}^n$, then by the transformation formula, for any non-negative measurable function $f\colon \mathbb{R}^n \to \mathbb{R}$,

$$\int_{D_1} (f \circ \phi)\, |\det J(\phi)| \, dx = \int_{D_2} f \, dx$$

with $J(\phi)$ the Jacobian of ϕ. Both observations together show that (2.2.10) is independent of the choice of the chart. Indeed, through (2.2.10), vol is first well-defined on Borel sets contained in the domain of a chart, and then also on all of $\mathscr{B}(M)$. The Riemannian volume measure on $(\mathbb{R}^n, \text{eucl})$ is obviously the n-dimensional Lebesgue measure.

On a complete Riemannian manifold (M, g) the Laplacian generates a canonical semigroup of operators on the space $B(M)$ of bounded measurable functions on M in the sense of a family of linear operators

$$P_t\colon B(M) \to B(M), \quad t \geq 0, \tag{2.2.11}$$

with the properties:

(a) $P_s P_t f = P_{s+t} f$ for $f \in B(M)$.
(b) $P_t f \geq 0$ for $0 \leq f \in B(M)$, as well as $P_t 1 \leq 1$.
(c) $(P_t f)(x) - f(x) = \frac{1}{2}\int_0^t (P_s \Delta f)(x)\, ds$ for any test function $f \in C_c^\infty(M)$.
(d) $(P_t)_{t\geq 0}$ is minimal, i.e., for any other family $(Q_t)_{t\geq 0}$ of positive linear operators on $B(M)$ satisfying (a), (b), (c), it holds

$$P_t f \leq Q_t f, \quad 0 \leq f \in B(M),\ t \geq 0.$$

In addition, $(P_t)_{t\geq 0}$ possesses a C^∞-kernel $p \in C^\infty(]0,\infty[\times M \times M)$ such that

$$(P_t f)(x) = \int p(t,x,y)\, f(y)\ \mathrm{vol}(dy), \quad f \in B(M),\ t > 0, \tag{2.2.12}$$

and $u(t,x) := (P_t f)(x)$ defines a classical solution of the heat equation

$$\begin{cases} \frac{\partial}{\partial t} u - \frac{1}{2}\Delta u = 0 \\ u|_{t=0} = f. \end{cases} \tag{2.2.13}$$

These are well-known facts from Spectral Theory of the heat kernel (see for instance [**4**], p. 187 ff.). We want briefly sketch the relation to Brownian motion.

THEOREM 2.2.8. *Let (M,g) be a metrically complete Riemannian manifold and $(P_t)_{t\geq 0}$ the minimal semigroup* (2.2.11) *generated by $\frac{1}{2}\Delta$. Then*

$$(P_t f)(x) = \mathbb{E}\big[f(X_t^x)\, 1_{\{t<\zeta^x\}}\big], \quad f \in B(M), \tag{2.2.14}$$

where X^x denotes a Brownian motion with lifetime ζ^x, starting in x. In particular, (M,g) is BM-*complete if and only if $P_t 1 = 1$.*

PROOF. Let $(Q_t f)(x) := \mathbb{E}\big[f(X_t^x)\, 1_{\{t<\zeta^x\}}\big]$. We fix a non-negative function $f \in B(M)$, as well as $t \geq 0$. Since $u(t,x) := (P_t f)(x)$ solves the heat equation (2.2.13), it follows from Itô's formula that

$$(Y_s)_{0\leq s<t\wedge\zeta^x}, \quad Y_s := (P_{t-s} f)(X_s^x),$$

defines a non-negative local martingale. Hence there exists a localizing sequence of stopping times $(\zeta_n^x)_{n\in\mathbb{N}}$ with $\zeta_n^x \uparrow \zeta^x$ such that

$$(P_t f)(x) = Y_0 = \mathbb{E}[Y_{t\wedge\zeta_n^x}] \geq \mathbb{E}[\liminf_{n\to\infty} Y_{t\wedge\zeta_n^x}] \geq \mathbb{E}[Y_t\, 1_{\{t<\zeta^x\}}] = (Q_t f)(x).$$

Now also $(Q_t)_{t\geq 0}$ satisfies the conditions (a), (b), (c) from above, where for instance (a) follows from the strong Markov property (Remark 2.2.5). We then conclude from the minimality of $(P_t)_{t\geq 0}$ that $P_t = Q_t$. □

COROLLARY 2.2.9. *For a metrically complete Riemannian manifold (M,g) are equivalent:*

(i) *Bounded solutions u of the heat equation $\frac{\partial}{\partial t} u - \frac{1}{2}\Delta u = 0$ are uniquely determined by the initial condition $u(0,\cdot)$.*
(ii) *(M,g) is* BM-*complete.*

PROOF. (i) ⇒ (ii): For $x \in M$ let X^x be again a Brownian motion starting at x with lifetime ζ^x. Then $u(t,x) := (P_t 1)(x) = \mathbb{P}\{\zeta^x > t\}$ solves the heat equation to the initial condition $u(0,\cdot) \equiv 1$. By means of the unique solvability we have $\mathbb{P}\{\zeta^x > t\} = 1$ for any $t \geq 0$ and hence $\mathbb{P}\{\zeta^x = \infty\} = 1$.

(ii) ⇒ (i): Conversely, let u be a bounded solution of the heat equation with initial condition $f = u(0,\cdot)$. For fixed $t > 0$ and $x \in M$ then

$$(Y_s)_{0\leq s<t}, \quad Y_s := u(t-s, X_s^x),$$

defines a bounded martingale; hence $u(t,x) = \mathbb{E}[Y_0] = \mathbb{E}[Y_t] = \mathbb{E}[f \circ X_t^x]$ which gives the claim. $\square$

THEOREM 2.2.10. *On a metrically complete Riemannian manifold* (M,g) *the cut locus* cut(x) *of* M *with respect to any point* x *is a nullset of the Riemannian volume measure.*

PROOF. For $x \in M$ we have cut$(x) = \exp_x(C_x)$ (according to Definition 2.1.10) with

$$C_x = \{s(v)\, v \colon v \in T_xM,\ |v| = 1,\ s(v) < \infty\}$$

and the strictly positive continuous function $s\colon \{v \in T_xM \colon |v| = 1\} \to \overline{\mathbb{R}}_+$ defined by $s(v) = \sup\bigl\{t \geq 0 \colon d\bigl(x, \exp_x(tv)\bigr) = t\bigr\}$. Now $C_x \subset T_xM$ is a Lebesgue nullset, as graph in polar coordinates of a (continuous) function. Then also cut$(x) \subset M$, as image of the differentiable map $\exp_x$ defined on T_xM, is a nullset with respect to the Riemannian volume measure, which is an immediate consequence of the definition of the Riemannian volume measure and the fact that Lebesgue nullsets are preserved under differentiable transformations of $\mathbb{R}^n$. $\square$

COROLLARY 2.2.11. *The occupation time of a Brownian motion on the cut locus* cut(x) *of a metrically complete Riemannian manifold* (M,g) *with respect to any point* $x \in M$ *is zero, i.e., for each Brownian motion* X *on* (M,g) *with lifetime* ζ *it holds:*

$$\int_0^\zeta 1_{\{t : X_t \in \mathrm{cut}(x)\}}\, dt = 0 \quad \text{a.s.}$$

PROOF. Let X be a Brownian motion on (M,g); by the Markov property 2.2.9 (with $\tau = 0$) without restriction with deterministic starting point. Then

$$\mathbb{E}\left[\int_0^\zeta 1_{\{t : X_t \in \mathrm{cut}(x)\}}\, dt\right] = \int_0^\infty \mathbb{E}\bigl[1_{\mathrm{cut}(x)}(X_t)\, 1_{\{t<\zeta\}}\bigr]\, dt = 0;$$

because of (2.2.12) and (2.2.14) the last equality is a consequence of Theorem 2.2.10. $\square$

We want to investigate now for Brownian motions X on a Riemannian manifold (M,g) properties of the distance process $d_M(o,X)$ (with respect to a given point $o \in M$) beyond the first entrance time of the Brownian motion into the cut locus cut(o) of M with respect to o. The main difficulty hereby, namely the distance function $d_M(o,\cdot)$ being differentiable only on $M \backslash (\mathrm{cut}(o) \cup \{o\})$, with the consequence that it is not even clear whether $d_M(o,X)$ represents a globally defined semimartingale, can be approached in different ways. On one hand, it is well-known that estimates for $\Delta d_M(o,\cdot)$ on $M \backslash (\mathrm{cut}(o) \cup \{o\})$ extend globally to all of M if interpreted in the distributional sense (see [**47**], p. 669-70). We follow in contrast the approach of W.S. Kendall [**23**] and use the observation above that in general Brownian motions may in fact hit the cut locus but "spend no time on it" (see Corollary 2.2.11).

THEOREM 2.2.12. *Let* (M,g) *be a metrically complete Riemannian manifold with* $n = \dim M \geq 2$ *and let* $r = d_M(o,\cdot)$ *be the distance function to a given point* $o \in M$*. Let* X *be a Brownian motion on* (M,g) *with starting point* $x_0 \in M$ *and lifetime* ζ*, as well as* U *a horizontal lift of* X *to* O(TM) *and* W *the* $\mathbb{R}^n$*-valued* BM*given as anti-development of* X *(with respect to the initial basis* U_0*). Then, for* $t < \zeta$*,*

$$r(X_t) - r(x_0) = \sum_{i=1}^n \int_0^t dr(X)\,(Ue_i)\, dW^i + \frac{1}{2}\int_0^t \Delta r(X)\, dt - L_t^{(o)}, \tag{2.2.15}$$

where dr and Δr are set zero on $\mathrm{Cut}(o) \cup \{o\}$*; here $L^{(o)}$ is an adapted isotone process which increases only when X hits the cut locus* $\mathrm{cut}(o)$*, i.e.,*

$$\int_0^\zeta 1_{\{t\colon X_t \notin \mathrm{cut}(o)\}}\, dL_t^{(o)} = 0 \quad \textit{a.s.} \tag{2.2.16}$$

Note that the convention $dr = 0$ and $\Delta r = 0$ at places where r is not differentiable, is inessential by Corollary 2.2.11.

Theorem 2.2.12 generalizes the Geometric Itô formula (Theorem 1.6.45) for the radial part of a Brownian motion X to its whole life interval including the hitting times of the cut locus. The necessary subtraction of a "correction term" in the form of an isotone process $L^{(o)}$ which grows only when X hits the cut locus, can be interpreted as local time of the Brownian motion on the cut locus $\mathrm{cut}(o)$, see [**5**] for a detailed analysis of the geometric and stochastic nature of $L^{(o)}$.

PROOF OF THEOREM 2.2.12. (see [**23**]) (1) The lifetime of X (considered as continuous process taking values in the one-point-compactification of M) is given by

$$\zeta = \sup\big\{t > 0\colon r(X) \text{ is bounded on } [0,t]\big\}.$$

It is hence sufficient to verify (2.2.15) up to the first exit from a geodesic ball, that is, up to the first time $r(X)$ exceeds a certain value. The claim to verify is then only concerns a sufficiently large geodesic ball B. An elementary consideration thus shows that (M, g) may be modified outside of B to a compact Riemannian manifold. For simplicity we may hence assume without restriction of generality M to be already compact; in particular then $\mathrm{Riem}^M \geq -c^2$ for some $c > 0$, and the injectivity radius of M being strictly positive, i.e., $\varrho = \inf\big\{d\big(x, \mathrm{cut}(x)\big) : x \in B\big\} > 0$.

(2) We verify first that $r(X)$ defines a semimartingale. To this end, we show that for a suitable function V on M the process

$$r(X_t) - r(x_0) - \int_0^t V(X_s)\, ds, \quad t \geq 0, \tag{2.2.17}$$

is a supermartingale. This is sufficient since by the general Doob-Meyer decomposition (e.g., [**28**], section 3.7) each supermartingale is in particular a semimartingale. By (2.2.17) then trivially also $r(X)$ is a semimartingale and can be decomposed as

$$r(X) = r(x_0) + N + A, \quad N \in \mathscr{M}_0,\ A \in \mathscr{A}_0. \tag{2.2.18}$$

We consider

$$V\colon M\backslash\{o\} \to \mathbb{R}, \quad V(x) := \begin{cases} \frac{n-1}{2}\, c \coth cr(x) & \text{for } r(x) \leq \varrho/3, \\ \frac{n-1}{2}\, c \coth c\varrho/3 & \text{for } r(x) \geq \varrho/3. \end{cases}$$

Comparison with the n-dimensional hyperbolic space of constant curvature $-c^2$, i.e. the model with radial function $f(t) = (1/c)\sinh ct$, gives by Theorem 2.1.55 (Laplacian Comparison Theorem) and Theorem 2.1.57,

$$\frac{1}{2}\,(\Delta r)(x) \leq V(x) \quad \text{for } x \notin \mathrm{cut}(o) \cup \{o\}. \tag{2.2.19}$$

As already verified in the proof of Theorem 2.2.1, the Brownian motion $X = (X_t)_{t\geq 0}$ does not hit the given point o for $t > 0$ a.s. For arbitrary $0 < t_1 \leq t_2$, we have to show that

$$\mathbb{E}^{\mathscr{F}_{t_1}}\left[r(X_{t_2}) - r(X_{t_1}) - \int_{t_1}^{t_2} V(X_s)\, ds\right] \leq 0. \tag{2.2.20}$$

By the strong Markov property of Brownian motion (Remark 2.2.5) it is then sufficient to show that for each Brownian motion X on (M, g) with deterministic starting point (different to o) in M

$$\mathbb{E}\left[r(X_t) - r(X_0) - \int_0^t V(X_s)\,ds\right] \le 0, \quad t \ge 0. \tag{2.2.21}$$

We divide the proof into several steps.

(3) Let $x_0 \in \text{cut}(o)$ be an arbitrary point, and

$$\gamma_v(t) = \exp_o(tv), \quad 0 \le t \le s(v),\ v \in T_oM,\ |v| = 1,$$

be a minimal geodesic from o to x_0. Then $\gamma_v(\varrho/3) \notin \text{cut}(x_0)$, or equivalently $x_0 \notin \text{cut}\big(\gamma_v(\varrho/3)\big)$. Hence the following two subsets of $M \times M$,

$$\begin{aligned} \text{cut} &:= \big\{(x,y) \in M \times M : y \in \text{cut}(x)\big\} \\ C &:= \big\{\big(\gamma_v(s(v)), \gamma_v(\varrho/3)\big) : v \in T_oM,\ |v| = 1\big\} \end{aligned}$$

are disjoint and have positive distance with respect to the product metric on $M \times M$. Hence there exists $\delta > 0$ with the following property: If $x_0 = \gamma_v(s(v)) \in \text{cut}(o)$, then $x \notin \text{cut}\big(\gamma_v(\varrho/3)\big)$ for each $x \in M$ such that $d(x, x_0) < \delta$. We choose such a $\delta > 0$ such that in addition $\delta < \varrho/3$. This leads to the following

Claim: If X is a Brownian motion with $X_0 = x_0 \in \text{cut}(o)$ and $\tau := \inf\big\{t \ge 0 : d(X_0, X_t) = \delta\big\}$, then

$$\mathbb{E}\Big[r(X_{t\wedge\tau}) - r(X_0) - \int_0^{t\wedge\tau} V(X_s)\,ds\Big] \le 0, \quad t \ge 0. \tag{2.2.22}$$

Indeed, fixing a minimal geodesic $\gamma_v(t) = \exp_o(tv)$ from o to x_0, then with $\hat{o} := \gamma_v(\varrho/3)$, according to the choice of δ, die function

$$\hat{r}(x) := d(x, \hat{o})$$

is differentiable on the geodesic ball $B_\delta(x_0)$ about x_0 of radius δ, and by Theorem 2.2.1 we have

$$\mathbb{E}\left[\hat{r}(X_{t\wedge\tau}) - \hat{r}(X_0) - \frac{1}{2}\int_0^{t\wedge\tau} (\Delta\hat{r})(X_s)\,ds\right] = 0, \quad t \ge 0. \tag{2.2.23}$$

The same comparison argument leading to (2.2.19) now gives

$$\frac{1}{2}\,(\Delta\hat{r})(x) \le \hat{V}(x) \quad \text{for } x \notin \text{cut}(\hat{o}) \cup \{\hat{o}\} \tag{2.2.24}$$

with the modified function

$$\hat{V} : M\backslash\{\hat{o}\} \to \mathbb{R}, \quad \hat{V}(x) := \begin{cases} \frac{n-1}{2}\,c\,\coth c\hat{r}(x) & \text{for } \hat{r}(x) \le \varrho/3, \\ \frac{n-1}{2}\,c\,\coth c\varrho/3 & \text{for } \hat{r}(x) \ge \varrho/3, \end{cases}$$

where for $x \in B_\delta(x_0)$ by definition $\hat{V}(x) = V(x)$ holds according to $\delta < \varrho/3$. Considering finally the function

$$r_+(x) := \hat{r}(x) + \varrho/3$$

we observe that $r_+(x) \ge r(x)$ by the triangle inequality where $r_+(x_0) = r(x_0)$. Hence it holds

$$\frac{1}{2}\,(\Delta r_+)(x) = \frac{1}{2}\,(\Delta\hat{r})(x) \le \hat{V}(x) = V(x), \quad x \in B_\delta(x_0),$$

and (2.2.22) follows:

$$\begin{aligned}
&\mathbb{E}\left[r(X_{t\wedge\tau}) - r(X_0) - \int_0^{t\wedge\tau} V(X_s)\,ds\right]\\
&\qquad \le \mathbb{E}\left[r_+(X_{t\wedge\tau}) - r_+(X_0) - \int_0^{t\wedge\tau} V(X_s)\,ds\right]\\
&\qquad = \mathbb{E}\left[\hat r(X_{t\wedge\tau}) - \hat r(X_0) - \int_0^{t\wedge\tau} \hat V(X_s)\,ds\right] \le 0;
\end{aligned}$$

the last inequality holds by (2.2.23) und (2.2.24). This completes the proof of the Claim.

(4) Assertion (2.2.21) can now be verified by means of the Claim in part (3): For each Brownian motion X on (M,g) with $X_0 = x_0 \neq o$, it holds

$$\mathbb{E}\left[r(X_t) - r(X_0) - \int_0^t V(X_s)\,ds\right] \le 0, \quad t \ge 0. \tag{2.2.25}$$

Indeed, defining inductively sequences of stopping times $(\tau_n)_{n\ge 0}$ and $(\sigma_n)_{n\ge 1}$ by $\tau_0 = 0$ and

$$\begin{aligned}
\sigma_n &= \inf\bigl\{t \ge \tau_{n-1} : X_t \in \mathrm{cut}(o)\bigr\},\\
\tau_n &= \inf\bigl\{t \ge \sigma_n : d(X_t, X_{\sigma_n}) = \delta\bigr\}, \quad n \ge 1,
\end{aligned} \tag{2.2.26}$$

one obtains by using the strong Markov property (Remark (2.2.5)), for any $n \in \mathbb{N}$,

$$\mathbb{E}^{\mathscr{F}_{\tau_{n-1}}}\left[r(X_{t\wedge\sigma_n}) - r(X_{t\wedge\tau_{n-1}}) - \int_{t\wedge\tau_{n-1}}^{t\wedge\sigma_n} V(X_s)\,ds\right] \le 0, \tag{2.2.27}$$

$$\mathbb{E}^{\mathscr{F}_{\sigma_n}}\left[r(X_{t\wedge\tau_n}) - r(X_{t\wedge\sigma_n}) - \int_{t\wedge\sigma_n}^{t\wedge\tau_n} V(X_s)\,ds\right] \le 0. \tag{2.2.28}$$

Here (2.2.27) is a consequence of Theorem 2.2.1 (on the radial part of a Brownian motion) and estimate (2.2.19), whereas (2.2.28) reduces to the Claim by means of the strong Markov property. To complete the proof of (2.2.25) only $\tau_n \uparrow \infty$ a.s. needs to be verified.

