The Bonnet Plancherel formula for monomial representations for classes of completely solvable Lie groups

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Abstract

We compute the Bonnet Plancherel formula associated to a monomial representation of a nilpotent Lie group. We give also the corresponding formula for finite multiplicity monomial representation for a class of completely solvable Lie groups.

0. Introduction

Let $G$ be a connected Lie group having a smooth dual. Given a unitary representation $\pi$ of $G$ acting in a Hilbert space $H_\pi$, we denote by $H^\infty_\pi$ the Fréchet space of smooth vectors for $\pi$, and $H^{-\infty}_\pi$ the space of continuous anti-linear functionals on $H^\infty_\pi$. Let $\alpha$ be any positive distribution on $G$ of finite order. Bonnet’s Plancherel formula ([Bon.]) tells us that for $\varphi \in D(G)$

$$\alpha(\varphi) = \int_G \operatorname{tr}(\pi(\varphi) U_\pi) d\nu(\pi)$$

where for $\nu$ almost everywhere, $U_\pi : H^\infty_\pi \to H^{-\infty}_\pi$, $\pi \in \hat{G}$, is a certain uniquely determined nuclear operator (see [Bon.] Theorem 4-1).

We recall that Penney’s and Bonnet’s Plancherel formulas have been described for nilpotent groups and exponential groups in ([Pen], [Fu.1,3,4], [F.Y.], [Gr,1,2], [Li.2,3], [B.L.2]). Furthermore, Fujiwara has given an explicit expression by duality of Bonnet’s operators in the case of monomial representations of nilpotent Lie groups.

In the first part of this paper we take a closed connected subgroup $H = \exp(\mathfrak{h})$ of a nilpotent connected simply connected Lie group $G = \exp(\mathfrak{g})$, a unitary character
\[ \chi = \chi_f \text{ of } H \text{ (where } f \in \mathfrak{g}^* \text{ is such that } \langle f, [h, h] \rangle = 0 \text{) and we consider the positive distribution} \]

\[ \langle S_{H,f}, \varphi \rangle = \int_{H} \varphi(h)\chi_f(h)dh, \quad \varphi \in \mathcal{D}(G). \]

To describe the measure \( \nu \) given in (1), we use the result of [B.L. 1] where it has been shown that there exists a certain affine subspace \( V \) of \( (f + \mathfrak{h}^\perp) \) such that \( \text{Ind}_{G}^{H} \chi_f \simeq \int_{V} \pi_{\phi} d\phi \) (\( d\phi \) denotes the Lebesgue measure on \( V \)). There exists a Borel cross-section \( \Sigma \) of \( G \)-orbits in \( G \cdot V \) and it turns out that the measure \( \nu \) of Bonnet’s formula is supported on \( \Sigma \). We show in (6) that for \( \sigma \in \Sigma \) the operator \( U_{\sigma} \) is an integral of rank one operators:

\[ U_{\sigma} = \int_{\Gamma_{\sigma}} Q_{s,\sigma} d\lambda_{\sigma}(s) \]

where \( \Gamma_{\sigma} \) is defined in paragraph (1.1.b), the operators \( Q_{s,\sigma} \) and the measure \( d\lambda_{\sigma} \) in 1.3.

In the exponential case, the determination of Bonnet’s operators \( U_{\sigma} \) is difficult. One of the reasons is that there exists no easy way to determine explicitly the \( C^\infty \) vectors of a representation.

Several authors have studied in the past the disintegration of induced representations for exponential solvable Lie groups. In ([D.R.]) Duflo and Rais computed the Plancherel formula for \( L^2(G) \) of an exponential solvable Lie group. Bonnet’s operators have been explicitly described for a normal monomial representation induced from a normal subgroup of an exponential solvable Lie group in [G.H.L.S.].

