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★**Stochastic calculus of variations in mathematical finance.**

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The book under review is on the applications of the Malliavin calculus to financial mathematics. The Malliavin calculus, which was developed in the 1970s, is a differential calculus on the probability space with the objective of studying the regularity of laws of random variables. The book by Malliavin and Thalmaier gives a concise account on the power of this calculus applied to a variety of financial problems, a development which began rather recently with the Clark-Ocone formula [see I. Karatzas, D. L. Ocone and J. Li, Stochastics Stochastics Rep. **37** (1991), no. 3, 127–131; MR1148344 (93f:60078)], and later the papers by E. Fournié et al. [Finance Stoch. **3** (1999), no. 4, 391–412; MR1842285 (2002e:91062); Finance Stoch. **5** (2001), no. 2, 201–236; MR1841717 (2002e:91063)].

The authors have written a short book introducing the reader efficiently to the key points of the Malliavin calculus in mathematical finance. Although the book is theoretically demanding, the reviewer believes that the applications presented may also tempt less mathematically trained readers to make the effort to learn the basics of the Malliavin calculus. This calculus is presented in a very clear and motivating way in the first chapter of the book. In fact, the authors introduce the Malliavin calculus from a discrete-time point of view, analogous to discrete-time trading in financial markets, and build the bridge over to the theory for continuous-time processes. Key concepts are the Malliavin derivative, divergence operators and the integration-by-parts formula.

Following the first chapter, the book consists of seven more chapters and three appendices devoted to financial applications. Also, the list of references is comprehensive and updated, and gives a clear picture of the activity and relevance of this approach to many financial problems. The book is only 142 pages long, however, it is still rigourous and readable. The compact form is to the advantage of the reader, who is led to the applications rather quickly. There are few concrete examples, but such can be easily found in the many cited papers.

One of the basic questions in mathematical finance is the sensitivity, or the “Greek”, of an option price with respect to one of the parameters in the model. Typical parameters of interest are the underlying stock price, the interest rate or the volatility. This is the topic of Chapter 2, which is based on the work by Fournié et al. [op. cit.]. The sensitivities are important for the practitioners, because they are the central inputs to the question of hedging the option, and the risk involved in trading options. The sensitivity can be expressed in terms of the Malliavin derivative of the underlying stock price process, and using the integration-by-parts formula one can derive expressions for the sensitivities which do not involve any differentiation of the pay-off function of the option. This is of crucial importance, for example, for digital options, where the derivative of the pay-off function is non-existing. Also, for other options like barrier or Asian types, the approach using the Malliavin calculus in conjunction with the integration by parts is powerful and leads to new formulas for sensitivity calculations. These are expressed in terms of expectations involving certain random weight functions, which are feasible for Monte Carlo valuation.

Chapter 3 deals with the market equilibrium and price-volatility feedback rate, based on recent work by the present authors and their collaborators [E. Barucci et al., Math. Finance **13** (2003), no. 1, 17–35; MR1968094 (2004d:91099)]. The price-volatility feedback rate is a function which describes the market's stability. The authors call it a liquidity index, and it is defined as the logarithmic derivative of the pathwise stock price process sensitivity rescaled by the volatility. The feedback rate is possible to estimate from data by a succession of three iterative volatility estimations. In Appendix C this is demonstrated in practice for the IBM stock, where the rate is calculated for two typical trading days. When the market is unstable, the feedback rate displays large positive values, whereas in a more stable market the feedback rate is negative for longer periods. In fact, the authors prove in Chapter 3 that a consistently negative feedback rate along with a sufficiently smooth volatility gives ergodicity of the market.

Frequently, problems in mathematical finance end up with the calculation of conditional expectations (like, e.g., the price of an American option). Chapter 4 deals with deriving expressions for conditional expectations applicable for Monte Carlo simulation where one does not need to use the crude approach of rejecting any trajectory not satisfying the conditioning. The problem of calculating the conditional expectation is turned into an expectation of a function of the random variable, a divergence operator and differentiation. The authors develop a theory for multivariate conditioning by using the tools from the Malliavin calculus, but also propose to use the Riesz transform leading to expressions where one avoids differentiating divergences. The theory in this chapter has widened the possible range of problems which can be solved by Monte Carlo methods in finance.

Interest rate models and problems with hedging instabilities are the topics of Chapter 5. The condition of ellipticity has to be left due to the high dimensions of the market, along with a low-dimensional variance. The authors consider hypoelliptic models instead, and study in particular a Heath-Jarrow-Morton model for the interest-rate market. They make use of the pathwise compartment principle, which says that the covariance matrix of the model has a strictly increasing range as a function of time, implying that the norms for different times are not equivalent. The corollary to this is the existence of digital options where hedging becomes unstable.

An insider in a financial market is a trader who has access to more information than the market. In mathematical terms this is modelled by an enlargement of the filtration generating the market information and involves anticipative stochastic calculus. Chapter 6 is devoted to the application of the Malliavin calculus to the analysis of the additional utility for an insider investing in the financial market. The insider information is modelled as the knowledge of the value of a random variable, and this is shown to correspond to an additional information drift on the driving Wiener process with respect to the insider's filtration. This in turn adds to the insider's maximal utility when acting optimally upon this information. The theory quantifies the value of having additional information in the market.

Chapter 7 deals with the asymptotic expansion and convergence of the Euler scheme for stochastic differential equations. The first question is analyzed using the Watanabe theory which leads to an asymptotic expansion of the price of a digital option depending on a parameter. Further, a strong functional convergence of the Euler scheme in the sense of convergence in Sobolev norms on the Wiener space is shown. A weak convergence of the same scheme is also derived.

The final chapter is devoted to a development of a stochastic calculus of variations for markets with jumps, that is, for markets where the randomness is modelled by compound Poisson processes. On the Poisson space, differentiation is defined with respect to the jump times of the Poisson process, and an integration-by-parts formula is proven. This paves the way for the calculation of Greeks in the market, however only for those options where the payoff functional belongs to the domain of the derivative operator, which is not the case for plain vanilla options. The application in this chapter is to mean-variance minimal hedging, which is calculated explicitly.

This book is recommended to all researchers in mathematical finance. It shows how advanced mathematics can play an important role in solving practical financial problems as well as developing new understanding and concepts.

Reviewed by *Fred Espen Benth*