To this end, consider to a fixed $\varepsilon > 0$ the independent sequence of events

$$A_n := \{\tau_n - \sigma_n \ge \varepsilon\}, \quad n \in \mathbb{N}.$$

By the Lemma of Borel-Cantelli it is sufficient to show $\sum_{n=1}^\infty \mathbb{P}(A_n) = \infty$. We may compare with the n-dimensional hyperbolic space $\mathbb{H}^n(-c^2)$ of constant curvature $-c^2$: If B denotes the geodesic ball in $\mathbb{H}^n(-c^2)$ about 0 of Radius δ, then we get by Theorem 2.2.2 (Comparison Theorem for Brownian motion), using again the strong Markov property of the Brownian motion X on M,

$$\begin{aligned}
&\mathbb{E}^{\mathscr{F}_{\sigma_n}}[1_{\{\tau_n - \sigma_n \ge \varepsilon\}}]\\
&\ge \mathbb{P}\bigl\{\text{exit time of BM}(\mathbb{H}^n(-c^2)) \text{ from } B \text{ when starting in 0 is at least } \varepsilon\bigr\}\\
&\ge 1/2 \quad \text{for } \varepsilon > 0 \text{ sufficiently small.}
\end{aligned}$$

This shows that $r(X)$ can be written as sum of a supermartingale and an isotone process; hence, in particular, $r(X)$ is a semimartingale. We want to continue by giving a more detailed description of the terms in (2.2.18).

(5) Adopting the convention $dr = 0$ on $\mathrm{cut}(o) \cup \{o\}$, the process

$$\sum_{i=1}^n \int_0^t dr(X)\,(Ue_i)\,dW^i \equiv \int_0^t dr(X)\,U dW, \quad t \ge 0,$$

is seen to be the martingale part of $r(X)$.

Indeed, denoting by $r(X) = r(x_0) + N + A$ the decomposition of $r(X)$ as semimartingale, the martingale part N allows an integral representation of the form

$$N_t = \int_0^t F\, dW \equiv \int_0^t \tilde{F}\, U dW$$

with a uniquely determined predictable $\mathbb{R}^n$-valued process F, respectively $\tilde{F} := FU^{-1}$ the corresponding T^*M-valued process over X. Considering the difference

$$\bar{N}_t := \int_0^t \tilde{F}\, U dW - \int_0^t dr(X)\, U dW,$$

one observes that the local martingale $\bar{N}$ is constant on each stochastic interval, on which X doesn't hit the cut locus cut(o); since X avoids the point o almost surely, we have on such an interval by the geometric Itô formula

$$d(r(X)) = (dr)(X)\,(UdW) + \frac{1}{2}(\Delta r \circ X)\, dt.$$

For $\delta > 0$ sufficiently small, we consider again the stopping times (2.2.26) and set

$$I_\delta := \bigcup\nolimits_{n\in\mathbb{N}}]\tau_{n-1}, \sigma_n[\;\uparrow\; I_* \quad \text{for } \delta \searrow 0.$$

Obviously, it holds $I_* = \big\{(t,\omega) : X_t(\omega) \notin \text{cut}(o)\big\}$. As already noted, $[\bar{N}, \bar{N}]$ is constant on each I_δ, and hence $\int_{I_\delta} d[\bar{N}, \bar{N}] = 0$, which implies $\int_{I_*} d[\bar{N}, \bar{N}] = 0$ almost surely. In addition also $\int_0^\infty 1_{\{X_t \in \text{cut}(o)\}}\, d[\bar{N}, \bar{N}]$ almost surely, since $[\bar{N}, \bar{N}]$ is absolutely continuous with respect to the Lebesgue measure and since $\int_0^\infty 1_{\{t: X_t \in \text{cut}(0)\}}\, dt = 0$ holds almost surely by Corollary 2.2.11. Together it shows $[\bar{N}, \bar{N}] = 0$ almost surely and hence $\bar{N} = 0$ modulo indistinguishability. This gives

$$N = \int dr(X)\, U dW,$$

as wanted.

(6) Following the convention $\Delta r = 0$ on cut$(o) \cup \{o\}$, the $L^{(o)}$,

$$L_t^{(o)} = \int_0^t dr(X)\, U dW + \frac{1}{2}\int_0^t \Delta r(X)\, ds - \big(r(X_t) - r(x_0)\big),$$

is an isotone process with the property $\int_0^\infty 1_{\{t: X_t \notin \text{cut}(o)\}}\, dL_t^{(o)} = 0$ almost surely.

Let I_δ be as in (5) with $\delta = 1/n$, $n \in \mathbb{N}$. For sufficiently large n,

$$L_t^{(o,n)} := \int_0^t dr(X)\, U dW + \frac{1}{2}\int_{[0,t]\cap I_{1/n}} \Delta r(X)\, ds + \int_{[0,t]\setminus I_{1/n}} V(X)\, ds - \big(r(X_t) - r(x_0)\big)$$

determines an isotone process $L^{(o,n)}$. By (2.2.19) it holds $L_t^{(o,n)} \geq L_t^{(o,n+1)}$, and hence

$$L_t^{(o,\infty)} := \lim_{n\to\infty} L_t^{(o,n)}$$

defines an isotone process $L^{(o,\infty)}$ such that

$$\begin{aligned} L_t^{(o,\infty)} &= \int_0^t dr(X)\, U dW + \frac{1}{2}\int_{[0,t]\cap I_*} \Delta r(X)\, ds + \int_{[0,t]\setminus I_*} V(X)\, ds - \big(r(X_t) - r(x_0)\big) \\ &= \int_0^t dr(X)\, U dW + \frac{1}{2}\int_0^t \Delta r(X)\, ds - \big(r(X_t) - r(x_0)\big); \end{aligned}$$

for the last equality we used $I_* = \big\{(t,\omega) : X_t(\omega) \notin \mathrm{cut}(o)\big\}$ together with Corollary 2.2.10. This shows $L^{(o,\infty)} = L^{(o)}$.

The still missing property $\int_0^\infty 1_{\{t: X_t \notin \mathrm{cut}(o)\}}\, dL_t^{(o)} = 0$ a.s. comes from the equation $\int_0^\infty 1_{I_\delta}(t,\cdot)\, dL_t^{(o)} = 0$ a.s. which holds for each $\delta > 0$. $\square$

Theorem 2.2.12 allows to sharpen the Comparison Theorem for Brownian motion (Theorem 2.2.2) in the case of lower curvature bounds: in this case one may consider arbitrary geodesic balls, also balls which intersect the cut locus.

THEOREM 2.2.13 (Comparison Theorem for Brownian motion; strong version). *Let (M,g) be a metrically complete Riemannian manifold of dimension $n \geq 2$ and let $B_\rho(o)$ be the open geodesic ball of radius $\rho > 0$ about some given point $o \in M$. Suppose that there exists a model $\mathbb{M}$ of same dimension with center 0 and radial curvature function $k_{\mathbb{M}}$ such that for any $x \in M \backslash (\mathrm{cut}(o) \cup \{o\})$ with $0 < d_M(o,x) = r < \rho$ it holds:*

$$\mathrm{Ric}_x^M(\partial^M, \partial^M) \geq (n-1)\, k_{\mathbb{M}}(r).$$

Let X be a Brownian motion on (M,g), starting in a point $x_0 \in B_\rho(o)$, and τ_ρ be its exit time from $B_\rho(o)$. Accordingly, let $\tilde X$ be a Brownian motion on $\mathbb{M}$, starting in $\tilde x_0 \in \mathbb{M}$ with $d_{\mathbb{M}}(0,\tilde x_0) = d_M(o,x_0)$, and $\tilde\tau_\rho$ the exit time of $\tilde X$ from the open geodesic ρ-ball about 0. Then, for any antitone function $\varphi\colon [0,\rho[\, \to \mathbb{R}$,

$$\mathbb{E}\big[\big(\varphi \circ d_M(o,X_t)\big)\, 1_{\{t<\tau_\rho\}}\big] \geq \mathbb{E}\big[\big(\varphi \circ d_{\mathbb{M}}(0,\tilde X_t)\big)\, 1_{\{t<\tilde\tau_\rho\}}\big].$$

In particular, for $0 < \rho' < \rho$, one has the inequalities:

$$\mathbb{P}\big\{d_M(o,X_t) < \rho' \text{ and } t < \tau_\rho\big\} \geq \mathbb{P}\big\{d_{\mathbb{M}}(0,\tilde X_t) < \rho' \text{ and } t < \tilde\tau_\rho\big\}.$$

PROOF. According to (2.2.15) we have for $t < \tau_\rho$,

$$r_M(X_t) \leq r_M(x_0) + \hat W_t + \frac{1}{2}\int_0^t \Delta r_M(X)\, dt$$

where $\hat W_t := \sum_{i=1}^n \int_0^t dr_M(X)\,(Ue_i)\, dW^i$ is a one-dimensional Brownian motion (stopped at ζ). The remainder of the proof of Theorem 2.2.2 then carries over verbatim. $\square$

Before continuing the discussion on further asymptotic properties of Brownian motions, we want to note some fundamental facts about harmonic functions.

LEMMA 2.2.14. *Let (M,g) be a Riemannian manifold and $h\colon M \to \mathbb{R}$ a bounded measurable function. The following conditions are equivalent:*

(i) *h is harmonic (i.e., $h \in C^\infty(M)$ and $\Delta h = 0$).*

(ii) *$h(x) = \mathbb{E}[h \circ X_\tau^x]$ for any $x \in M$ and any stopping time τ such that $0 \leq \tau < \zeta$ a.s.*

(iii) *h has the mean-value property, i.e., for any $x_0 \in M$ and any sufficiently small geodesic ε-ball $B_\varepsilon(x_0) \subset M$ about x_0,*

$$h(x) = \mathbb{E}[h \circ X_{\tau^x}^x], \quad x \in B_\varepsilon(x_0),$$

where $\tau^x = \inf\big\{t \geq 0 : X_t^x \notin B_\varepsilon(x_0)\big\}$ is the first exit time from $B_\varepsilon(x_0)$.

PROOF. (i) $\Rightarrow$ (ii) is a direct consequence of Itô's formula combined with the Optional Sampling Theorem.

(ii) $\Rightarrow$ (iii) is a weakening; the almost sure finiteness of the exit time of Brownian motions from small geodesic balls follows immediately from Theorem 2.2.1.

(iii) $\Rightarrow$ (i): We exploit the solvability of the Dirichlet problem for small geodesic balls in the following sense (see e.g. [**2**]): To each $x_0 \in M$ and sufficiently small $\varepsilon > 0$

there exists a family $\big(k_\varepsilon(x,dy)\big)_{x\in B_\varepsilon(x_0)}$ of "harmonic" measures $k_\varepsilon(x,dy)$ on $S_\varepsilon(x_0)=\partial B_\varepsilon(x_0)$, namely $k_\varepsilon(x,dy)=\mathbb{P}\circ(X^x_{\tau^x})^{-1}(dy)$, such that for each bounded measurable boundary function $f\colon S_\varepsilon(x_0)\to\mathbb{R}$ a harmonic function φ_f is defined on $B_\varepsilon(x_0)$ by

$$\varphi_f(x)=\int k_\varepsilon(x,dy)\,f(y)\equiv\mathbb{E}[f\circ X^x_{\tau^x}].$$

For $f\in C\big(S_\varepsilon(x_0);\mathbb{R}\big)$ the function φ_f is the unique harmonic continuation of f to $B_\varepsilon(x_0)$. The support of $\mathbb{P}\circ(X^x_{\tau^x})^{-1}$ is $S_\varepsilon(x_0)$, i.e., $\mathbb{P}\{X^x_{\tau^x}\in U\}>0$ for each non-empty open subset $U\subset S_\varepsilon(x_0)$.

Applied to our situation this means that $h=\varphi_h$ on $B_\varepsilon(x_0)$ for each $x_0\in M$ and each sufficiently small $\varepsilon>0$; in particular h is harmonic. □

Note that the equivalence (i) ⇔ (iii) in Lemma 2.2.14 also holds for not necessarily bounded functions.

COROLLARY 2.2.15 (Maximum principle). *Let (M,g) be a Riemannian manifold, $h\colon M\to\mathbb{R}$ a harmonic function, $m=\sup_{x\in M}h(x)\in\overline{\mathbb{R}}$. If $h(x_0)=m$ for some $x_0\in M$, then h is constant.*

PROOF. The set $M_0:=\{x\in M: h(x)=m\}$ is open in M as a consequence of the mean value property; trivially, M_0 is closed by the continuity of h. Since all manifolds are assumed to be connected, the claim follows. □

The next Theorem shows how on a Riemannian manifold (M,g) asymptotic properties of BM(M,g) and richness of harmonic functions on M correspond to each other.

THEOREM 2.2.16. *For a Riemannian manifold (M,g) the following two items are equivalent:*

(i) BM(M,g) *has only trivial exit sets, i.e., if X is a Brownian motion on (M,g) starting from a deterministic initial point and $U\subset\hat M$ an open subset of the one-point-compactification $\hat M$ of M, then*

$$\mathbb{P}\{X_t\in U\text{ eventually}\}\in\{0,1\}.$$

(ii) (M,g) *is a Liouville manifold, i.e., all bounded harmonic functions on M are constant.*

For a Brownian motion X on (M,g) with lifetime ζ, we use again the convention $X_t(\omega)=\infty$ in $\hat M$ for $t>\zeta(\omega)$. If $X_0=x\in M$, we write $X=X^x$ and denote by ζ^x the corresponding lifetime.

PROOF OF THEOREM 2.2.16. For $U\subset\hat M$ let

$$\mathscr{H}_U:=\big\{\alpha\in C(\mathbb{R}_+;\hat M):\alpha(t)\in U\text{ eventually}\big\},$$

so that

$$\begin{aligned}X_\bullet^{-1}(\mathscr{H}_U)&=\{X_t\in U\text{ eventually}\}\\&=\big\{\omega\in\Omega:\exists\,t_0(\omega)>0\text{ such that }X_t(\omega)\in U\text{ for all }t\ge t_0(\omega)\big\}.\end{aligned}$$

We first note that for an open $U\subset\hat M$ the function $h_U\colon M\to\mathbb{R}$,

$$h_U(x):=\mathbb{P}\{X^x_t\in U\text{ eventually}\}=\mathbb{E}[1_{\mathscr{H}_U}\circ X^x_\bullet],$$

is harmonic. Indeed, by Remark 2.2.5 (strong Markov property of Brownian motion) it holds for each stopping time τ with $0\le\tau<\zeta^x$ that

$$h_U(X^x_\tau)=\mathbb{E}[1_{\mathscr{H}_U}(X^y_\bullet)]|_{y=X^x_\tau}=\mathbb{E}^{\mathscr{F}_\tau}[1_{\mathscr{H}_U}(X^x_{\tau+\bullet})]=\mathbb{E}^{\mathscr{F}_\tau}[1_{\mathscr{H}_U}(X^x_\bullet)]\quad\text{a.s.}$$

and hence $h_U(x) = \mathbb{E}[h_U(X^x_\tau)]$. Thus, by Lemma 2.2.14 (ii), the function h_U is harmonic. In particular, if in addition $h_U(x) \in \{0,1\}$ for some $x \in M$, then already $h_U \equiv 0$ or $h_U \equiv 1$ according to the maximum principle.

(i) ⇒ (ii): Let h be a bounded harmonic function on M. Then $h(X^x)$ is a bounded and hence almost surely convergent martingale; let $\xi^x := \lim_{t\uparrow\zeta^x} h(X^x_t)$. To $\alpha \in \mathbb{R}$ we consider the open set $U_\alpha := \{h > \alpha\}$. By assumption and the maximum principle, for each of the harmonic functions h_{U_α} on M,

$$h_{U_\alpha}(x) = \mathbb{P}\{X^x_t \in U_\alpha \text{ eventually}\} = \mathbb{P}\{h \circ X^x_t > \alpha \text{ eventually}\},$$

it follows that either $h_{U_\alpha} \equiv 0$ or $h_{U_\alpha} \equiv 1$. For any real α, hence $\mathbb{P}\{\xi^x \le \alpha\} \in \{0,1\}$, independently of x. This shows that $\xi^x \equiv \lambda$ a.s. with a constant λ independent of x. Hence h is constant, namely $h(x) = \mathbb{E}[\xi^x] \equiv \lambda$.

(ii) ⇒ (i): Since by assumption bounded harmonic functions on M are constant, it holds in particular for any open subset $U \subset \hat{M}$ that

$$h_U(x) = \mathbb{P}\{X^x_t \in U \text{ eventually}\} \equiv \lambda \in [0,1].$$

We need to show that $h_U(M) \subset \{0,1\}$. But we have

$$\lambda \equiv h_U(X^x_t) = \mathbb{E}^{\mathscr{F}_t}[1_{\mathscr{H}_U}(X^x_\bullet)] \to 1_{\mathscr{H}_U} \circ X^x_\bullet \quad \text{a.s. as } t \to \infty,$$

and hence $\lambda \in \{0,1\}$. □

THEOREM 2.2.17. BM(M,g) *is either recurrent or transient, i.e., for any Brownian motion X on a metrically complete Riemannian manifold (M,g) the following dichotomy holds: Either it holds*

(i) $\liminf_{t\uparrow\zeta} d(X_0, X_t) = 0$ *a.s.* *or* (ii) $\liminf_{t\uparrow\zeta} d(X_0, X_t) = \infty$ *a.s.*

PROOF. For $x \in M$ let X^x be a Brownian motion on (M,g) starting from x and

$$A_x := \big\{\liminf_{t\uparrow\zeta^x} d(X^x_0, X^x_t) = 0\big\} \subset \{\zeta^x = \infty\}.$$

The function h_1 on M defined by $h_1(x) := \mathbb{P}(A_x)$ is independent of the choice of the Brownian motion starting in x, and as consequence of the strong Markov property (Remark 2.2.5) harmonic on M by Lemma 2.2.14 (ii). From

$$\mathbb{P}\big\{\liminf_{t\uparrow\zeta^x} d(X^x_0, X^x_t) = 0\big\} > 0 \tag{2.2.29}$$

for one $x \in M$, hence (2.2.29) already follows for all $x \in M$. In addition, given a non-empty open subset $U \subset M$, then for any $x \in M$,

$$\mathbb{P}\{X^x_t \in U \text{ infinitely often}\} > 0,$$

since also $h_2(x) := \mathbb{P}\{X^x_t \in U \text{ infinitely often}\}$ is harmonic and $h_2|U > 0$ by (2.2.29). But then, for each non-empty open subset $U \subset M$, it must already hold that

$$\mathbb{P}\{X^x_t \in U \text{ infinitely often}\} = 1$$

for any $x \in M$, since $h_2(X)$ is an almost surely convergent martingale and X^x enters with positive probability every non-empty open subset infinitely often, from where it follows that h_2 is constant and hence $h_2 \equiv 1$.

The alternative to the condition $\liminf_{t\uparrow\zeta^x} d(X^x_0, X^x_t) = 0$ a.s. for each $x \in M$ is $\liminf_{t\uparrow\zeta^x} d(X^x_0, X^x_t) > 0$ a.s. for one (and then each) $x \in M$. By the strong Markov property of the Brownian motion however the condition $\liminf_{t\uparrow\zeta^x} d(X^x_0, X^x_t) > 0$ a.s. implies $X^x_t \to \infty$ a.s. as $t \uparrow \zeta^x$, and since (M,g) is metrically complete by assumption, this means $\liminf_{t\uparrow\zeta^x} d(X^x_0, X^x_t) = \infty$ a.s.

The generalization of the dichotomy above to Brownian motions with not necessarily deterministic starting values is again a consequence of the strong Markov property. □

Recurrence implies that Brownian motions return to any fixed non-empty open set infinitely often; their lifetime is hence infinite. In contrast, transience means that Brownian motions eventually leave every compact set. Historically, Riemannian manifolds with transient Brownian motion are called *hyperbolic*, whereas Riemannian manifolds with recurrent Brownian motion are called *parabolic*.