In the second part of this paper we take the semi-direct product \( G = NH \); where \( N = \exp(n) \) is nilpotent and normal in \( G \), and \( H = \exp(\mathfrak{h}) \) is abelian and acts semi-simply on \( N \) with real eigenvalues. Let \( \chi = \chi_f \) be a unitary character of \( H \) (where \( f \in \mathfrak{g}^* \)). We consider the representation \( \tau_f = \text{Ind}_{H}^{G} \chi_f \) and we assume that \( \tau_f \) has finite multiplicity. The first precise formulas in this case have been given by Currey in ([Cu.2]). To describe the measure \( \nu \) given in (1) we use the main results of this reference, where it has been shown that the set of generic \( H \)-orbits in the disintegration of \( \tau_f \) admits a natural smooth algebraic cross-section \( \Sigma \). We derive a cross-section \( \Gamma \) of \( G \)-orbits in \( G(\ell + \Sigma) \), and the measure \( \nu \) of Bonnet’s formula will be explicitly described as a measure on \( \Gamma \). We take \( \sigma \in (\ell + \Sigma) \) and for \( l \in G \cdot \sigma \cap (\ell + \Sigma) \) we define an operator \( \beta_{l}^{\sigma} \) on the space of the smooth vectors \( \mathcal{H}_{\sigma}^{\ell} \) of \( \pi_{\sigma} \). We show in (13) that the operators \( U_{\sigma} = U_{\sigma} \) in Bonnet’s formula are determined as a finite sum of rank one operators: \( P_{\beta_{l}^{\sigma}, \beta_{l}^{\sigma}} \).
1. The Bonnet Plancherel formula for nilpotent Lie group

1.1 Notations and definitions

1.1.a Quotient measures

Let $G$ be a connected simply connected nilpotent Lie group with Lie algebra $g$ and let $K = \exp(\mathfrak{k})$ be a closed subgroup of $g$. We choose a Jordan-Hölder basis $\mathcal{Z} = \{Z_1, \cdots, Z_n\}$ of $g$. Let $B = \{X_1, \cdots, X_r\}$ be a Malcev-basis relative to $\mathfrak{k}$, i.e. $g = \bigoplus_{1 \leq i \leq r} \mathbb{R}X_i \oplus \mathfrak{k}$ and for any $j = 1, \cdots, r$, the subspace $g_j = \text{span}\{X_j, \cdots, X_r, \mathfrak{k}\}$ is a subalgebra. The mapping $E_B : \mathbb{R}^r \to G/K : E_B(t_1, \cdots, t_r) = E_B'(t_1, \cdots, t_r)K$, where $E_B'(t_1, \cdots, t_r) = \exp(t_1X_1)\cdots\exp(t_rX_r)$, is then a diffeomorphism. We obtain a $G$-invariant measure $d\hat{g}$ on the quotient space $G/K$ by setting

$$\int_{G/K} \xi(g)d\hat{g} = \int_{\mathbb{R}^r} \xi(E_B(T))dT, \xi \in C_c(G/K),$$

where $C_c(G/K)$ denotes the space of the continuous functions with compact support on $G/K$.

It is not difficult to see the following:

1.1.a.1 Proposition Let $g$ be a nilpotent Lie algebra of dimension $n$. Let $\mathfrak{k}$ be a subalgebra of $g$, $B_1$ and $B_2$ be two Malcev-basis of $g$ relative to $\mathfrak{k}$. Then $E_B^{-1} \circ E_B'$ is a polynomial mapping from $\mathbb{R}^r$ to $\mathbb{R}^r$ (where $r$ is the codimension of $\mathfrak{k}$ in $g$) whose total degree is bounded by a constant $M$ which depends only on the dimension of $g$.

1.1.b Induced representation

Let $G$ be a nilpotent connected simply connected Lie group with Lie algebra $g$. Let $h$ be a subalgebra of $g$: $f \in g^*$ such that $\{f, [h, h]\} = 0$ and let $\chi_f$ be the unitary character of $H = \exp(h)$ associated to $f$. Let $\tau = Ind_H^G \chi_f$. It has been shown in [B.L.1] that there exists a certain affine subspace $\mathcal{V}$ of $\Gamma_f = f + h^0 \subset g^*$, such that

$$\tau = Ind_H^G \chi_f \simeq \int_{\mathcal{V}} \pi_\phi d\mu(\phi) \quad (2)$$

where $d\mu$ denotes Lebesgue measure on $\mathcal{V}$ and where $\pi_\phi$ is the irreducible representation associated to $\phi$ ($\phi \in \mathcal{V}$).

One has:
Lemma ([Bour], [Fuj. 3])

\[ \mu = \int_{G \cdot V / G} \nu_\Omega d\nu(\Omega) \]

where \( \nu_\Omega \) is a certain measure on \( \Omega \cap V \).

Let \( \Sigma \) be a borel cross-section of the \( G \)-orbits in \( G \cdot V \). We can consider the measure \( \nu \) as a measure on \( \Sigma \) and write \( \mu = \int_\Sigma \nu_{G \sigma} d\nu(\sigma) \).