We now turn the discussion to the asymptotics of Brownian motions on model manifolds which is a type spaces typically used as comparison manifolds. From a probabilistic point of view, they have the property that in polar coordinates their radial and angular process can be decoupled by a time transformation and completely characterized (see also [**18, 34**]). In this connection the angular behaviour is described by a martingale on the sphere, and hence the martingale convergence theorem allows to decide whether Brownian motion takes eventually an asymptotic direction or leaves every angular sector infinitely often. The same problem, namely existence of an asymptotic angle of Brownian motion, has been studied also for simply connected negatively curved Riemannian manifolds such that $-a^2 \leq \mathrm{Riem}^M \leq -b^2 < 0$ in [**44**], [**1**]; see also [**30**].

Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 2$ and $x \in M$ such that the exponential map at x defines a diffeomorphism; we may pull back the metric to T_xM and identify $T_xM \cong \mathbb{R}^n$ by choosing an orthonormal basis in T_xM:

$$(\mathbb{R}^n, \exp_x^* g) \cong (T_xM, \exp_x^* g) \xrightarrow[\exp_x]{\sim} (M, g)\,. \tag{2.2.30}$$

In this way we identify M and $\mathbb{R}^n$ also as sets. In geodesic polar coordinates on $M\backslash\{0\} = \,]0,\infty[\, \times \mathbb{S}^{n-1}$ about $0 \in M$ we then have $g = dr \otimes dr + h_r$ with a Riemannian metric h_r on $\mathbb{S}^{n-1}$ depending on r, where we consider the following special cases:

(a) $g = dr \otimes dr + f^2(r, \cdot)\, h$ where $f\colon \,]0,\infty[\, \times \mathbb{S}^{n-1} \to \,]0,\infty[$ is a scalar function and h a Riemannian metric on $\mathbb{S}^{n-1}$ which is independent of r.
(b) $g = dr \otimes dr + f^2(r)\, d\vartheta^2$ where $f\colon \,]0,\infty[\, \to \,]0,\infty[$ is a scalar function and $d\vartheta^2$ the standard metric on $\mathbb{S}^{n-1}$.

The situation (b) corresponds to the already treated model manifolds, whereas in (a) the induced metric on $\mathbb{S}^{n-1}$ is allowed to vary with the angle via the function $f\colon \,]0,\infty[\, \times \mathbb{S}^{n-1} \to \,]0,\infty[$. In order to study the angular behaviour of $\mathrm{BM}(M, g)$ on such manifolds relative to 0, we first investigate geometric properties of the angular map

$$q\colon M\backslash\{0\} \to \mathbb{S}^{n-1}, \quad (r, \vartheta) \mapsto \vartheta \tag{2.2.31}$$

induced by (2.2.30).

LEMMA 2.2.18. *Let* $q\colon M\backslash\{0\} \to \mathbb{S}^{n-1}$ *be the angular map defined in* (2.2.31).

(i) *In situation* (a) *the map* $q\colon (M\backslash\{0\},\, g) \to (\mathbb{S}^{n-1}, h)$ *is harmonic if and only if*
$$(n-3)\ \mathrm{grad}\, f_r = 0$$
where $\mathrm{grad}\, f_r$ *denotes the gradient vector field of* $f_r = f(r, \cdot)$ *on* (S^{n-1}, h).
(ii) *In situation* (b) *the map* $q\colon (M\backslash\{0\},\, g) \to (\mathbb{S}^{n-1}, d\vartheta^2)$ *is affine and in addition a harmonic morphism.*

PROOF. We want to calculate the second fundamental form of $q\colon (M\backslash\{0\},\, g) \to (\mathbb{S}^{n-1}, h)$ with respect to the fixed Riemannian metric on $\mathbb{S}^{n-1}$. As can be seen from formula (1.7.2), in charts (φ, U) for $M\backslash\{0\}$ and (ψ, V) for $\mathbb{S}^{n-1}$ such that $\varphi(U) \subset V$ the following general representation in coordinates holds:

$$(\nabla dq)^k_{ij} = \partial_i \partial_j q^k - \sum_\alpha \Gamma^\alpha_{ij} \, (\partial_\alpha q^k) + \sum_{\alpha,\beta} \bar\Gamma^k_{\alpha\beta} \, (\partial_i q^\alpha)(\partial_j q^\beta)$$

with indices $1 \le i, j \le n$, $1 \le k \le n-1$ and Γ resp. $\bar\Gamma$ the Christoffel symbols with respect to the Levi-Civita connection on $(M\backslash\{0\}, g)$, resp. on $(\mathbb{S}^{n-1}, h)$. Thus choosing coordinates of the form $\varphi = (\theta^1, \ldots, \theta^{n-1}, r)$ for $M\backslash\{0\}$ with $r(\cdot) = d(0, \cdot)$ and $\psi = (\theta^1, \ldots, \theta^{n-1})$ for $\mathbb{S}^{n-1}$, one obtains

$$(\nabla dq)^k_{ij} = \begin{cases} -\Gamma^k_{ij} & i = n \text{ or } j = n \\ -\Gamma^k_{ij} + \bar\Gamma^k_{ij} & 1 \le i, j \le n-1. \end{cases}$$

As by (1.5.2) the Christoffel symbols of the Levi-Civita connection can be expressed via the Riemannian metric through

$$\Gamma^k_{ij} = \frac{1}{2} \sum_\ell g^{k\ell} \big\{ \partial_i \, g_{\ell j} + \partial_j \, g_{i\ell} - \partial_\ell \, g_{ij} \big\},$$

we observe that $(\nabla dq)^k_{ij} = 0$ for $i = n$ or $j = n$, as well as

$$(\nabla dq)^k_{ij} = (1/f) \sum_{\ell=1}^{n-1} h^{k\ell} \big\{ h_{\ell j} \, \partial_i f + h_{i\ell} \, \partial_j f - h_{ij} \partial_\ell f \big\}, \quad 1 \le i, j \le n-1.$$

This shows at one hand that q is affine in case (b), whereas in case (a)

$$\tau(q)^k = \sum_{i,j=1}^n g^{ij} (\nabla dq)^k_{ij} = -(n-3) \, f^{-3} \sum_{i=1}^{n-1} h^{ki} \, \partial_i f$$

holds, and hence $\tau(q)\,(r, \cdot) = -(n-3)\, f_r^{-3}$ grad f_r. It remains to verify that q in case (b) defines in addition an harmonic morphism. Denoting by Δ the Laplacian on (M, g) and accordingly by $\bar\Delta$ the Laplacian on $(\mathbb{S}^{n-1}, h)$, it is immediate to check that

$$f^2 \, \Delta(\varphi \circ q) = (\bar\Delta \varphi) \circ q$$

for every differentiable function $\varphi \in C^\infty(\mathbb{S}^{n-1})$, which according to Theorem 1.7.19 implies that q is an harmonic morphism with Dilatation f^{-1}. $\square$

From the probabilistic point of view, Lemma 2.2.18 shows in particular that the angular process of a Brownian motion on a model is an $\mathbb{S}^{n-1}$-valued martingale with respect to the standard metric. This observation enables a complete description of the asymptotics of the angular behaviour.

THEOREM 2.2.19 (Brownian motion on models). *Let M be a n-dimensional model with center $0 \in M$ and Riemannian metric $g = dr \otimes dr + f^2(r)\, d\vartheta^2$. Let X be a Brownian motion on (M, g) with $X_0 = x_0 \ne 0$, decomposed according to $M\backslash\{0\} = \,]0, \infty[\, \times \mathbb{S}^{n-1}$ in its radial and angular part $X = (R, \Theta)$.*

(i) *For the radial process, $R_t \to \infty$ almost surely (i.e., X is transient) if and only if*

$$\int_1^\infty f^{1-n}(r) \, dr < \infty.$$

(ii) *The lifetime ζ of X is either a.s. finite or a.s. infinite, and a.s. finite if and only if*

$$\int_1^\infty f^{n-1}(r) \left\{ \int_r^\infty f^{1-n}(\rho) \, d\rho \right\} dr < \infty.$$

(iii) *The angular process Θ converges on $\mathbb{S}^{n-1}$ for $t \uparrow \zeta$ a.s., if and only if*

$$\int_1^\infty f^{n-3}(r) \left\{ \int_r^\infty f^{1-n}(\rho)\, d\rho \right\} dr < \infty.$$

The latter is equivalent to M being no Liouville manifold.

Note that in case $d = 2$, a model (M, g) is a Liouville manifold if and only if $\mathrm{BM}(M, g)$ is recurrent.

PROOF. (1) By Theorem 2.2.1 it holds that $dR = dW + \frac{1}{2}\,\Delta r(X)\,dt$ where W is a one-dimensional Brownian motion. Since $\Delta r(X) = (n-1)\,(f'/f)(R)$, the radial process R satisfies the SDE

$$dR = dW + \frac{1}{2}\,(n-1)\,(f'/f)(R)\,dt.$$

Thus R is a one-dimensional diffusion with infinitesimal generator

$$\frac{1}{2}\left\{D^2 + (n-1)\,(f'/f)\,D\right\}.$$

We want to calculate the Riemannian quadratic variation of the martingales Θ. To this end, we first note that in each chart (φ, U) of the form $\varphi = (r, \theta)$ and $U = \left]0, \infty\right[\times U'$ where (θ, U') is a chart for $\mathbb{S}^{n-1}$ obviously

$$\begin{aligned} 1_{\{X \in U\}}\, d[X, X] &= 1_{\{X \in U\}} \sum_{i,j=1}^{n} g_{ij}(X)\, d[X^i, X^j] \\ &= 1_{\{X \in U\}} \left\{ d[R, R] + f^2(R)\, d[\Theta, \Theta] \right\} \end{aligned}$$

holds, from where we conclude that

$$d[X, X] = d[R, R] + f^2(R)\, d[\Theta, \Theta].$$

Taking into account that $d[X, X] = (\dim M)\,dt = n\,dt$ and $d[R, R] = dt$, we finally obtain

$$d[\Theta, \Theta] = (n-1)\, f^{-2}(R)\, dt. \tag{2.2.32}$$

By Theorem 1.8.8 (convergence theorem of Darling-Zheng), hence Θ converges on $\mathbb{S}^{n-1}$ for $t \uparrow \zeta$ almost surely if and only if $\int_0^\zeta f^{-2}(R_t)\,dt < \infty$ almost surely. We let

$$T(t) = \textstyle\int_0^t f^{-2}(R_s)\,ds, \quad t < \zeta,$$

and consider for $t < T_\zeta$ the continuous time change (τ_t) where

$$\tau_t := T^{-1}(t) \equiv \inf\{s \in \mathbb{R}_+ : T(s) \geq t\}.$$

Since T_ζ is obviously the maximal lifetime of the time transformed radial process $\tilde{R}_t := R_{\tau_t}$, we get as consequence that Θ converges for $t \uparrow \zeta$ if and only the lifetime of $\tilde{R}$ is finite, almost surely.

By Lemma 2.2.18 the map q is an harmonic morphism with dilatation f^{-1}; consequently X decomposes as $X_t = (R_t, B_{T(t)})$ with B a Brownian motion on $(\mathbb{S}^{n-1}, d\vartheta^2)$. By the time-change (τ_t) the radial and angular process decompose as $X_{\tau_t} = (R_{\tau_t}, B_t)$ for $t < T_\zeta$; in the new clock the angular component is described by a $\mathrm{BM}(\mathbb{S}^{n-1}, d\vartheta^2)$ which runs up to time T_ζ; hence it converges if and only if T_ζ is almost surely finite.

(2) By Eq. (2.2.1) and Theorem 2.1.57 (ii) the radial process R solves the one-dimensional SDE

$$dR = d\hat{W} + \frac{1}{2}\,(n-1)\,(f'/f)(R)\,dt\,;$$

hence R is a diffusion on $]0,\infty[$ with infinitesimal generator

$$\frac{1}{2}\,\big(D^2+(n-1)\,(f'/f)\,D\big)\,.$$

Correspondingly the time-changed process $\tilde R$ is a diffusion on $]0,\infty[$ with generator

$$\frac{1}{2}f^2\,\big(D^2+(n-1)\,(f'/f)\,D\big)\,.$$

The questions that interest us here concerning transience and lifetime of R resp. $\tilde R$ can hence be answered by means of Theorem A.1.9. Set $c_1=0$, $c_2=\infty$ and without restrictions $c=1$. Consider

$$H(r)=\exp\left(-(n-1)\int_1^r\frac{f'}{f}(s)\,ds\right)=f^{1-n}(r)\,f^{n-1}(1),\quad 0<r<\infty.$$

As seen in the proof of Theorem 2.2.1, almost surely, Brownian motion does not hit any point fixed in advance; hence 0 is both for R and $\tilde R$ a non-accessible boundary point, and it is thus sufficient to investigate the respective behaviour at the right-hand boundary point $c_2=\infty$. In detail we find:

$$\begin{aligned}
R_t\to\infty \text{ a.s. (transient)} &\iff \int_1^\infty f^{1-n}(r)\,dr<\infty\,;\\
R \text{ has finite lifetime a. s.} &\iff \int_1^\infty H(r)\left[\int_1^r\frac{d\rho}{H(\rho)}\right]dr<\infty\\
&\iff \int_1^\infty \frac{1}{H(r)}\left[\int_r^\infty H(\rho)\,d\rho\right]dr<\infty\\
&\iff \int_1^\infty f^{n-1}(r)\left[\int_r^\infty f^{1-n}(\rho)\,d\rho\right]dr<\infty\,;\\
\tilde R \text{ has finite lifetime a. s.} &\iff \int_1^\infty f^{n-3}(r)\left[\int_r^\infty f^{1-n}(\rho)\,d\rho\right]dr<\infty.
\end{aligned}$$

(3) It remains to show that M supports con-constant bounded harmonic functions exactly if the angular process Θ converges almost surely on the sphere $\mathbb{S}^{n-1}$ as $t\uparrow\zeta$.

We consider first the case that M supports non-constant bounded harmonic functions. Then (M,g) is not a Liouville manifold, and by Theorem 2.2.16 there exist non-trivial exit set for $\mathrm{BM}(M,g)$. It is easy to see that among them some must be of the form $\mathbb{R}_+\times V$ where $V\subset\mathbb{S}^{n-1}$ is an open subset. This excludes the possibility that $T_\zeta\equiv\infty$ almost surely, as a consequence of the recurrence of $\mathrm{BM}(\mathbb{S}^{n-1},d\vartheta^2)$. Recall that T_ζ is the lifetime of the time-changed radial process $\tilde R$ and hence almost surely infinite or almost surely finite. Thus T_ζ almost surely finite must hold, which, as already shown, is equivalent to convergence of the angular process Θ on $\mathbb{S}^{n-1}$.

Conversely, suppose that Θ converges almost surely on the sphere $\mathbb{S}^{n-1}$ as $t\uparrow\zeta$. Write Θ^x for the angular part of a Brownian motion X^x starting at $x\in M$. For each function $\varphi\in C(\mathbb{S}^{n-1})$ then

$$u(x):=\mathbb{E}\big[\varphi(\Theta^x_\zeta)\big]$$

defines a bounded harmonic function u on M. Since

$$u(X^x_{t\wedge\zeta})\to\varphi(\Theta^x_\zeta)\quad\text{almost surely as } t\to\infty,$$

the considered function u is non-constant if and only if $\varphi(\Theta^x_\zeta)$ is non-degenerate on $\mathbb{S}^{n-1}$, i.e. not almost surely constant. If however the "exit measure" $\mathbb{P}\circ(\Theta^x_\zeta)^{-1}$ equals the Dirac measure δ of a point on $\mathbb{S}^{n-1}$, then as an easy application of the maximum principle shows, the measures $\mathbb{P} \circ (\Theta_\zeta^x)^{-1}$ for $x \in M$ must be identical, hence $\mathbb{P} \circ (\Theta_\zeta^x)^{-1} = \delta$, independently of x. This is in contradiction to the rotational invariance of $\mathbb{P} \circ (\Theta_\zeta^0)^{-1}$ on $\mathbb{S}^{n-1}$ which comes from the fact that (M, g) is a model. $\square$

It is straight-forward to translate the integral conditions in Theorem 2.2.19 into curvature bounds. This then enables a geometric characterization of the Liouville property.

THEOREM 2.2.20. *Let (M, g) be an n-dimensional model with center $0 \in M$ and Riemannian metric $g = dr \otimes dr + f^2(r)\, d\vartheta^2$. For the radial curvature function $k_M(r) = -f''(r)/f(r)$ of (M, g) assume that $k_M(\cdot) \le 0$. Furthermore let $c = 1$ in case $n = 2$, respectively $c = 1/2$ in case $n \ge 3$. Then there exist non-constant harmonic functions on M, if $k_M(r) \le -\dfrac{(c+\varepsilon)}{r^2 \log r}$ for some $\varepsilon > 0$ and sufficiently large r. In contrary, if $k_M(r) \ge -\dfrac{(c-\varepsilon)}{r^2 \log r}$ for some $\varepsilon > 0$ and sufficiently large r, then M is a Liouville manifold.*

Note that constant negative curvature outside a compact set is not sufficient for the existence of non-constant bounded harmonic functions. The condition is not even enough for transience of Brownian motion, as the following example shows. Let $M = \mathbb{R}^2$ be a two-dimensional rotationally symmetric manifold, for instance, with radial function $f(r) = \exp(-r)$ for $r > 1$ and a differentiable interpolation for $0 \le r \le 1$, such that $f(0) = 0$ and $f'(0) = 1$ holds. Then (M, g) has constant negative curvature outside the unit disk, but according to Theorem 2.2.19, Brownian motion on (M, g) is recurrent; M is hence a Liouville manifold.

We proceed with an elementary Lemma before turning to the proof of Theorem 2.2.20.

LEMMA 2.2.21. *Let $n \ge 2$ and (M, g) be an n-dimensional model with Riemannian metric $g = dr \otimes dr + f^2(r)\, d\vartheta^2$. Let $k = -f''/f$ be the radial curvature function of (M, g) and*

$$I(f) = \int_1^\infty f^{n-3}(r) \left[\int_r^\infty f^{1-n}(\rho)\, d\rho \right] dr.$$

Furthermore, let $(\tilde{M}, \tilde{g})$ another n-dimensional model with metric $\tilde{g} = dr \otimes dr + \tilde{f}^2(r)\, d\vartheta^2$, and define $\tilde{k} = -\tilde{f}\,''/\tilde{f}$ and $I(\tilde{f})$ correspondingly.

- (i) *If $\tilde{k} \le k$ on $]0, \infty[$ then $f \le \tilde{f}$ and $f'/f \le \tilde{f}\,'/\tilde{f}$ on $]0, \infty[$.*
- (ii) *If $\tilde{k} \le k$ on $]\rho_0, \infty[$ for some $\rho_0 > 0$, then $(f'/f)(\rho_0) \le (\tilde{f}'/\tilde{f})(\rho_0)$ implies $f'/f \le \tilde{f}\,'/\tilde{f}$ on the interval $[\rho_0, \infty[$.*
- (iii) *If $k, \tilde{k} \le 0$ and $\tilde{k} \le k$ on $]\rho_0, \infty[$ for some $\rho_0 > 0$, then there exists a constant $c > 0$ such that $f \le c\,\tilde{f}$ on $[0, \infty[,$.*
- (iv) *If $k, \tilde{k} \le 0$ and $\tilde{k} \le k$ on $]\rho_0, \infty[$ for some $\rho_0 > 0$, then with $I(f) < \infty$ also $I(\tilde{f}) < \infty$.*

PROOF. (1) The claims in (i) and (ii) follow immediately from

$$\big(f^2\, (\tilde{f}/f)\,'\big)' = (f\tilde{f}\,' - \tilde{f} f\,')' = \tilde{f} f \left(\tilde{f}\,''/\tilde{f} - f\,''/f\right) = \tilde{f} f\,(k - \tilde{k}). \tag{2.2.33}$$

(2) Let now $k, \tilde{k} \le 0$. From $k \le 0$ we deduce by (i) that $r \le f(r)$ and $1/r \le f'(r)/f(r)$ for $0 < r < \infty$; correspondingly for $\tilde{f}$. In particular, f' and $\tilde{f}'$ are strictly positive on $]0, \infty[$. We set $k_+ := k \vee \tilde{k}$, $k_- := k \wedge \tilde{k}$ and consider the C^2 solutions $u_\pm$ of

$$u'' + k_\pm\, u = 0 \quad \text{with } u(0) = 0,\ u'(0) = 1.$$

As above, we conclude that $u_\pm$ and $u'_\pm$ are strictly positive on $]0,\infty[$; in particular

$$\begin{aligned} c_1\, u_+(\rho_0) &\le f(\rho_0) \le c_2\, u_+(\rho_0) \\ c_1\, u'_+(\rho_0) &\le f'(\rho_0) \le c_2\, u'_+(\rho_0) \end{aligned}$$

with appropriate constants $c_1, c_2 > 0$. But u_+ and f satisfy identical differential equations on $]\rho_0, \infty[$ which implies

$$c_1\, u_+(r) \le f(r) \le c_2\, u_+(r), \quad r \ge \rho_0. \tag{2.2.34}$$

Analogously, for suitable constants $\tilde c_1, \tilde c_2 > 0$, we obtain the inequality

$$\tilde c_1\, u_-(r) \le \tilde f(r) \le \tilde c_2\, u_-(r), \quad r \ge \rho_0. \tag{2.2.35}$$

By (2.2.33), using u_+, u_- instead of $f, \tilde f$, we conclude $u_+ \le u_-$ on $[0,\infty[$; the combination of (2.2.34) and (2.2.35) then gives $f \le c\, \tilde f$ for some constant $c > 0$; at first on $[\rho_0, \infty[$ and after eventually enlarging c then on all of $[0,\infty[$.

(3) The claim in part (iv) can be easily reduced to the case $\rho_0 = 0$ by following the arguments used in (2): By (2.2.34) the condition $I(f) < \infty$ is equivalent $I(u_+) < \infty$; thus one may replace f by u_+ with the consequence that then $\tilde k \le k_+$ holds on all of $]0,\infty[$. Without restrictions we may thus assume that $\rho_0 = 0$. We let

$$I_s(f) = \int_1^s f^{n-3}(r)\, \Big[\int_r^\infty f^{1-n}(\rho)\, d\rho\Big]\, dr, \quad s \ge 1$$

and will show that under the condition $\tilde k \le k \le 0$ it holds that

$$I_s(\tilde f\,) \le I_s(f), \quad s \ge 1\,.$$

To this end we may assume that $\int_1^\infty f^{\,1-n}(\rho)\, d\rho < \infty$ and $n \ge 3$. At first we then have

$$\frac{d}{ds} I_s(f\,) = f^{\,n-3}(s)\, \Big[\int_s^\infty f^{\,1-n}(\rho)\, d\rho\Big] \ge 0$$

and hence

$$f^{\,3-n}(s)\, \frac{d}{ds} I_s(f\,) = \int_s^\infty f^{\,1-n}(\rho)\, d\rho \searrow 0 \quad \text{as } s \uparrow \infty. \tag{2.2.36}$$

Differentiation of (2.2.36) gives

$$\frac{d}{ds}\Big[f^{\,3-n}(s)\, \frac{d}{ds} I_s(f\,)\Big] = -f^{\,1-n}(s)$$

and then

$$\frac{d^2}{ds^2} I_s(f) - (n-3)\, \frac{f^{\,\prime}(s)}{f(s)}\, \frac{d}{ds} I_s(f) + \frac{1}{f(s)^2} = 0.$$

Using the assumptions along with part (i), we get

$$\frac{d^2}{ds^2} I_s(f\,) - (n-3)\, \frac{\tilde f^{\,\prime}(s)}{\tilde f(s)}\, \frac{d}{ds} I_s(f\,) + \frac{1}{\tilde f(s)^{\,2}} \le 0$$

and hence

$$-\tilde f^{\,1-n}(s) \ge \tilde f^{\,3-n}(s)\, \frac{d^2}{ds^2} I_s(f) - (n-3)\, \frac{\tilde f^{\,\prime}(s)}{\tilde f^{\,n-2}(s)}\, \frac{d}{ds} I_s(f) = \frac{d}{ds}\Big[\tilde f^{\,3-n}(s)\, \frac{d}{ds} I_s(f\,)\Big].$$

Integration then gives

$$-\int_r^\infty \tilde f^{\,1-n}(\rho)\, d\rho \ge \lim_{u\to\infty} \Big[\tilde f^{\,3-n}(s)\, \frac{d}{ds} I_s(f\,)\Big]_{s=r}^{s=u}$$

and thus

$$\tilde f^{\,n-3}(r)\int_r^\infty \tilde f^{1-n}(\rho)\,d\rho \le \frac{d}{dr} I_r(f\,).$$

Integrating again finally gives for $s \ge 1$

$$I_s(\tilde f\,) = \int_1^s \tilde f^{\,n-3}(r)\,\left[\int_r^\infty \tilde f^{\,1-n}(\rho)\,d\rho\right]\,dr \le I_s(f) - I_1(f) = I_s(f),$$

which is the claim. $\square$

PROOF OF THEOREM 2.2.20. For $\alpha > 0$ and $\varrho > 1$ let $\Phi(r) := r\,(\log r)^\alpha$ and $F(r) := \big(\Phi(r+\varrho) - \Phi(\varrho)\big)/\Phi'(\varrho)$. Then it holds that $F(0) = 0$, $F'(0) = 1$ and $-(F''/F)(r) \le 0$ for $r > 0$, as well as $-(F''/F)(r) \sim -\alpha/(r^2\log r)$ as $r \to \infty$. Using that

$$\int_2^\infty \frac{dr}{r\,(\log r)^a} < \infty \iff a > 1,$$

we obtain in case $n = 2$ that $I(F) < \infty$ holds if and only if $\alpha > 1$. Analogously one sees in case $n \ge 3$, by using

$$\int_r^\infty F(\rho)^{1-n} d\rho \;\sim\; \big[(n-2)r^{n-2}(\log r)^{(n-1)\alpha}\big]^{-1} \quad \text{as } r \to \infty,$$

that $I(F) < \infty$ holds if and only if $\alpha > 1/2$. $\square$

Through the dependence on curvature of the exit time of Brownian motions from geodesic balls, in connection with the explicit knowledge of the asymptotics of Brownian motions on models, comparison theorems play an important role for applications (see [**19, 20**]).

THEOREM 2.2.22 (Comparison criterion for BM-completeness and transience).
A. *Let (M, g) be a metrically complete Riemannian manifold of dimension $n \ge 2$. Suppose there is a point $o \in M$ and a model $(\mathbb{M}, \tilde g)$ of equal dimension with center 0 such that*

$$\mathrm{Ric}^M_x(\partial^M, \partial^M) \ge (n-1)\,k_{\mathbb{M}}(r)$$

for $x \in M\backslash\mathrm{cut}(o)$ and $0 < r = d_{\mathbb{M}}(0, x)$. Then if $(\mathbb{M}, \tilde g)$ is BM*-complete also (M, g) is* BM*-complete; if* BM$(\mathbb{M}, \tilde g)$ *is recurrent, then also* BM(M, g).
B. *Let (M, g) be a metrically complete Riemannian manifold of dimension $n \ge 2$. Suppose there is a point $o \in M$ and a model $(\mathbb{M}, \tilde g)$ of equal dimension with center such that*

$$\mathrm{Riem}^M_x(E) \le k_{\mathbb{M}}(r)$$

for each radial plane E in T_xM with $x \in M\backslash\mathrm{cut}(o)$ and $0 < r = d_{\mathbb{M}}(0, x)$. If M is simply connected, then with $(\mathbb{M}, \tilde g)$ also (M, g) is not BM*-complete and* BM(M, g) *has finite lifetime almost surely; in this case* BM(M, g) *is transient if* BM$(\mathbb{M}, \tilde g)$ *is transient.*

PROOF. Part A follows from Theorem 2.2.13; Part B follows from Theorem 2.2.2, along with the observation that $\mathrm{cut}(o) = \varnothing$ under the given assumptions. Indeed, by the given curvature assumptions in case B, along normal minimal geodesic curves $\gamma\colon [0, a] \to M$ with $\gamma(0) = o$ one has

$$\mathrm{Riem}^M_{\gamma(r)}(E) \le k_{\mathbb{M}}(r)$$

for each plane $E \subset T_{\gamma(r)}M$ such that $\dot\gamma(r) \in E$. The Comparison principle (Corollary 2.1.40) then gives $\mathrm{Conj}(o) = \varnothing$. Thus $\exp_o\colon (T_oM, \exp_o^* g) \to (M, g)$ is a local isometry and hence a covering (as already justified in the proof of Theorem 2.1.41). For simply connected M hence $\exp_o\colon T_oM \to M$ defines a diffeomorphism which in particular shows

$\operatorname{cut}(o) = \varnothing$. The claims then follow from the Comparison Theorem for Brownian motion (Theorem 2.2.2). Indeed, if for instance $(\mathbb{M}, \tilde{g})$ is not BM-complete, then Brownian motions on $(\mathbb{M}, \tilde{g})$ have only a finite lifetime almost surely, and one concludes by Theorem 2.2.2 (and the notation there) that also the lifetime $\zeta \equiv \sup_\rho \tau_\rho$ for each Brownian motion on (M, g) must be finite almost surely; almost sure explosion follows first for Brownian motions with deterministic starting point, but then also in general by means of the Markov property 2.2.9. $\square$

COROLLARY 2.2.23 (Test for BM-completeness). *Let (M, g) be a metrically complete Riemannian manifold of dimension $n \geq 2$. For a given point $o \in M$ suppose that the Ricci curvature in the radial direction is bounded below by quadratic function of the distance $r_M = d_M(o, \cdot)$ to o, i.e.*

$$\operatorname{Ric}_x^M(\partial^M, \partial^M) \geq -c_1 - c_2\, r_M^2(x), \quad c_1,\, c_2 > 0.$$

Then (M, g) is BM-*complete, i.e., Brownian motions on (M, g) have infinite lifetime.*

PROOF. We compare (M, g) to the model $(\mathbb{M}, \tilde{g}) = (\mathbb{R}^n, dr \otimes dr + f^2(r)\, d\vartheta^2)$ where $f(r) = r \exp(cr^2)$ with $c > 0$. Then it holds that $-(f''/f)\,(r) = -(6c + 4c^2r^2)$, and then

$$\operatorname{Ric}_x^M(\partial^M, \partial^M) \geq -(n-1)\,(f''/f)\,\big(r_M(x)\big)$$

by choosing the constant c appropriately. The claim then follows from

$$\int_1^\infty f^{n-1}(r)\, \left\{\int_r^\infty f^{1-n}(\rho)\, d\rho\right\}\, dr = \infty$$

(which is easy to verify) by Theorem 2.2.22 A. $\square$

2.3. Heat Equation on Sections of a Vector Bundle

In this section we use martingale arguments to derive stochastic formulae for the differential of diffusion semigroups on functions and the codifferential on 1-forms. In particular, for a semigroup generated by some elliptic operator, we aim at probabilistic representations for dP_tf and $P_t(Vf)$, not involving derivatives of f, where V is a vector field. Such formulae are typically referred to under the name of Bismut type formulae [**3**]. For non-symmetric generators, such formulae for dP_tf and $P_t(Vf)$ are related to the derivative of the heat kernel in the backward, resp. forward variable.

2.3.1. Notations. Let (M, g) be an n-dimensional Riemannian manifold, ∇ be the Levi-Civita connection on M and $\pi\colon\ \mathrm{O}(TM) \to M$ be the orthonormal frame bundle over M. Let $E \to M$ be an associated vector bundle with fiber V and structure group $G = \mathrm{O}(n)$. The induced covariant derivative

$$\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$$

determines the so-called *connection Laplacian* (or *rough Laplacian*) $\square$ on $\Gamma(E)$,

$$\square a = \operatorname{trace} \nabla^2 a.$$

Note that $\nabla^2 a \in \Gamma(T^*M \otimes T^*M \otimes E)$ and hence $(\square a)_x = \sum_i \nabla^2 a(v_i, v_i) \in E_x$ where v_i runs through an orthonormal basis of T_xM. For $a, b \in \Gamma(E)$ of compact support it is immediate to check that

$$\langle \square a, b\rangle_{L^2(E)} = -\langle \nabla a, \nabla b\rangle_{L^2(T^*M\otimes E)}.$$

In this sense we have $\square = -\nabla^*\nabla$.

Let H be the horizontal subbundle of the G-invariant splitting of $T\,\mathrm{O}(TM)$ and

$$h\colon\ \pi^*TM \xrightarrow{\sim} H \hookrightarrow T\,\mathrm{O}(TM)$$

be the *horizontal lift* of the G-connection; fiberwise this bundle isomorphism reads as reads as

$$h_u\colon\ T_{\pi(u)}M \xrightarrow{\sim} H_u, \quad u \in \mathrm{O}(TM).$$

In terms of the *standard-horizontal vector fields* $H_1, \ldots, H_n$ on $\mathrm{O}(TM)$,

$$H_i(u) := h_u(ue_i), \quad u \in \mathrm{O}(TM),$$

Bochner's horizontal Laplacian Δ^{hor}, acting on smooth functions on $\mathrm{O}(TM)$, is given as

$$\Delta^{\mathrm{hor}} = \sum_{i=1}^n H_i^2.$$

To formulate the relation between $\square$ and Δ^{hor}, it is convenient to write sections $a \in \Gamma(E)$ as "equivariant functions" $F_a\colon\ \mathrm{O}(TM) \to V$ via

$$F_a(u) = u^{-1}a_{\pi(u)}$$

where we read $u \in \mathrm{O}(TM)$ as isomorphism $u\colon V \xrightarrow{\sim} E_{\pi(u)}$. Equivariance means that

$$F_a(ug) = g^{-1}F_a(u), \quad u \in \mathrm{O}(TM),\ g \in G = \mathrm{O}(n).$$

LEMMA 2.3.1 (see [**26**], p. 115). *For $a \in \Gamma(E)$ and F_a the corresponding equivariant function on $\mathrm{O}(TM)$, we have*

$$(H_iF_a)(u) = F_{\nabla_{ue_i}a}(u), \quad u \in \mathrm{O}(TM).$$

Hence

$$\Delta^{\mathrm{hor}}F_a = F_{\square a},$$

where as above

$$\square: \Gamma(E) \xrightarrow{\nabla} \Gamma(T^*M \otimes E) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M \otimes E) \xrightarrow{\mathrm{trace}} \Gamma(E).$$

PROOF. Fix $u \in \mathrm{O}(TM)$ and choose a curve γ in M such that $\gamma(0) = \pi(u)$ and $\dot\gamma = ue_i$. Let $t \mapsto u(t)$ be the horizontal lift of γ to $\mathrm{O}(TM)$ such that $u(0) = u$. Note that $\dot u(t) = h_{u(t)}\,(\dot\gamma(t))$, and in particular $\dot u(0) = h_u(ue_i) = H_i(u)$. Hence, denoting the parallel transport along γ by $/\!/_\varepsilon = u(\varepsilon)u(0)^{-1}$, we get

$$\begin{aligned} F_{\nabla_{ue_i}a}(u) &= u^{-1}\left(\nabla_{ue_i}a\right)_{\pi(u)} \\ &= u^{-1}\lim_{\varepsilon\downarrow 0}\frac{/\!/_\varepsilon^{-1}a_{\gamma(\varepsilon)} - a_{\gamma(0)}}{\varepsilon} \\ &= \lim_{\varepsilon\downarrow 0}\frac{u(\varepsilon)^{-1}a_{\gamma(\varepsilon)} - u(0)^{-1}a_{\gamma(0)}}{\varepsilon} \\ &= \lim_{\varepsilon\downarrow 0}\frac{F_a(u(\varepsilon)) - F_a(u(0))}{\varepsilon} \\ &= (H_i)_uF_a = (H_iF_a)(u). \end{aligned}$$

$\square$

2.3.2. Preliminaries. Consider diffusion processes X_t on M generated by the operator

$$L = \Delta + Z \tag{2.3.1}$$

where $Z \in \Gamma(TM)$ is a smooth vector field on M. We assume that X_t is non-explosive. Such diffusions on M may be constructed from the corresponding horizontal diffusions on $\mathrm{O}(TM)$ generated by

$$\Delta^{\text{hor}} + \bar{Z}$$

where the vector field $\bar{Z}$ is the horizontal lift of Z to $\mathrm{O}(TM)$, i.e. $\bar{Z}_u = h_u(Z_{\pi(u)})$, $u \in \mathrm{O}(TM)$. More precisely, we start from the Stratonovich stochastic differential equation on $\mathrm{O}(TM)$,

$$dU = \sum_{i=1}^n H_i(U) \circ dB^i + \bar{Z}(U)dt, \quad U_0 = u \in \mathrm{O}(TM).$$

Then for $X_t = \pi(U_t)$, the following equation holds:

$$dX = \sum_{i=1}^n Ue_i \circ dB^i + Z(X)dt, \quad X_0 = x := \pi u.$$

Here B is a Brownian motion on $\mathbb{R}^n$, speeded up by the factor 2, i.e., $dB^i dB^j = 2\delta_{ij}dt$; recall that B is the martingale part of the anti-development $\int_U \vartheta$ of X where ϑ denotes the canonical 1-form ϑ on $\mathrm{O}(TM)$, i.e. $\vartheta_u(e) = u^{-1}e_{\pi(u)}$, $e \in T_u\,\mathrm{O}(TM)$.

In particular, for $F \in C^\infty(\mathrm{O}(TM))$, resp. $f \in C^\infty(M)$, we get

$$\begin{aligned} d(F \circ U) &= \sum_{i=1}^n (H_iF)(U) \circ dB^i + (\bar{Z}F)(U)dt \\ &= \sum_{i=1}^n (H_iF)(U)dB^i + \left(\Delta^{\text{hor}} + \bar{Z}\right)(F)(U)dt, \end{aligned} \tag{2.3.2}$$

resp.

$$\begin{aligned} d(f \circ X) &= \sum_{i=1}^n df(Ue_i) \circ dB^i + (Zf)(X)dt \\ &= \sum_{i=1}^n df(Ue_i)dB^i + (\Delta + Z)(f)(X)dt. \end{aligned} \tag{2.3.3}$$

PROPOSITION 2.3.2. *Let* $/\!/_t \colon E_{X_0} \to E_{X_t}$ *be parallel transport in* E *along* X, *induced by the parallel transport on* M,

$$/\!/_t = U_t U_0^{-1} \colon T_{X_0}M \to T_{X_t}M.$$

Then, for $a \in \Gamma(E)$, *we have*

$$d\left(/\!/_t^{-1} a(X_t)\right) = \sum_{i=1}^n /\!/_t^{-1} \left(\nabla_{U_te_i} a\right) \circ dB^i + /\!/_t^{-1} \left(\nabla_Z a\right)(X_t)\,dt, \tag{2.3.4}$$

resp. in Itô form,

$$d\left(/\!/_t^{-1} a(X_t)\right) = \sum_{i=1}^n /\!/_t^{-1} \left(\nabla_{U_te_i} a\right) dB^i + /\!/_t^{-1} \left(\Box a + \nabla_Z a\right)(X_t)dt. \tag{2.3.5}$$

In short terms, the last two equations may be written as

$$d\left(/\!/_t^{-1} a(X_t)\right) = /\!/_t^{-1} \nabla_{\circ dX} a,$$

resp.

$$d\left(/\!/_t^{-1} a(X_t)\right) = /\!/_t^{-1} \nabla_{dX} a + /\!/_t^{-1} (\square a)(X_t) dt.$$

PROOF. We have $/\!/_t^{-1} a(X_t) = U_0 U_t^{-1} a(X_t) = U_0 F_a(U_t)$. It is easily checked that

$$\bar{Z} F_a = F_{\nabla_Z a}.$$

Thus, we obtain from Eq. (2.3.2)

$$\begin{aligned} dF_a(U) &= \sum_{i=1}^n (H_i F_a)(U) dB^i + \left(\Delta^{\mathrm{hor}} F_a + \bar{Z} F_a\right)(U) dt \\ &= \sum_{i=1}^n \left(F_{\nabla_{Ue_i} a}\right)(U) dB^i + \left(F_{\square a} + F_{\nabla_Z a}\right)(U) dt \\ &= \sum_{i=1}^n U^{-1}\left(\nabla_{Ue_i} a\right)(X) dB^i + U^{-1}\left(\square a + \nabla_Z a\right)(X) dt. \end{aligned}$$

□

COROLLARY 2.3.3. *Fix $T > 0$ and let $a_t \in \Gamma(E)$ solve the equation*

$$\frac{\partial}{\partial t} a_t = \square a_t + \nabla_Z a_t \quad \text{on } [0,T] \times M.$$

Then

$$N_t := /\!/_t^{-1} a_{T-t}(X_t), \quad 0 \le t \le T,$$

is a local martingale.

PROOF. Indeed we have

$$dN_t \overset{\mathrm{m}}{=} /\!/_t^{-1} \underbrace{\left(\square a_{T-t} + \nabla_Z a_t + \frac{\partial}{\partial t} a_{T-t}\right)}_{=0} (X_t)\, dt = 0,$$

where $\overset{\mathrm{m}}{=}$ denotes equality modulo differentials of local martingales. □

We are now going to look at operators $L^{\mathscr{R}}$ on $\Gamma(E)$ which differ from $\square$ by a zero-order term, in other words,

(2.3.6) $$\square - L^{\mathscr{R}} = \mathscr{R} \quad \text{where } \mathscr{R} \in \Gamma(\mathrm{End} E).$$

Thus, by definition, the action $\mathscr{R}_x \colon E_x \to E_x$ is linear for each $x \in M$.

EXAMPLE 2.3.4. A typical example is $E = \Lambda^p T^*M$ and $A^p(M) = \Gamma(\Lambda^p T^*M)$ with $p \ge 1$. The *de Rham-Hodge Laplacian*

$$\Delta^{(p)} = -(d^*d + dd^*) : A^p(M) \to A^p(M)$$

then takes the form

$$\Delta^{(p)} \alpha = \square \alpha - \mathscr{R} \alpha$$

where $\mathscr{R}$ is given by the Weitzenböck decomposition. In the special case $p = 1$, one obtains $\mathscr{R}\alpha = \mathrm{Ric}(\cdot, \alpha^\sharp)$ where $\mathrm{Ric} \colon TM \oplus TM \to \mathbb{R}$ is the Ricci tensor.

DEFINITION 2.3.5. Fix $x \in M$ and let X_t be a diffusion to $L = \Delta + Z$, starting at x. Let Q_t be the $\mathrm{Aut}(E_x)$-valued process defined by the following linear pathwise differential equation

$$\frac{d}{dt}Q_t = -Q_t \mathscr{R}_{//_t}, \quad Q_0 = \mathrm{id}_{E_x},$$

where

$$\mathscr{R}_{//_t} := //_t^{-1} \circ \mathscr{R}_{X_t} \circ //_t \in \mathrm{End}(E_x)$$

and $//_t$ is parallel transport in E along X.

PROPOSITION 2.3.6. *Let $L^{\mathscr{R}} = \square - \mathscr{R}$ be as in Eq.* (2.3.6) *and X_t be a diffusion to $L = \Delta + Z$, starting at x. Then, for any $a \in \Gamma(E)$,*

$$d\left(Q_t //_t^{-1} a(X_t)\right) = \sum_{i=1}^n Q_t //_t^{-1} \left(\nabla_{U_t e_i} a\right) dB_t^i + Q_t //_t^{-1} \left(\square a + \nabla_Z a - \mathscr{R} a\right)(X_t)\, dt.$$

PROOF. Let $n_t := //_t^{-1} a(X_t)$. Then

$$\begin{aligned} d(Q_t n_t) &= (dQ_t)\, n_t + Q_t\, dn_t \\ &= -Q_t //_t^{-1} \mathscr{R}_{X_t} //_t n_t\, dt + Q_t\, dn_t \\ &= -Q_t //_t^{-1} (\mathscr{R} a)(X_t)\, dt + Q_t\, dn_t. \end{aligned}$$

The claim thus follows from Proposition 2.3.2. $\square$

COROLLARY 2.3.7. *Fix $T > 0$ and let $X_t(x)$ be a diffusion to $L = \Delta + Z$, starting at x. Suppose that a_t solves*

$$\begin{cases} \dfrac{\partial}{\partial t} a_t = \left(\square - \mathscr{R} + \nabla_Z\right) a_t \quad \textit{on } [0,T] \times M, \\ a_t|_{t=0} = a \in \Gamma(E). \end{cases}$$

Then

$$N_t := Q_t //_t^{-1} a_{T-t}\left(X_t(x)\right), \quad 0 \le t \le T, \tag{2.3.7}$$

is a local martingale, starting at $a_T(x)$.

In particular, if Eq. (2.3.7) *is a true martingale, we arrive at the formula*

$$a_T(x) = \mathbb{E}\big[Q_T //_T^{-1} a(X_T(x))\big], \quad a \in \Gamma(E).$$

PROOF. Indeed, we have

$$dN_t \overset{\mathrm{m}}{=} Q_t //_t^{-1} \underbrace{\left((\square + \nabla_Z - \mathscr{R}) a_{T-t} + \frac{\partial}{\partial t} a_{T-t}\right)}_{=0}(X_t)\, dt = 0. \qquad \square$$

REMARK 2.3.8. Note that

$$\frac{d}{dt}Q_t = -Q_t \mathscr{R}_{//_t}, \quad \text{with } Q_0 = \mathrm{id}_{E_x},$$

implies the obvious estimate

$$\|Q_t\|_{\mathrm{op}} \le \exp\left(-\int_0^t \underline{\mathscr{R}}(X_s(x))ds\right)$$

where $\underline{\mathscr{R}}(x) = \inf\{\langle \mathscr{R}_x v, w\rangle \colon v, w \in E_x,\ \|v\| = \|w\| = 1\}$.

2.3.3. A formula for the differential. In the sequel, we consider the special case $E = T^*M$. Thus $\Gamma(E)$ is the space of differential 1-forms on M. We identify vector fields $V \in \Gamma(TM)$ and 1-forms $\alpha \in \Gamma(T^*M)$ via the metric:

$$V \longleftrightarrow V^\flat, \quad \alpha \longleftrightarrow \alpha^\#,$$

hence our results immediately apply to vector fields as well.

DEFINITION 2.3.9. Let $Z \in \Gamma(TM)$ be a vector field on M.

(1) The divergence of Z is denoted by

$$\operatorname{div} Z \in \mathcal{S}^\infty(M), \quad \operatorname{div} Z = \operatorname{trace}(v \mapsto \nabla_v Z).$$

Thus, $(\operatorname{div} Z)(x) = \sum_i \langle \nabla_{e_i} X, e_i \rangle$ where (e_i) is an orthonormal base for T_xM.

(2) The adjoint Z^* of Z is given by the relation

$$Z^* f = -Zf - (\operatorname{div} Z) f, \quad f \in C^\infty(M).$$

We have the identities:

$$\begin{aligned} \langle Z, \nabla f \rangle_{L^2(TM)} &= -\langle \operatorname{div} Z, f \rangle_{L^2(M)}, \quad \text{if } f \text{ is compactly supported,} \\ \langle Zf, h \rangle_{L^2(M)} &= \langle f, Z^* h \rangle_{L^2(M)}, \quad \text{if } f \text{ or } h \text{ is compactly supported.} \end{aligned}$$

(3) Finally,

$$\operatorname{Ric}_Z(X,Y) := \operatorname{Ric}(X,Y) - \langle \nabla_X Z, Y \rangle, \quad X, Y \in \Gamma(TM).$$

Similarly, for $\alpha \in \Gamma(T^*M)$, let

$$(\operatorname{div} \alpha)(x) = \operatorname{trace}\big(T_xM \xrightarrow{\nabla \alpha} T_x^*M \xrightarrow{\#} T_xM\big).$$

Thus $\operatorname{div} Y = \operatorname{div} Y^\flat$ and $\operatorname{div} \alpha = \operatorname{div} \alpha^\#$. In the same way, let $\operatorname{Ric}_Z(\alpha) := \operatorname{Ric}_Z(\cdot, \alpha^\sharp)$ for any $\alpha \in \Gamma(T^*M)$.

LEMMA 2.3.10 (Commutation rules). *Let $Z \in \Gamma(TM)$.*

(1) *For the exterior differential d, we have*

$$d\big(\Delta + Z\big) = \big(\Box - \operatorname{Ric}_Z + \nabla_Z\big) d$$

(2) *For the codifferential $d^* = -\operatorname{div}$, we have*

$$\big(\Delta + Z^*\big) \operatorname{div} = \operatorname{div}\big(\Box - \operatorname{Ric}_Z^* + \nabla_Z^*\big),$$

where the formal adjoint of ∇_Z (acting on 1-forms) is $\nabla_Z^* \alpha = -\nabla_Z \alpha - (\operatorname{div} Z)\alpha$.

PROOF. Indeed, we have

$$\begin{aligned} d\big(\Delta + Z\big) f &= d\big(-d^* df + (df) Z\big) \\ &= \Delta^{(1)} df + \nabla_Z df + \langle \nabla_{\bullet} Z, \nabla f \rangle \\ &= (\Box + \nabla_Z)(df) - \operatorname{Ric}_Z(\cdot, \nabla f) \\ &= \big(\Box - \operatorname{Ric}_Z + \nabla_Z\big)(df). \end{aligned}$$

The formula in (2) is just dual to (1). $\Box$

Now, let $X = X(x)$ be a diffusion to $\Delta + Z$ on M, starting at $X_0(x) = x$, U be a horizontal lift of X to $\mathrm{O}(TM)$, and $B = U_0 \int_U \vartheta$ (taking values in T_xM) the martingale part of the anti-development of X.

By Itô's formula, we have

$$d(f\circ X)=\sum_{i=1}^n df(Ue_i)\,U_0^{-1}dB^i+(\Delta f+Zf)(X)\,dt$$
$$d(F\circ U)=\sum_{i=1}^n (H_iF)(U)\,U_0^{-1}dB^i+(\Delta^{\text{hor}}F+\bar ZF)(U)\,dt$$

for $f\in C^2(M)$, resp. $F\in C^2(\mathrm{O}(TM))$.

PROPOSITION 2.3.11. *Let Q_t be the $\mathrm{Aut}(T_x^*M)$-valued process defined by*

$$\frac{d}{dt}Q_t=-Q_t\,(\mathrm{Ric}_Z)_{//_t}\,,\quad Q_0=\mathrm{id}_{T_x^*M}\,.$$

Let

$$P_tf(x)=\mathbb{E}[f(X_t(x)],\quad f\in C_b^\infty(M),$$

be the semigroup generated by $\Delta+Z$, acting on bounded functions on M.

Fix $T>0$. Then,

$$Q_t//_t^{-1}\,(dP_{T-t}f)_{X_t(x)}\,,\quad 0\le t\le T, \tag{2.3.8}$$

is a local martingale in T_x^*M, starting at $(dP_Tf)_x$.

From the fact that $N_t:=Q_t//_t^{-1}\,(dP_{T-t}f)_{X_t(x)}$ is a local martingale, we deduce that, for any adapted process ℓ with paths in the Cameron-Martin space $\mathbb{H}(\mathbb{R}_+;T_xM)$, the process

$$\begin{aligned}n_t&=(N_t,\ell_t)-\int_0^t(N_s,d\ell_s)\\&=\big((dP_{T-t}f)_{X_t(x)},//_t\,Q_t^{\text{tr}}\,\ell_t\big)-\int_0^t\big((dP_{T-s}f)_{X_s(x)},//_s\,Q_s^{\text{tr}}\,\dot\ell_s\big)\,ds\end{aligned}$$

is a local martingale as well; here $(\cdot,\cdot)$ denotes the pairing between T_x^*M and T_xM. Interpreting the last term on the right as quadratic covariation of two (local) martingales, we see that also

$$\tilde n_t=(dP_{T-t}f)_{X_t(x)}\,//_t\,Q_t^{\text{tr}}\,\ell_t-\int_0^t\big((dP_{T-s}f)_{X_s(x)},//_s\,dB_s\big)\int_0^t\langle Q_t^{\text{tr}}\,\dot\ell_s,dB_s\rangle$$

is a local martingale. However taking into account that

$$(P_{T-t}f)(X_t(x))=\int_0^t(dP_{T-s}f)_{X_s(x)}//_s\,dB_s,$$

we finally see that

$$\tilde n_t=(dP_{T-t}f)_{X_t(x)}\,//_t\,Q_t^{\text{tr}}\,\ell_t-(P_{T-t}f)(X_t(x))\int_0^t\langle Q_s^{\text{tr}}\,\dot\ell_s,dB_s\rangle,\quad 0\le t\le T, \tag{2.3.9}$$

is a local martingale. The idea now is to choose ℓ_t such that first the local martingale $\tilde n_t$ is a true martingale, and secondly such that

$$\ell_0=v\quad\text{and}\quad\ell_T=0.$$

Then, the equality $\mathbb{E}[\tilde n_0]=\mathbb{E}[\tilde n_T]$ gives the formula

$$(dP_Tf)_x\,v=\mathbb{E}\left[f(X_T(x))\int_0^T\langle Q_s^{\text{tr}}\,\dot\ell_s,dB_s\rangle\right].$$

This gives the following result.

THEOREM 2.3.12 (Gradient formula). *Let u be the solution to the heat equation*

$$\frac{\partial}{\partial t}u = \frac{1}{2}\Delta u, \quad u|_{t=0} = f.$$

Let $X = X(x)$ be the diffusion to $\Delta + Z$ starting from x, which is assumed to be non-explosive. Then, for $v \in T_xM$,

$$(dP_T f)_x\, v = -\mathbb{E}\left[f(X_T(x))\int_0^{\tau\wedge T}\left\langle \Theta_s\dot{\ell}_s, dB_s\right\rangle\right]$$

where

- Θ_t is the Aut(T_xM)-valued process defined by

$$\frac{d}{dt}\Theta_t = -\operatorname{Ric}_{//_t}(\bullet, \Theta_t)^\sharp + (\nabla_\bullet Z)_{//_t}\Theta_t, \quad \Theta_0 = \mathrm{id}_{T_xM};$$

- $\tau = \tau_D(x)$ is the first exit time of X from some relatively compact neighbourhood D of x;
- B is a Brownian motion in T_xM;
- ℓ_t is any adapted process in T_xM with absolutely continuous paths such that (for some $\varepsilon > 0$)

$$\ell_0 = v, \quad \ell_\tau = 0 \quad \text{and} \quad \Big(\int_0^{\tau\wedge T}|\dot{\ell}_t|^2\,dt\Big)^{1/2} \in L^{1+\varepsilon}.$$

2.3.4. A formula for the codifferential. Let now $X = X(x)$ be a diffusion to $\Delta - Z$ on M, starting at $X_0(x) = x$, U be a horizontal lift of X to $\mathrm{O}(TM)$, and $B = U_0\int_U \vartheta$ (taking values in T_xM) the martingale part of the anti-development of X.

By Itô's formula, we have

$$d(f\circ X) = \sum_{i=1}^n df(Ue_i)\,U_0^{-1}dB^i + (\Delta f - Zf)(X)\,dt$$

$$d(F\circ U) = \sum_{i=1}^n (H_iF)(U)\,U_0^{-1}dB^i + (\Delta^{\mathrm{hor}}F - \bar{Z}F)(U)\,dt$$

for $f \in C^2(M)$, resp. $F \in C^2(\mathrm{O}(TM))$.

PROPOSITION 2.3.13. *Let $X = X(x)$ be a diffusion to $\Delta - Z$ on M, starting at $X_0(x) = x$, which for simplicity is assumed to be non-explosive. In terms of*

$$\mathscr{R}_Z := \operatorname{Ric}_Z^* + \operatorname{div} Z \in \operatorname{End}(T^*M)$$

where div Z *acts fiberwise as multiplication operator, let Q_t be the* Aut(T_x^*M)*-valued process defined by*

$$\frac{d}{dt}Q_t = -Q_t\,(\mathscr{R}_Z)_{//_t}, \quad Q_0 = \mathrm{id}_{T_x^*M}\,.$$

Fix $T > 0$ and let α_t solve

$$\begin{cases} \dfrac{\partial}{\partial t}\alpha_t = (\Box - \mathscr{R}_Z + \nabla_Z)\,\alpha_t & \text{on } [0,T]\times M,\\ \alpha_t|_{t=0} = \alpha \in \Gamma(T^*M). \end{cases}$$

Then, for given $x \in M$,

$$n_t := (\operatorname{div} \alpha_{T-t})(X_t(x)) \, \exp\left(\int_0^t (\operatorname{div} Z)(X_s(x))\, ds\right), \quad 0 \le t \le T, \tag{2.3.10}$$

is a real local martingale, starting at $(\operatorname{div} \alpha_T)(x)$.

Let ℓ be an adapted process with paths in $\mathbb{H}(\mathbb{R}_+;\mathbb{R})$ which takes the form

$$\ell_t = \exp\left(-\int_0^t (\operatorname{div} Z)(X_s(x))\, ds\right) h_t \tag{2.3.11}$$

where $h_0 = 0$ and $h_T = 1$. Starting with the local martingale n_t given by (2.3.10), we observe that also

$$\tilde n_t = n_t \ell_t - \int_0^t n_s \, d\ell_s \equiv \int_0^t \ell_s \, dn_s + n_0 \ell_0, \quad 0 \le t \le T, \tag{2.3.12}$$

is a local martingale. In explicit terms it writes as

$$\tilde n_t = (\operatorname{div} \alpha_{T-t})(X_t(x))\, h_t + \int_0^t (\operatorname{div} \alpha_{T-s})(X_s(x)) \left[(\operatorname{div} Z)(X_s(x))\, h_s - \dot h_s\right] ds. \tag{2.3.13}$$

Assuming that (2.3.12) is actually a true martingale, by taking expectations, we get from $\mathbb{E}[\tilde n_T] = 0$ the formula

$$\mathbb{E}\left[(\operatorname{div} \alpha)(X_T(x))\right] = \mathbb{E}\left[\int_0^T n_t \, d\ell_t\right]. \tag{2.3.14}$$

It remains to evaluate the right-hand-side of Eq. (2.3.14). In this way we obtain

$$\begin{aligned}
n_t \, d\ell_t &= -(\operatorname{div} \alpha_{T-t})(X_t(x)) \left[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\right] dt \\
&= -\sum_{i=1}^n \left\langle \nabla_{U_t e_i} \alpha_{T-t}, U_t e_i \right\rangle \left[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\right] dt \\
&= -\sum_{i=1}^n \left\langle //_t^{-1} \nabla_{U_t e_i} \alpha_{T-t}, U_0 e_i \right\rangle \left[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\right] dt \\
&= -\sum_{i=1}^n \left\langle Q_t //_t^{-1} \nabla_{U_t e_i} \alpha_{T-t}, (Q_t^{-1})^* \, U_0 e_i \right\rangle \left[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\right] dt.
\end{aligned}$$

Using the fact that $N_t := Q_t //_t^{-1} a_{T-t}(X_t(x))$ is a local martingale, indeed

$$d\left(Q_t //_t^{-1} a_{T-t}(X_t(x))\right) = \sum_{i=1}^n Q_t //_t^{-1} \left(\nabla_{U_t e_i} a_{T-t}\right) dB_t^i,$$

we get

$$\begin{aligned}
&\int_0^T n_t \, d\ell_t \\
&= -\Big\langle \sum_{i=1}^n \int_0^T Q_t //_t^{-1} \left(\nabla_{U_t e_i} a_{T-t}\right) dB_t^i \, , \int_0^T \left[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\right] (Q_t^{-1})^* \, dB_t \Big\rangle \\
&= -\Big\langle N_T, \int_0^T \left[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\right] (Q_t^{-1})^* \, dB_t \Big\rangle
\end{aligned}$$

$$= -\Big\langle Q_T /\!/_T^{-1} \alpha(X_T), \int_0^T \Big[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\Big] (Q_t^{-1})^* \, dB_t \Big\rangle.$$

Using the identification of differential forms and vector fields via the metric, we obtain the following result.

THEOREM 2.3.14. *Let M be a Riemannian manifold and Z a smooth vector field on M. Let $X = X(x)$ be a diffusion to $\Delta - Z$ on M, starting at $X_0(x) = x$, which is assumed to be non-explosive. Let $T > 0$ and h be an adapted process with paths in $\mathbb{H}([0,T];\mathbb{R})$ such that $h_0 = 0$ and $h_T = 1$, and such that* (2.3.13) *is a true martingale. Then for all smooth vector fields V on M,*

$$\begin{aligned} &\mathbb{E}\big[(\operatorname{div} V)(X_T(x))\big] \\ &\quad = -\mathbb{E}\left[\Big\langle Q_T /\!/_T^{-1} V(X_T(x)), \int_0^T \Big[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\Big] (Q_t^{-1})^* dB_t \Big\rangle\right] \end{aligned}$$

where Q is the $\mathrm{Aut}(T_xM)$*-valued process defined by the following pathwise differential equation:*

$$\frac{d}{dt} Q_t = -Q_t \, (\mathscr{R}_Z)_{/\!/_t} \quad \textit{with } Q_0 = \mathrm{id}_{T_xM}.$$

COROLLARY 2.3.15. *We keep notations and assumptions of Theorem* 2.3.14*. Using the relation*

$$\operatorname{div}(fV) = Vf + f \operatorname{div} V, \quad f \in C^\infty(M),\ V \in \Gamma(TM),$$

we get the formula

(2.3.15)

$$\begin{aligned} &\mathbb{E}\big[(Vf)(X_T(x))\big] = -\mathbb{E}\big[f(X_T(x))(\operatorname{div} V)(X_T(x))\big] \\ &\quad - \mathbb{E}\left[f(X_T(x))\Big\langle Q_T /\!/_T^{-1} V(X_T(x)), \int_0^T \big[(\operatorname{div} Z)(X_t(x)) h_t - \dot h_t\big] (Q_t^{-1})^* \, dB_t \Big\rangle\right] \end{aligned}$$

where the right-hand side of Eq. (2.3.15) *does not contain any derivatives of f.*

2.3.5. General remarks. Our approach is based on martingale arguments; integration by parts is done at the level of local martingales. We get the wanted formulae then by taking expectations, under conditions which assure the local martingales to be true martingales.

REMARK 2.3.16. The formulae allow the choice of a finite energy process $(\ell_t)_{t\in[0,T]}$. Depending on the type of the wanted formulae, conditions are imposed on the left endpoint ℓ_0, or the right endpoint ℓ_t.

(i) The argument leading to the gradient formula of Theorem 2.3.12 is based on the fact that the local martingale (2.3.9) is a true martingale. Since the condition on ℓ_t is imposed on the left endpoint, this can always be achieved, for instance, by taking $\ell_s = 0$ for $s \geq \tau \wedge T$ where τ is the first exit time of some relatively compact neighbourhood of x. No bounds on the geometry are needed; even explosion in finite times of the underlying diffusion can be allowed.

(ii) Imposing however the conditions $\ell_0 = 0$ and $\ell_T = v$ in (2.3.9) would lead to a formula for

$$\mathbb{E}\left[(df)_{X_T(x)} /\!/_T \, Q_T^{\mathrm{tr}} \, v\right]$$

which clearly requires strong assumptions. Note that if the local martingale (2.3.8) is a true martingale, we get the formula

$$(dP_T f)_x = \mathbb{E}\left[Q_T //_T^{-1} (df)_{X_T(x)}\right].$$

For such a formula to hold, clearly $X_t(x)$ needs to be non-explosive.

(iii) For the formula in Theorem 2.3.14 the conditions on ℓ_t, resp. h_t, given by Eq. (2.3.11) are on the right-hand-side. Thus assumptions are needed to make (2.3.13) a true martingale.

2.4. Brownian Bridges and Brownian Loops

Let (M, g) be a complete connected Riemannian manifold of dimension n. Given $x \in M$ we denote by $X_\bullet(x)$ Brownian motion on M starting from x at time zero. Now fix $t > 0$. Given $x, y \in M$ we want to define a *Brownian bridge* $X^t(x, y)$ from x to y of lifetime t. Intuitively, a Brownian bridge $X^t(x, y)$ is a Brownian motion on M starting from x at time 0 but "conditioned to hit y at time t". In particular, for $s < t$ we require the properties:

$$\mathbb{P}\{X_s(x) \in A\} = \int 1_A(z)\, \mathbb{P}\{X_s(x) \in dz\} = \int 1_A(z)\, p(s, x, z)\, \mathrm{vol}(dz),$$
$$\mathbb{P}\{X_s^t(x, y) \in A\} = \int 1_A(z)\, \frac{p(s, x, z) p(t - s, z, y)}{p(t, x, y)}\, \mathrm{vol}(dz),$$

where $p(t, x, y)$ denotes the heat kernel on M.

LEMMA 2.4.1 (Local heat kernel estimates). *Let M be complete.*

(1) *For each point $x_0 \in M$ and each $R > 0$, there exist constants c_i, $k_i > 0$ (for $i = 1, 2$) such that*

$$k_1 s^{-n/2} \exp\left(-c_1 \frac{d(x, y)^2}{s}\right) \leq p(s, x, y) \leq k_2 s^{-n/2} \exp\left(-c_2 \frac{d(x, y)^2}{s}\right)$$

for all $(s, x, y) \in \,]0, R[\, \times B(x_0, R) \times B(x_0, R)$.

(2) *For each point $x_0 \in M$ and each $R > 0$, there exist constant $c > 0$ such that*

$$\left|(\nabla \log p(s, \cdot, y))(x)\right| \leq c \left(\frac{d(x, y)}{s} + \frac{1}{\sqrt{s}}\right),$$

for all $(s, x, y) \in \,]0, R[\, \times B(x_0, R) \times B(x_0, R)$.

Let $(C(\mathbb{R}_+, \mathbb{R}^n), \mathscr{F}, \mathbb{P})$ be the standard Wiener space and $(\mathscr{F}_s)_{s \geq 0}$ be the standard filtration (with the usual conditions). Denote by

$$B_s := \mathrm{pr}_s \colon C(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}^n, \quad s \geq 0,$$

the standard Wiener process. Construct Brownian motion X on (M, g) starting from x in the usual way as $X = \pi \circ U$, where

$$\begin{cases} dU = \displaystyle\sum_{i=1}^n L_i(U) \circ dB^i \\ U_0 = u \in \pi^{-1} x \in \mathrm{O}(TM). \end{cases}$$

Let

$$h : [0, t[\, \times M \to \,]0, \infty[$$

be a smooth *space-time harmonic* function, in the sense that

$$\left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta\right) h \equiv 0.$$

In the sequel we always take

$$h(s, \cdot) := \frac{p(t-s, \cdot, y)}{p(t, x, y)}. \tag{2.4.1}$$

Furthermore let $\mathscr{F}_s^0 := \sigma\{\mathrm{pr}_r : r \le s\} \subset \mathscr{F}_s$.

LEMMA 2.4.2. *Let* $h(s, \cdot)$ *be given by* (2.4.1). *Then*

$$Z_s := h(s, X_s(x)), \quad 0 \le s < t,$$

is a martingale and by Kolmogorov's theorem there exists a unique probability measure $\mathbb{P}^h$ *on* $C([0,t[, \mathbb{R}^n), \sigma\{\mathrm{pr}_s \colon s < t\}$ *such that*

$$\frac{d\mathbb{P}^h}{d\mathbb{P}}\Big|_{\mathscr{F}_s^0} = Z_s \quad \mathbb{P}\text{-a.s.} \quad (0 \le s < t).$$

COROLLARY 2.4.3. *Let*

$$h(s, \cdot) := \frac{p(t-s, \cdot, y)}{p(t, x, y)},$$

and let $\mathbb{P}_{x,y}^t := \mathbb{P}^h$ *on* $C([0,t[, \mathbb{R}^n)$ *as constructed above. The following properties hold:*

(1) *For each* $s < t$,
$$\mathbb{P}_{x,y}^t\{X_s(x) \in A\} = \mathbb{E}\left[(1_A \circ X_s(x))\, Z_s\right] = \mathbb{E}\left[1_A \circ X_s(x)\, \frac{p(t-s, X_s(x), y)}{p(t, x, y)}\right].$$
(2) *The process* $X(x)|_{[0,t[}$ *is a* $\mathbb{P}_{x,y}^t$*-semimartingale (with respect to the local completion of* $(\mathscr{F}_s^0)_{0 \le s < t}$ *with respect to* $\mathbb{P}_{x,y}^t$*).*
(3) *The process*
$$\left(\frac{1}{h(s, X_s(x))}\right)_{0 \le s < t}$$
is a positive $\mathbb{P}_{x,y}^t$*-martingale, and hence it has a* $\mathbb{P}_{x,y}^t$*-a.s. limit as* $s \uparrow t$.
(4) *Since for* $x \ne y$,
$$p(t-s, x, y) \to 0 \text{ as } s \uparrow t,$$
we observe that $X_s(x) \to y$ $\mathbb{P}_{x,y}^t$*-a.s. as* $s \uparrow t$.

NOTATION 2.4.4. The process $(X_s(x))_{0 \le s \le t}$ with respect to $\mathbb{P}_{x,y}^t$ is called Brownian bridge (pinned Brownian motion) on M from x to y with lifetime t. The distribution $\mu_{x,y}^t$ of $X(x)|_{[0,t]}$ on

$$\{\omega \in C([0,t], M) \colon w(0) = x,\, w(t) = y\}$$

is called Brownian bridge measure.

PROPOSITION 2.4.5.

(1) *By construction each Brownian bridge from* x *to* y *with lifetime* t *is a continuous Markov process* $(Y_s)_{0 \le s \le t}$ *such that*
$$\mathbb{P}\{Y_s \in dz | Y_0 = x\} = \frac{p(s, x, z) p(t-s, z, y)}{p(t, x, y)}\ \mathrm{vol}(dz).$$
Hence the time-reversed Markov process $\widehat{Y}_s := Y_{t-s}$ *is a bridge from from* y *to* x *in the sense that its distribution is given by* $\mu_{y,x}^t$.

(2) *Since $(X_s(x))_{s<t}$ is a $\mathbb{P}^t_{x,y}$-semimartingale on M, it is the stochastic development of an $\mathbb{R}^n$-valued semimartingale. Indeed, it is the stochastic development of the $\mathbb{P}$-Brownian motion B on $\mathbb{R}^n$ considered as $\mathbb{P}^t_{x,y}$-semimartingale.*

This leads to the problem of finding the Doob-Meyer decomposition of $(B_s)_{s<t}$ with respect to $\mathbb{P}^t_{x,y}$. First, recall that

$$Z_s = h(s, X_s(x)), \quad 0 \le s < t,$$

is a $\mathbb{P}$-martingale. Thus by Itô's formula,

$$dZ = \sum_i \langle \nabla h(s, X_s(x)), U_s e_i \rangle_{T_{X_s}M} dB^i$$

where

$$\nabla h(s, z) = (\nabla h(s, \cdot))_z.$$

Hence, we have

$$dZ = Z \, \langle c, dB \rangle_{\mathbb{R}^n}$$

where

$$c^i_s = \frac{\langle \nabla h(s, X_s(x)), U e_i \rangle}{h(s, X_s(x))}$$

or equivalently

$$c_s = \frac{U_s^{-1} \nabla h(s, \pi(U_s))}{h(s, \pi(U_s))}.$$

Thus by Girsanov's theorem we find that

$$\tilde{B}_s := B_s - \int_0^s c_r dr$$

is a $\mathbb{P}^t_{x,y}$-BM($\mathbb{R}^n$) on the interval $[0, t[$. This means in particular that

$$B_s = \tilde{B}_s + \int_0^s U_r^{-1} \nabla \log \ h(r, X_r(x)) dr$$

is the $\mathbb{P}^t_{x,y}$-Doob-Meyer decomposition of $(B_s)_{s<t}$. Note that on the right-hand-side $\tilde{B}_s$ has a limit $\mathbb{P}^t_{x,y}$-a.s., as $s \uparrow t$, and the integral converges absolutely $\mathbb{P}^t_{x,y}$-a.s. as $s \uparrow t$. Therefore B is a $\mathbb{P}^t_{x,y}$-semimartingale on $[0, t]$.

COROLLARY 2.4.6.

a) *The process $X(x)$ is a $\mathbb{P}^t_{x,y}$-semimartingale on $[0, t]$ with anti-development given by*

$$\mathscr{A}(X(x)) = \text{BM}(T_x M) + \int_0^{\cdot} //_{0,r}^{-1} (\nabla \log h)(r, X_r(x)) \, dr,$$

where

$$(\nabla \log h)(r, \cdot) = \frac{\nabla \, p(t - r, \cdot, y)}{p(t - r, \cdot, y)}.$$

b) *With respect to the Brownian bridge measure $\mathbb{P}^t_{x,y}$ we have*

$$\begin{aligned}
dU &= \sum_i L_i(U) \circ dB^i \\
&= \sum_i L_i(U) \circ (d\tilde{B}^i + c^i_s ds) \\
&= \sum_i L_i(U) \circ d\tilde{B}^i + L_0(s, U_s) ds, \quad \pi U_0 = x,
\end{aligned}$$

where $L_0(s,\cdot) \in \Gamma(T\,\mathrm{O}(TM))$ denotes the horizontal lift of the (time-dependent) vector field

$$\nabla \log\, p(t-s,\cdot,y) \in \Gamma(TM), \quad 0 \le s < t,$$

i.e., $L_0(s,u) = h_u(\nabla \log\, p(t-s,\pi(u),y)) \in T_u\,\mathrm{O}(TM)$.

REMARK 2.4.7. Let $(U_s)_{0\le s<t}$ be a solution of

$$dU = \sum_i L_i(U) \circ dW^i + L_0(s,U_s)ds, \quad U_0 = u \in \pi^{-1}\{x\}$$

where W is a Brownian motion on $\mathbb{R}^n$, the limit $U_t := \lim_{s\uparrow t} U_s$ exists a.s. and

$$X_s := \pi \circ U_s, \quad 0 \le s \le t,$$

is Brownian bridge on M from x to y of lifetime t (in particular, X_s is a semimartingale on the interval $[0,t]$ with the property $X_s \to y$ for $s \uparrow t$ almost surely). On M we have

$$dX_s = U_s \circ dW_s + \nabla \log\, p(t-s,X_s,y)\,ds, \quad X_0 = x.$$

REMARK 2.4.8. Let X be a Brownian motion on (M,g) starting from x (with respect to $\mathbb{P}$) and fix $t>0$.

(1) For all $A \in \sigma\{X_r : r \le t\}$, we have

$$\begin{aligned}\mathbb{P}(A) &= \int \mathbb{P}^t_{x,y}(A)(P \circ X_t^{-1})(dy)\\ &= \int \mathbb{P}^t_{x,y}(A)\, p(t,x,y)\ \mathrm{vol}(dy).\end{aligned}$$

Indeed, if $A \in \sigma\{X_r : r \le s\}$ where $s<t$, then

$$\mathbb{P}^t_{x,y}(A) = \mathbb{E}[1_A Z_s] = \mathbb{E}\left[1_A \frac{p(t-s,X_s,y)}{p(t,x,y)}\right]$$

and

$$\int P^t_{x,y}(A)\, p(t,x,y)\ \mathrm{vol}(dy) = \int \mathbb{E}[1_A\, p(t-s,X_s,y)]\ \mathrm{vol}(dy) = \mathbb{E}[1_A] = \mathbb{P}(A).$$

(2) The map

$$y \mapsto P^t_{x,y}$$

is a desintegration of $\mathbb{P}|_{\{\sigma\{X_r\colon r\le t\}}$ with respect to Brownian motion X starting at x according to its values y at time t. In an informal way, we write

$$\mathbb{P}^t_{x,y}\{\ldots\} = \mathbb{P}\{\ldots | X_t(x) = y\}.$$

EXAMPLE 2.4.9. Let $M = \mathbb{R}^n$ and

$$dX_s = dW_s + \frac{y - X_s}{t-s} ds, \quad X_0 = x.$$

Then X is a Brownian bridge from x to y of lifetime t.

APPENDIX A

Background on SDEs

A.1. One-dimensional Stochastic Differential Equations

In this Section we collect some facts about stochastic differential equations in the one-dimensional case and develop a qualitative theory of one-dimensional SDEs which provides a useful tool for many geometric comparison theorems. Most of the results of this Section go back to the work of William Feller and have originally been formulated for one-dimensional diffusion processes.

We consider the situation of an Itô SDE of the form

$$dY = \beta(t, Y)\, dt + \sigma(t, Y)\, dB \tag{A.1.1}$$

with continuous coefficients $\beta, \sigma\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and a one-dimensional Brownian motion B as driving process. We consider first the case of global solutions of (A.1.1) of infinite lifetime.

THEOREM A.1.1. *Let Y^1 and Y^2 be two solutions of* (A.1.1) *with $Y_0^1 = Y_0^2$. Then also $Y^1 \vee Y^2$ is a solution of* (A.1.1) *to the same initial condition if and only if the local time $L^0(Y^2 - Y^1)$ of $Y^2 - Y^1$ at 0 vanishes modulo indistinguishability.*

DEFINITION A.1.2. For any real a the local time of a continuous real semimartingale $X \in \mathscr{S}$ at a is given by

$$L_t^a(X) := |X_t - a| - |X_0 - a| - \int_0^t \operatorname{sign}(X_s - a)\, dX_s \tag{A.1.2}$$

where $\int_0^\infty \mathbb{1}_{\{|X_s| \neq a\}} dL_s^a(X) = 0$ almost surely. Recall that $\operatorname{sign} := -\mathbb{1}_{]-\infty,0]} + \mathbb{1}_{]0,\infty[}$. It holds that

$$(X_t - a)_+ = (X_0 - a)_+ + \int_0^t \mathbb{1}_{\{X_s > a\}}\, dX_s + \frac{1}{2} L_t^a(X) \tag{A.1.3}$$

$$(X_t - a)_- = (X_0 - a)_- - \int_0^t \mathbb{1}_{\{X_s \leq a\}}\, dX_s + \frac{1}{2} L_t^a(X) \tag{A.1.4}$$

These formulae are well-known (e.g. [**38**, p. 222]) and usually refered to under the name "Tanaka formulae".

PROOF (of Theorem A.1.1). Letting $L^0(Y^2 - Y^1)$ denote the local time of $Y^2 - Y^1$ at 0, we have

$$\begin{aligned}
d(Y^1 \vee Y^2) &= dY^1 + d(Y^2 - Y^1)_+ \\
&= \beta(t, Y^1)\, dt + \sigma(t, Y^1)\, dB + 1_{\{Y^2 > Y^1\}}\, d(Y^2 - Y^1) + \frac{1}{2}\, dL^0(Y^2 - Y^1) \\
&= \Big(\beta(t, Y^1) + \big(\beta(t, Y^2) - \beta(t, Y^1)\big) 1_{\{Y^2 > Y^1\}}\Big)\, dt \\
&\quad + \Big(\sigma(t, Y^1) + \big(\sigma(t, Y^2) - \sigma(t, Y^1)\big) 1_{\{Y^2 > Y^1\}}\Big)\, dB + \frac{1}{2}\, dL^0(Y^2 - Y^1)
\end{aligned}$$

$$= \beta(t, Y^1 \vee Y^2)\, dt + \sigma(t, Y^1 \vee Y^2)\, dB + \frac{1}{2}\, dL^0(Y^2 - Y^1),$$

from where the claim is seen. □

THEOREM A.1.3. *Suppose that for any two solutions Y^1 and Y^2 of* (A.1.1) *such that $Y_0^1 = Y_0^2$, it holds that $L^0(Y^1 - Y^2) = 0$. Then if solutions to* (A.1.1) *are unique in distribution, they are even pathwise unique.*

PROOF. Indeed, letting Y^1 and Y^2 be two solutions satisfying $Y_0^1 = Y_0^2$, by Theorem A.1.1 then also $Y^1 \vee Y^2$ is a solution. For $t \geq 0$, by the uniqueness of solutions in law, both Y_t^1 and $Y_t^1 \vee Y_t^2$ have the same law, with the consequence that $Y_t^1 \geq Y_t^2$ almost surely; analogously one obtains $Y_t^1 \leq Y_t^2$ almost surely. The claim thus follows by the continuity of the paths. □

The following Lemma gives a criterion for the vanishing of the local time $L^0(X)$ of a semimartingales $X \in \mathscr{S}$ at 0. To this end, we denote by ρ a measurable function $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^t \rho(u)^{-1}\, du = \infty$ for any $t > 0$. We note this property shortly as $\int_{0+} \rho(u)^{-1}\, du = \infty$.

LEMMA A.1.4. *Let $X \in \mathscr{S}$ and $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function such that $\int_{0+} \rho(u)^{-1}\, du = \infty$. Suppose there exists $\varepsilon > 0$ such that for any $t > 0$*

$$\int_0^t 1_{\{0 < X_s \leq \varepsilon\}}\, \rho(X_s)^{-1}\, d[X]_s < \infty \quad \text{almost surely.} \tag{A.1.5}$$

Then $L^0(X) = 0$ modulo indistinguishability.

PROOF. From the "occupation times formula" of the local time (e.g. [**38**, p. 224]) we have for fixed $t > 0$ the relation

$$\int_0^t 1_{\{0 < X_s \leq \varepsilon\}}\, \rho(X_s)^{-1}\, d[X]_s = \int_0^\varepsilon \rho(a)^{-1}\, L_t^a(X)\, da$$

where $L_t^a(X)$ denotes again the local time of X at a. Hence if $L_t^0(X)$ does not vanish almost surely, by means of the right continuity of $L_t^a(X)$ in a, the right-hand side of the last formula would be infinite with positive probability – in contradiction to assumption (A.1.5). □

THEOREM A.1.5 (Yamada-Watanabe). *Let $\beta, \sigma\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the following properties:*

(i) *σ is bounded, and there exists a measurable function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_{0+} \rho(u)^{-1}\, du = \infty$ such that*

$$\big|\sigma(s, x) - \sigma(s, y)\big|^2 \leq \rho\big(|x - y|\big)$$

for all $s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

(ii) *β is globally Lipschitz, i.e., for any $t \geq 0$ there is a constant L_t such that*

$$\big|\beta(s, x) - \beta(s, y)\big| \leq L_t\, |x - y|$$

for all $0 \leq s \leq t$ and $x, y \in \mathbb{R}$.

Then solutions of (A.1.1) *are pathwise unique.*

PROOF. Let Y^1 and Y^2 be two solutions of (A.1.1) satisfying $Y_0^1 = Y_0^2$. Then by $d(Y^1 - Y^2) = \big(\beta(t,Y^1) - \beta(t,Y^2)\big)\,dt + \big(\sigma(t,Y^1) - \sigma(t,Y^2)\big)\,dB$, we obtain

$$\begin{aligned}\int_0^t \rho(Y_s^1 - Y_s^2)^{-1}\, 1_{\{Y_s^1 > Y_s^2\}}\, d[Y^1 - Y^2]_s \\ = \int_0^t \rho(Y_s^1 - Y_s^2)^{-1}\, 1_{\{Y_s^1 > Y_s^2\}}\, \big(\sigma(s,Y_s^1) - \sigma(s,Y_s^2)\big)^2\, ds \le t,\end{aligned}$$

and by Lemma A.1.4 hence $L^0(Y^1 - Y^2) = 0$ modulo indistinguishability. Thus we have

$$\begin{aligned}|Y_t^1 - Y_t^2| &= \int_0^t \operatorname{sign}(Y_s^1 - Y_s^2)\, d(Y_s^1 - Y_s^2) \\ &= \int_0^t \operatorname{sign}(Y_s^1 - Y_s^2)\,\big(\beta(s,Y_s^1) - \beta(s,Y_s^2)\big)\, ds \\ &\quad + \int_0^t \operatorname{sign}(Y_s^1 - Y_s^2)\,\big(\sigma(s,Y_s^1) - \sigma(s,Y_s^2)\big)\, dB_s,\end{aligned}$$

and consequently

$$\mathbb{E}|Y_t^1 - Y_t^2| \le L_t \int_0^t \mathbb{E}|Y_s^1 - Y_s^2|\, ds.$$

From this inequality we get $|E|Y_t^1 - Y_t^2| = 0$ by means of Gronwall's lemma along with the usual continuity argument. $\square$

EXAMPLE A.1.6 (Girsanov). Solutions to the one-dimensional SDE

$$dY = \sigma_\alpha(Y)\, dB, \quad Y_0 = 0 \tag{A.1.6}$$

with $\sigma_\alpha(x) = |x|^\alpha \wedge 1$ are pathwise unique for $\alpha \ge 1/2$, and $Y \equiv 0$ is the only solution. This is an immediate consequence of Theorem A.1.5: for $1/2 \le \alpha \le 1$ it holds that $\big|\sigma_\alpha(x) - \sigma_\alpha(y)\big| \le |x-y|^\alpha$, whereas σ_α is globally Lipschitz continuous for $\alpha \ge 1$. We are going to verify that pathwise uniqueness in Equation (A.1.6) is violated for $0 < \alpha < 1/2$. To this end, let

$$T(t) := \int_0^t \big(|B_s|^{2\alpha} \wedge 1\big)^{-1}\, ds, \quad t \in \mathbb{R}_+.$$

As $\mathbb{E}[T(t)] < \infty$, each $T(t)$ is finite almost surely, $t \mapsto T(t)$ is almost surely strictly monotone increasing, and because of $T(t) \ge t$ trivially $T(\infty) = \infty$ holds. Hence $\tau_t := T^{-1}(t) \equiv \inf\{s \in \mathbb{R}_+ \colon T(s) > t\}$ defines a finite continuous time-change, and $Y_t := B_{\tau_t}$ with respect to the time-changed filtration $\big(\mathscr{F}_{\tau_t}\big)_{t\in\mathbb{R}_+}$ gives a (non-trivial) weak solution Y of (A.1.6). Indeed, with the $\big(\mathscr{F}_{\tau_t}\big)$-Brownian motion

$$\tilde{B}_t := \int_0^{\tau_t} \big(|B_s|^\alpha \wedge 1\big)^{-1}\, dB_s, \quad t \in \mathbb{R}_+$$

we have

$$Y_t = B_{\tau_t} = \int_0^{\tau_t} dB_s = \int_0^t \big(|B_{\tau_s}|^\alpha \wedge 1\big)\, d\tilde{B}_s = \int_0^t \big(|Y_s|^\alpha \wedge 1\big)\, d\tilde{B}_s,$$

which shows the claim. Hence uniqueness in distribution, and in particular pathwise uniqueness of solutions to (A.1.6), does not hold for $0 < \alpha < 1/2$.

REMARK A.1.7. It may be surprising in Example A.1.6 that unique solvability of (A.1.6) is given in cases where uniqueness of solutions in the analogous ordinary differential equation is violated. For instance, the equation $y(t) = \int_0^t \big(|y(s)|^\alpha \wedge 1\big)\, ds$ has for

$\alpha \geq 1$ only the trivial solution $y \equiv 0$, as can be seen by the Gronwall lemma, whereas for $0 < \alpha < 1$ a further solution is given by

$$y(t) := \begin{cases} (\beta\, t)^{1/\beta} & \text{for } 0 \leq t \leq 1/\beta \\ (1 - 1/\beta) + t & \text{for } t \geq 1/\beta, \end{cases}$$

where $\beta = 1 - \alpha$.

We want now to use the above techniques for the goal to derive comparison theorems of one-dimensional SDEs. To this end, we consider the situation of two SDEs

$$\begin{aligned} dY^1 &= \beta^1(t, Y^1)\, dt + \sigma(t, Y^1)\, dB \\ dY^2 &= \beta^2(t, Y^2)\, dt + \sigma(t, Y^2)\, dB \end{aligned} \tag{A.1.7}$$

with continuous functions $\beta^1, \beta^2, \sigma\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and the same one-dimensional Brownian motion B as driving process.

THEOREM A.1.8 (Comparison Theorem of Ikeda-Watanabe). *Let Y^1 and Y^2 be solutions of* (A.1.7) *under the following conditions:*

(i) *Either β^1 or β^2 is globally Lipschitz, and it holds $\beta^1 \geq \beta^2$,*
(ii) *σ satisfies Condition* (i) *of Theorem* A.1.5.

Then $Y_0^1 \geq Y_0^2$ almost surely already implies $Y_t^1 \geq Y_t^2$ almost surely for any $t > 0$.

PROOF. By the same argument as in the proof of Theorem A.1.5 we obtain again $L^0(Y^1 - Y^2) = 0$. Since $Y_0^1 \geq Y_0^2$ almost surely, we have by the Tanaka formula (A.1.3)

$$\begin{aligned} (Y_t^2 - Y_t^1)_+ = &\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left(\beta^2(s, Y_s^2) - \beta^1(s, Y_s^1)\right) ds \\ &+ \int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left(\sigma(s, Y_s^2) - \sigma(s, Y_s^1)\right) dB_s, \end{aligned}$$

and hence

$$\phi(t) := \mathbb{E}\big[(Y_t^2 - Y_t^1)_+\big] = \mathbb{E}\left[\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left[\beta^2(s, Y_s^2) - \beta^1(s, Y_s^1)\right] ds\right].$$

In case β^1 is globally Lipschitz, we obtain

$$\begin{aligned} \phi(t) &\leq \mathbb{E}\left[\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left[\beta^1(s, Y_s^2) - \beta^1(s, Y_s^1)\right] ds\right] \\ &\leq L_t\, \mathbb{E}\left[\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left|Y_s^2 - Y_s^1\right| ds\right] = L_t \int_0^t \phi(s)\, ds, \end{aligned}$$

and the claim follows in the usual way by Gronwall's Lemma. On the other hand, if β^2 is globally Lipschitz, then

$$\begin{aligned} \phi(t) &= \mathbb{E}\left[\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left(\beta^2(s, Y_s^2) - \beta^2(s, Y_s^1)\right) ds\right] \\ &\qquad + \mathbb{E}\left[\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left(\beta^2(s, Y_s^1) - \beta^1(s, Y_s^1)\right) ds\right] \\ &\leq \mathbb{E}\left[\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left(\beta^2(s, Y_s^2) - \beta^2(s, Y_s^1)\right) ds\right] \\ &\leq L_t\, \mathbb{E}\left[\int_0^t 1_{\{Y_s^2 > Y_s^1\}} \left|Y_s^2 - Y_s^1\right| ds\right], \end{aligned}$$

and the claim is derived as in the first case. □

We now want to discuss the asymptotic behaviour of solutions to one-dimensional SDEs. For such questions it is natural to consider SDEs on open real intervals and then solutions with lifetime.

Let $I =]c_1, c_2[$ with $-\infty \le c_1 < c_2 \le \infty$ be an interval in $\mathbb{R}$ and let $a, b\colon I \to \mathbb{R}$ be continuous functions where $a > 0$. We suppose that on I a diffusion process Y with infinitesimal generator $L = a\,D^2 + b\,D$ (where $D = d/dt$) is given, which we assume to be realized as maximal solution to an SDE of the form

$$dY = b(Y)\,dt + \sigma(Y)\,dB \tag{A.1.8}$$

with $\sigma^2 = 2a$, $\sigma > 0$ and B a one-dimensional BM. In particular, Y is then a continuous I-valued semimartingale of maximal, but not necessarily infinite lifetime ζ, such that

$$d(f(Y)) - (Lf)(Y)\,dt \in d\mathscr{M}.$$

for any C^2-function $f\colon I \to \mathbb{R}$ of compact support $\operatorname{supp}(f) \subset I$. Note that on the set $\{\zeta < \infty\}$ the limit $\lim_{t\uparrow\zeta} Y_t$ exists almost surely with values in $\{c_1, c_2\}$. Thus we may extend Y via $Y_t := \lim_{s\uparrow\zeta} Y_s$ on $\{\zeta < \infty\}$ for $t \ge \zeta$ to a continuous process defined globally on $\mathbb{R}_+$. For $x \in \bar{I}$ we denote by $\tau_x = \inf\{t \ge 0 : Y_t = x\}$ the hitting time of x and call the boundary point c_i *accessible* for Y if $\mathbb{P}\{\tau_{c_i} < \infty\} > 0$.

The problem is now to find conditions on the coefficients of (A.1.8) which characterize properties such as transience, recurrence, or infinite lifetime of Y. The process Y is called *transient* if Y eventually exits every compact subset in I almost surely, and *recurrent* if $\mathbb{P}\{\tau_x < \infty\} = 1$ for each $x \in I$. The lifetime of Y is obviously given by $\zeta = \tau_{c_1} \wedge \tau_{c_2}$; transience of Y means $Y_t \to \{c_1, c_2\}$ as $t \uparrow \zeta$, and is in particular satisfied if $\zeta < \infty$ almost surely.

Fixing c such that $c_1 < c < c_2$, we consider

$$H\colon I \to \mathbb{R}, \quad H(r) = \exp\left\{-\int_c^r \frac{b(\rho)}{a(\rho)}\,d\rho\right\}, \tag{A.1.9}$$

as well as the $\mathbb{R}$-valued functions s, m, k on I defined by

$$s(r) = \int_c^r H(\rho)\,d\rho, \quad m(r) = \int_c^r \frac{1}{a(\rho)H(\rho)}\,d\rho, \quad k(r) = \int_c^r m(\rho)\,s(d\rho),$$

which extend to $\mathbb{R} \cup \{\pm\infty\}$-valued functions on $\bar{I} = I \cup \{c_1, c_2\}$; here $s(d\rho)$ denotes the Borel measure on I with distribution function s.

THEOREM A.1.9. *Let $a, b\colon I \to \mathbb{R}$ be continuous functions on an interval $I =]c_1, c_2[$ where $-\infty \le c_1 < c_2 \le \infty$ and set $\sigma^2 = 2a$ with $\sigma > 0$. For a one-dimensional Brownian motion B and $y \in I$, let Y be the maximal solution to the* SDE

$$dY = b(Y)\,dt + \sigma(Y)\,dB, \quad Y_0 = y.$$

With respect to a fixed $c \in I$ let H be defined by (A.1.9)*. The following items hold true:*

(i) *The process Y is either recurrent or transient, and in fact transient if and only if $s(c_i) = \int_c^{c_i} H(r)\,dr$ is finite for $i = 1$ or 2.*
More precisely, one can distinguish the following four cases:
(1) *If $s(c_1) = -\infty$ and $s(c_2) = \infty$ then*

$$\mathbb{P}\{\zeta = \infty\} = \mathbb{P}\Big\{\inf_{0\le t<\infty} Y_t = c_1\Big\} = \mathbb{P}\Big\{\sup_{0\le t<\infty} Y_t = c_2\Big\} = 1.$$

(2) *If* $s(c_1) > -\infty$ *and* $s(c_2) = \infty$ *then*

$$\mathbb{P}\Big\{\lim_{t\uparrow\zeta} Y_t = c_1\Big\} = \mathbb{P}\Big\{\sup_{0\le t<\zeta} Y_t < c_2\Big\} = 1.$$

(3) *If* $s(c_1) = -\infty$ *and* $s(c_2) < \infty$ *then*

$$\mathbb{P}\Big\{\inf_{0\le t<\zeta} Y_t > c_1\Big\} = \mathbb{P}\Big\{\lim_{t\uparrow\zeta} Y_t = c_2\Big\} = 1.$$

(4) *If* $s(c_1) > -\infty$ *and* $s(c_2) < \infty$ *then*

$$\mathbb{P}\Big\{\lim_{t\uparrow\zeta} Y_t = c_1\Big\} = 1 - \mathbb{P}\Big\{\lim_{t\uparrow\zeta} Y_t = c_2\Big\} = \frac{s(c_2) - s(y)}{s(c_2) - s(c_1)}.$$

(ii) (Feller's test for explosion) Y *has almost surely infinite lifetime if and only if*

$$k(c_i) = \int_c^{c_i} H(r)\bigg(\int_c^r \frac{1}{a(\rho)H(\rho)}\,d\rho\bigg)\,dr = \infty, \quad i = 1 \textit{ and } 2.$$

Moreover c_i *is accessible for* Y *if* $k(c_i) < \infty$.

(iii) *The lifetime of* Y *is almost surely finite* (*i.e.*, $\mathbb{P}\{\zeta < \infty\} = 1$) *if and only if one of the following three cases is at hand:*
(1) $k(c_1) < \infty$ *and* $k(c_2) < \infty$, *or*
(2) $k(c_1) < \infty$ *and* $s(c_2) = \infty$, *or*
(3) $k(c_2) < \infty$ *and* $s(c_1) = -\infty$.
In case (1) *even* $\mathbb{E}[\zeta] < \infty$ *holds.*

The following implications hold trivially:

(A.1.10) $$s(c_1) = -\infty \;\Rightarrow\; k(c_1) = \infty, \quad s(c_2) = \infty \;\Rightarrow\; k(c_2) = \infty.$$

The crucial method to prove Theorem A.1.9 will be to "rescale" the process Y by composition with an isotone transformation φ in such a way that $\varphi(Y)$ becomes a local martingale. One speaks then of a "natural scale" for the diffusion Y and calls φ a *scale function* for Y. We start by verifying that actually s defines a scale function for Y.

PROOF OF THEOREM A.1.9. (a) The function s defines a C^2-diffeomorphism of $I =]c_1, c_2[$ onto $]s(c_1), s(c_2)[$ such that $\tilde{Y} = s(Y)$ is a local martingale with lifetime ζ. Indeed, by $Ls = as'' + bs' = aH' + bH = 0$, we have

$$d\tilde{Y} = d\big(s(Y)\big) = s'(Y)\,dY + \frac{1}{2}\,s''(Y)\,dY\,dY = H(Y)\,\sigma(Y)\,dB,$$

and thus $d\tilde{Y} = \phi(\tilde{Y})\,dB$ where $\phi := (H\sigma)\circ s^{-1}$. In particular, modulo a time change, $\tilde{Y}$ is a Brownian motion, i.e., there exists a stopped one-dimensional Brownian motion W such that $\tilde{Y}_t = W_{T_t}$ almost surely where $T_t := \int_0^t \phi(\tilde{Y}_s)^2\,ds$.

We want to note first that Y leaves each compact subinterval of I in finite time: it holds $\mathbb{P}\{\tau_{a,b} < \zeta\} = 1$ where

$$\tau_{a,b} := \inf\big\{t \ge 0 : Y_t \notin [a,b]\big\}, \quad c_1 < a < b < c_2.$$

Indeed, on $N := \{\tau_{a,b} = \zeta\}$ it holds hat $\zeta = \infty$ by the maximality of the solution; the paths of Y hence stay in the interval $[a,b]$, and the ones of $\tilde{Y}$ in $[s(a), s(b)]$. As consequence of $\tilde{Y}_t = W_{T_t}$, we get $T_\zeta < \infty$ on N almost surely. On the other hand, we have $\phi \ge \varepsilon > 0$ on the compact interval $[s(a), s(b)]$, and hence $T_\zeta = \infty$ on N. Both facts together imply $\mathbb{P}(N) = 0$.

From the discussion above the claims of part (i) of the Theorem follow immediately. Indeed, the maximal lifetime ζ of Y on $]c_1, c_2[$ coincides with the maximal lifetime of $\tilde{Y}$

on $]s(c_1), s(c_2)[$, and Y is transient if and only if $\tilde{Y}$ is transient, for which $T_\zeta < \infty$ must hold almost surely. Conversely, convergence of $\tilde{Y}_t$ as $t \uparrow \zeta$ holds on $\{T_\zeta < \infty\}$, that is convergence almost surely to $s(c_1)$ or $s(c_2)$, since otherwise the path of Y would stay in a compact subinterval of I with the consequence that $T_\zeta = \infty$ as seen above. Hence transience of Y is given exactly if $T_\zeta < \infty$ almost surely, and this is the case if and only if $\tilde{Y}_t$ converges almost surely to $s(c_1)$ or $s(c_2)$ as $t \uparrow \zeta$, hence if and only if $s(c_1)$ or $s(c_2)$ is finite.

Let $y \in I$ be the starting point of Y und let $x, z \in I$ such that $x < y < z$. Then

$$\mathbb{P}\{\tau_x \le \tau_z\} = \mathbb{P}\{\tau_x < \tau_z\} = \frac{s(z) - s(y)}{s(z) - s(x)}, \tag{A.1.11}$$

as can be seen from the equality

$$s(y) = \mathbb{E}[s(Y_0)] = \mathbb{E}[s(Y_{\tau_x \wedge \tau_z})] = s(x)\,\mathbb{P}\{\tau_x < \tau_z\} + s(z)\,\mathbb{P}\{\tau_x \ge \tau_z\}.$$

Recall that Y is recurrent if and only if for any $x \in I$,

$$P\{\tau_x < \tau_{c_i}\} = 1, \quad i = 1, 2.$$

From (A.1.11) we conclude that this is equivalent to $s(c_1) = -\infty$, $s(c_2) = \infty$; more precisely, we have

$$\mathbb{P}\{\tau_x < \tau_{c_2}\} = 1 \iff \lim_{z \uparrow c_2} \mathbb{P}\{\tau_x < \tau_z\} = 1 \iff s(c_2) = \infty, \quad c_1 < x < y\,;$$

$$\mathbb{P}\{\tau_x < \tau_{c_1}\} = 1 \iff \lim_{z \searrow c_1} \mathbb{P}\{\tau_z \le \tau_x\} = 0 \iff s(c_1) = -\infty, \quad y < x < c_2.$$

This shows in particular the claimed criterion for recurrence. The items (1) – (4) of part (i) are immediate combinations of the above.

(b) For the analysis of explosions of Y we construct a twice continuously differentiable function $\psi\colon I \to \mathbb{R}_+$ such that $1 + k \le \psi \le \exp(k)$ and such that $Z_t := e^{-t}\,(\psi \circ Y_t)$ defines a local martingale on $[0, \zeta[$.

More specifically, let $\psi\colon I \to \mathbb{R}$ be the unique solution to the linear SDE

$$L\psi = \psi \text{ on } I \text{ with } \psi'(c) = 0 \text{ and } \psi(c) = 1. \tag{A.1.12}$$

We want to show that ψ has the intended properties. For a continuous function u on I, let $M(u)$ be the function on I defined by

$$M(u)(r) := \int_c^r s(d\rho)\left(\int_c^\rho u(t)\, m(dt)\right) = \int_c^r H(\rho)\left(\int_c^\rho \frac{u(t)}{H(t)a(t)}\, dt\right) d\rho.$$

From the equation $aH' + bH \equiv 0$ it follows immediately that $L\,M(u) = u$ on I; in particular, condition (A.1.12) is equivalent to the validity of the equation $\psi = 1 + M(\psi)$ on I. This leads to the presentation

$$\psi = \textstyle\sum_{n=0}^\infty M^n(1) \tag{A.1.13}$$

where $M^0(1) := 1$ and $M^{n+1}(1) = M\big(M^n(1)\big)$. We have $k = M(1)$ by definition, trivially $M^n(1) \ge 0$, and one verifies inductively

$$M^n(1) \le k^n/n!, \quad n = 1, 2, \dots \tag{A.1.14}$$

Indeed if $M^n(1) \le \frac{k^n}{n!}$ then also

$$\begin{aligned} M^{n+1}(1)(r) &= \int_c^r s(d\rho)\left(\int_c^\rho M^n(1)(t)\, m(dt)\right) \le \frac{1}{n!}\int_c^r s(d\rho)\left(\int_c^\rho k^n(t)\, m(dt)\right) \\ &\le \frac{1}{n!}\int_c^r s(d\rho)\, k^n(\rho)\,\left(\int_c^\rho m(dt)\right) = \frac{1}{n!}\int_c^r k^n(\rho)\, k(d\rho) \end{aligned}$$

$$= \frac{1}{n!} \int_c^r k^n(\rho)\, k'(\rho)\, d\rho = \frac{k^{n+1}(r)}{(n+1)!}.$$

By (A.1.14) the right-hand side $\psi^* := \sum_{n=0}^{\infty} M^n(1)$ of (A.1.13) is well-defined and satisfies the estimate $1 + k \le \psi^* \le \exp(k)$; on the other hand, as $1 + M(\psi^*) = \psi^*$ on I, we get $\psi = \psi^*$. In particular, the representation (A.1.13) shows that ψ is decreasing on $]c_1, c]$ and increasing on $[c, c_2[$.

The claim that $(Z_t)_{t<\zeta}$ with $Z_t = e^{-t}\, \psi(Y_t)$ is a local martingale is finally seen from Itô's formula using the fact that $a\psi'' + b\psi' = \psi$.

(c) If $k(c_1) = k(c_2) = \infty$, then $\mathbb{P}\{\zeta = \infty\} = 1$ holds for the lifetime $\zeta \equiv \tau_{c_1} \wedge \tau_{c_2}$.

To see this, we chose a compact exhaustion $[a_n, b_n] \uparrow]c_1, c_2[$ where $a_n < y < b_n$ and consider $\sigma_n = \inf\big\{t \ge 0 \colon Y_t \notin [a_n, b_n]\big\} \equiv \tau_{a_n} \wedge \tau_{b_n}$. Since Z^{σ_n} is a martingale, we have for each $t \in \mathbb{R}_+$ the equality $\psi(y) = E\big[e^{-(\sigma_n \wedge t)}\, \psi(Y_{\sigma_n \wedge t})\big]$ and thus

$$\mathbb{P}\{\sigma_n < t\} \le \frac{e^t\, \psi(y)}{\psi(a_n) \wedge \psi(b_n)} \to 0 \quad \text{for } n \to \infty;$$

here we use the assumption $k(c_1) = k(c_2) = \infty$ along with the estimate $1 + k \le \psi$.

(d) If $k(c_i) < \infty$, then c_i is accessible for Y, i.e., it holds that $\mathbb{P}\{\tau_{c_i} < \infty\} > 0$. In particular, $\mathbb{P}\{\zeta < \infty\} > 0$ holds if $k(c_i)$ is finite for $i = 1$ or 2.

Suppose for instance that $k(c_1)$ is finite; because of $\psi \le \exp(k)$ then $\psi(x)$ stays bounded as $x \to c_1+$. Without restrictions we may assume that $c_1 < y < c$. In addition to the sequence $(\sigma_n)_{n\in\mathbb{N}}$ of exit times described in (c), we consider the hitting time $\sigma_0 := \tau_c$ of c. Taking into account that $\sigma_n < \zeta$, we have

$$\begin{aligned} 1 < \psi(y) &= \mathbb{E}\big[e^{-(\sigma_n \wedge \sigma_0)}\, (\psi \circ Y_{\sigma_n \wedge \sigma_0})\big] \\ &= \mathbb{E}\big[1_{\{\sigma_n \ge \sigma_0\}}\, e^{-\sigma_0}\, \psi(c) + 1_{\{\sigma_n < \sigma_0\}}\, e^{-\sigma_n}\, (\psi \circ Y_{\sigma_n})\big] \\ &\le 1 + \psi(c_1+)\, \mathbb{E}\big[1_{\{\sigma_n < \sigma_0\}}\, e^{-\sigma_n}\big] \\ &\downarrow 1 + \psi(c_1+)\, E\big[1_{\{\tau_{c_1} < \sigma_0\}}\, e^{-\tau_{c_1}}\big] \quad \text{as } n \to \infty, \end{aligned}$$

where we used that the function ψ is decreasing on the subinterval $]c_1, c]$; in particular, then $\mathbb{P}\{\tau_{c_1} < \sigma_0\} > 0$ holds true. The case $k(c_2) < \infty$ is treated analogously. This completes the proof of part (ii) of the Theorem.

(e) In order to verify (iii) we show first: If $\mathbb{P}\{\zeta < \infty\} = 1$ holds, then one of the cases (1), (2) or (3) is in force.

If $\mathbb{P}\{\zeta < \infty\} = 1$, then $k(c_1) < \infty$ or $k(c_2) < \infty$ by (ii). Assume for instance $k(c_1) < \infty$, and suppose that none of the cases (1), (2), (3) is given. Taking (A.1.10) into account, we see then that

$$s(c_1) > -\infty \quad \text{and} \quad s(c_2) < \infty, \quad k(c_2) = \infty.$$

Hence we are in situation (4) of part (i), and $\mathbb{P}\big\{\lim_{t\uparrow\zeta} Y_t = c_2\big\} > 0$ holds true. We consider the time-changed process $\big(Z_{\tau_t} := e^{-\tau_t}\, \psi(Y_{\tau_t})\big)_{t\in\mathbb{R}_+}$ where the time change $(\tau_t)_{t\in\mathbb{R}_+}$ stretches the stochastic interval $[0, \zeta[$ to $\mathbb{R}_+ \times \Omega$. As Z_{τ_t} is a non-negative continuous local martingale, the limit

$$Z_{\tau_\infty} = \lim_{t\uparrow\zeta} e^{-t}\, \psi(Y_t)$$

almost surely exists in $\mathbb{R}$ and takes the value $e^{-\zeta}\, \psi(c_2+)$ on $\big\{\lim_{t\uparrow\zeta} Y_t = c_2\big\}$. Since $1 + k \le \psi \le \exp(k)$ and $k(c_2) = \infty$, it holds that $\psi(c_2-) = \infty$, with the consequence that $\zeta = \infty$ on $\big\{\lim_{t\uparrow\zeta} Y_t = c_2\big\}$. This is however in contradiction to $\mathbb{P}\{\zeta < \infty\} = 1$; hence either (1), (2) or (3) must hold true. One argues analogously in the case $k(c_2) < \infty$.

(f) It remains to show that $\mathbb{P}\{\zeta < \infty\} = 1$ if in (iii) one of the cases (1), (2), (3) is given. We verify first that $\mathbb{E}[\zeta] < \infty$ under the assumption $k(c_1) < \infty$, $k(c_2) < \infty$.

To this end, we construct under the condition $k(c_1) < \infty$, $k(c_2) < \infty$ a twice continuously differentiable non-negative function $u\colon I \to \mathbb{R}$ such that

$$Lu \equiv -1 \quad \text{and} \quad u(c_1+) = u(c_2-) = 0.$$

We set $u(r) = \int_I G(r,t)\, m(dt)$ where

$$G(r,t) := \frac{\big(s(r \wedge t) - s(c_1)\big)\,\big(s(c_2) - s(r \vee t)\big)}{s(c_2) - s(c_1)}, \quad (r,t) \in I \times I.$$

Under the assumption that $k(c_1) < \infty$, $k(c_2) < \infty$ we have $s(c_i) \in \mathbb{R}$ for $i = 1, 2$ and hence G is bounded. Furthermore, we have

$$\begin{aligned}
u(r) &= \frac{s(c_2) - s(r)}{s(c_2) - s(c_1)} \int_{c_1}^{r} \big(s(t) - s(c_1)\big) m(dt) + \frac{s(r) - s(c_1)}{s(c_2) - s(c_1)} \int_{r}^{c_2} \big(s(c_2) - s(t)\big) m(dt)\\
&= \frac{s(c_2) - s(r)}{s(c_2) - s(c_1)} \int_{c_1}^{r} \big(m(r) - m(\rho)\big) s(d\rho) + \frac{s(r) - s(c_1)}{s(c_2) - s(c_1)} \int_{r}^{c_2} \big(m(\rho) - m(r)\big) s(d\rho)\\
&= -\frac{s(c_2) - s(r)}{s(c_2) - s(c_1)} \int_{c_1}^{r} m(\rho)\, s(d\rho) + \frac{s(r) - s(c_1)}{s(c_2) - s(c_1)} \int_{r}^{c_2} m(\rho)\, s(d\rho),
\end{aligned}$$

where we used in the second line the conversion

$$\begin{aligned}
\int_r^{c_2} \big(s(c_2) - s(t)\big)\, m(dt) &= \int_r^{c_2} \left(\int_t^{c_2} H(\rho)\, d\rho \right) m(dt)\\
&= \int_r^{c_2} \int_r^{c_2} H(\rho)\, 1_{\{t \le \rho\}}(t, \rho)\, d\rho\; m(dt)\\
&= \int_r^{c_2} \left(\int_r^{\rho} m(dt) \right) H(\rho)\, d\rho = \int_r^{c_2} \big(m(\rho) - m(r)\big)\, s(d\rho).
\end{aligned}$$

The computation above shows that under the assumption that $k(c_1) < \infty$, $k(c_2) < \infty$ the function u is finite on I; evidently even bounded and twice continuously differentiable. In addition, one verifies $Lu \equiv -1$; trivially $u(c_1+) = u(c_2-) = 0$ holds. Itô's formula then gives

$$d\big(u(Y)\big) = u'(Y)\, \sigma(Y)\, dB - dt.$$

Choosing now as in part (c) a compact exhaustion $[a_n, b_n] \uparrow \,]c_1, c_2[$ such that $a_n < y < b_n$ and considering the stopping times $\sigma_n = \inf\big\{t \ge 0\colon Y_t \notin [a_n, b_n]\big\}$, we obtain

$$\mathbb{E}[u(Y_{t \wedge \sigma_n})] = u(y) - \mathbb{E}[t \wedge \sigma_n].$$

Hence we get $\mathbb{E}[t \wedge \sigma_n] \le u(y)$, and as $t \to \infty$, $n \to \infty$, we conclude $\mathbb{E}\zeta \le u(y) < \infty$.

(g) We show: the conditions $k(c_1) < \infty$ and $s(c_2) = \infty$ imply $\mathbb{P}\{\zeta < \infty\} = 1$.

If $(a_n)_{n \in \mathbb{N}}$ is a sequence in I such that $a_n \uparrow c_2$, then it holds $\mathbb{P}\{\tau_{c_1} \wedge \tau_{a_n} < \infty\} = 1$ according to (f); on the other hand, $k(c_1) < \infty$ implies $s(c_1) > -\infty$, so that situation (2) of part (i) is given, which implies $\lim_{n \to \infty} \mathbb{P}\{\tau_{a_n} > \tau_{c_1}\} = 1$. By the obvious identity $\{\tau_{c_1} < \infty\} = \bigcup_n \{\tau_{c_1} < \tau_{a_n}\}$ one obtains then $\mathbb{P}\{\tau_{c_1} < \infty\} = 1$ and $\mathbb{P}\{\zeta < \infty\} = 1$ as wanted.

Analogously, one verifies $\mathbb{P}\{\zeta < \infty\} = 1$ in the remaining case $k(c_2) < \infty$ and $s(c_1) = -\infty$. $\square$

A.2. Derivative Flows

Let M be an n-dimensional smooth manifold and, for some $m \in \mathbb{N}$, let

$$A\colon M \times \mathbb{R}^m \to TM, \quad (x,e) \mapsto A(x)e,$$

be a homomorphism of vector bundles over M. Thus, $A \in \Gamma(\mathbb{R}^m \otimes TM)$, i.e., the map $A(x)\colon \mathbb{R}^m \to T_xM$ is linear for $x \in M$, and $A(\,\cdot\,)e \in \Gamma(TM)$ is a smooth vector field on M for $e \in \mathbb{R}^m$. Consider the Stratonovich stochastic differential equation

$$dX = A(X) \circ dB + A_0(X)\,dt \tag{A.2.1}$$

where $A_0 \in \Gamma(TM)$ is an additional vector field, and B an $\mathbb{R}^m$-valued Brownian motion on a filtered probability space $\big(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\in\mathbb{R}_+}\big)$ satisfying the usual completeness conditions. There is a partial flow $X_t(\,\cdot\,), \zeta(\,\cdot\,)$ associated to (A.2.1) (see [**29**] for details) such that for each $x \in M$ the process $X_t(x)$, $0 \le t < \zeta(x)$, is the maximal strong solution to (A.2.1) with starting point $X_0(x) = x$, defined up to the explosion time $\zeta(x)$; moreover, using the notation $X_t(x,\omega) = X_t(x)(\omega)$ and $\zeta(x,\omega) = \zeta(x)(\omega)$, if

$$M_t(\omega) = \{x \in M : t < \zeta(x,\omega)\}$$

then there exists a set $\Omega_0 \subset \Omega$ of full measure such that for all $\omega \in \Omega_0$:

(i) $M_t(\omega)$ is open in M for each $t \ge 0$, i.e., $\zeta(\,\cdot\,,\omega)$ is lower semicontinuous on M.
(ii) $X_t(\,\cdot\,,\omega)\colon M_t(\omega) \to M$ is a diffeomorphism onto an open subset of M.
(iii) The map $s \mapsto X_s(\,\cdot\,,\omega)$ is continuous from $[0,t]$ into $C^\infty\big(M_t(\omega), M\big)$ with its C^∞-topology, for each $t > 0$.

The solution processes $X = X(x)$ to A.2.1 are diffusions on M with generator

$$L = A_0 + \tfrac{1}{2} \sum_{i=1}^m A_i^2$$

where $A_i = A(\,\cdot\,)e_i \in \Gamma(TM)$, $i = 1, \dots, m$.

Consider the special case that the system A.2.1 is *non-degenerate* (elliptic), in te sense that $A(x)\colon \mathbb{R}^m \to T_xM$ is surjective for each x, or equivalently that L is an elliptic operator. This non-degeneracy provides a Riemannian metric on M such that $A(x)A(x)^*\colon T_xM \to T_xM$ is the identity on T_xM for $x \in M$. Then $A(x)^*\colon T_xM \to \mathbb{R}^m$ defines an isometric inclusion for each $x \in M$, i.e.,

$$\langle u, v\rangle_{T_xM} = \langle A(x)^*u, A(x)^*v\rangle_{\mathbb{R}^m} \quad \text{for all } u, v \in T_xM.$$

With respect to this Riemannian metric, $L = \frac{1}{2}\Delta_M + Z$ where Z is of first order, i.e., a vector field on M. Standard examples are the *gradient Brownian systems* when M is immersed into some Euclidean space $\mathbb{R}^m$, and $A(x)\colon \mathbb{R}^m \to T_xM$ is the orthogonal projection; for $A_0 = 0$ this construction gives Brownian motion on M with respect to the induced metric, see [**11**].

For $x \in M$, let $T_xX_t\colon T_xM \to T_{X_t(x)}M$ be the differential of $X_t(\,\cdot\,)$ at x (well-defined for all $\omega \in \Omega$ such that $x \in M_t(\omega)$) and $V_t = V_t(v) = (T_xX_t)v$ the derivative process to $X_t(\,\cdot\,)$ at x in the direction $v \in T_xM$. It is well-known that V on TM solves the formally differentiated SDE (A.2.1), i.e.,

$$dV = (T_XA)\,V \circ dB + (T_XA_0)\,V\,dt, \quad V_0 = v, \tag{A.2.2}$$

with the same lifetime as $X(x)$, if $v \ne 0$. Using the metric and the corresponding Levi-Civita connection on M, Eq. (A.2.2) is most concisely written as a covariant equation along X

$$DV = (\nabla A)\,V \circ dB + (\nabla A_0)\,V\,dt \tag{A.2.3}$$

(see [**11**]); by definition, (A.2.3) means

$$d\tilde{V} = //_0^{-1}t(\nabla A)\,//_0t\tilde{V} \circ dB + //_0^{-1}t(\nabla A_0)\,//_0t\tilde{V}\,dt$$

for $\tilde{V}_t = //_0^{-1}tV_t$ where $//_0t\colon T_{X_0}M \to T_{X_t}M$ is parallel transport along the paths of X.

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