Hence for a continuous function \( F \) with compact support on \( V \) we get

\[ \int_V F(\phi) d\mu(\phi) = \int_\Sigma \int_{G \cdot \sigma \cap V} F(l) d\nu_{G \cdot \sigma}(l) d\nu(\sigma). \]

We identify \( G \cdot \sigma \cap V \) with the space \( G_{\sigma} / G(\sigma) \), where

\[ G_{\sigma} = \{ g \in G ; \ g \cdot \sigma \in \mathcal{V} \} \]

and

\[ G(\sigma) = \{ g \in G , \ g \cdot \sigma = \sigma \} \]

which corresponds to the measure \( \nu_{G \cdot \sigma} \), we write:

\[ \int_V F(\phi) d\mu(\phi) = \int_\Sigma \int_{\Gamma_{\sigma}} F(s \cdot \sigma) d\lambda_{\sigma}(s) d\nu(\sigma). \quad (3) \]

Let now \( S(G/H, f) \) be the space of all \( C^\infty \)-function \( \xi \) on \( G \), such that \( \xi(gh) = \chi_f(h^{-1})\xi(g) \) for all \( g \in G, h \in H \) and such that the function \( T \mapsto \xi(E_B(T)) \) is a Schwartz-function on \( \mathbb{R}^r \). Let \( S(G) \) denote the Schwartz-space of \( G \), i.e. the space of all complex valued functions \( \varphi \) on \( G \), such that \( \varphi \circ \exp \) is an ordinary Schwartz-function on the vector space \( \mathfrak{g} \).

Denote for \( \varphi \in S(G) \) \( P_{H,f}(\varphi)(g) = \int_H \varphi(gh)\chi_f(h)dh \) and let \( S_{H,f} \) be the tempered distribution on \( G \) defined by the projection \( P_{H,f}(\varphi) \) of \( \varphi \) on \( S(G/H, f) \), i.e:

\[ \langle S_{H,f}, \varphi \rangle = \int_H \varphi(h)\chi_f(h)dh = P_{H,f}(\varphi). \]

Let for \( \phi \in V \) \( B(\phi) \) denote the Vergne polarization at \( \phi \) for the basis \( Z \). It has been shown in [B.L.1] that there exists for \( \phi \in V \) an invariant measure \( \tilde{d}B \) on \( B(\phi) / B(\phi) \cap H \) such that for the mapping

\[ T_\phi : S(G/H, f) \to S(G/B(\phi), \phi) \quad (\phi \in V) \]

given by

\[ T_\phi(\xi)(g) = \int_{B(\phi) / B(\phi) \cap H} \xi(bg)\chi_{\phi}(b) \tilde{d}b, \quad \xi \in S(G/H, f), g \in G, \]

and for \( \xi \in S(G/H, f) \) we have:

\[ \int_V \langle T_\phi(\xi), T_\phi(\xi) \rangle_{\mathcal{H}_\tau} d\phi = \| \xi \|^2_{\mathcal{H}_\tau}. \quad (4) \]
We recall also that from [B.L.2] $S_{H,f}$ is disintegrated as an integral $\int_V S_\phi d\mu(\phi)$, where $S_\phi$ denotes the tempered distribution on $S(G)$ defined by:

$$\langle S_\phi, \varphi \rangle = \int_{H/H \cap B(\phi)} T_\phi(P_{H,f}(\varphi))(h)\chi_f(h) dh.$$  \hspace{1cm} (5)

1.2 Main results

This section is based on the paragraph 7.5 in [L.M.].

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Let $\mathcal{B}$ be an algebraic subset of finite dimensional real vector space $W$, the pair $(\mathfrak{g}, \mathcal{B})$ is a rationally variable nilpotent Lie algebra (or r.v.n.) if the following holds true:

For every $b \in \mathcal{B}$, a Lie bracket $[\cdot, \cdot]_b$ on $\mathfrak{g}$ is given such that $(\mathfrak{g}, [\cdot, \cdot]_b)$ forms a nilpotent Lie algebra. Moreover there exists a fixed basis $Z = \{Z_1, \ldots, Z_n\}$ of $\mathfrak{g}$, so that the structure constants $(a_{ij}^k(b))$, given by $[Z_i, Z_j]_b = \sum_{k=1}^n a_{ij}^k(b)Z_k$, are rational functions in $b$, satisfying $a_{ij}^k(b) = 0$ for $i < j, k \leq j$ (so that $Z$ is a Jordan-Hölder-basis for $(\mathfrak{g}, [\cdot, \cdot]_b)$ (see [L.M]).

A mapping on $\mathcal{B}$ is called polynomial if it is the restriction of a polynomial mapping on $W$ to $\mathcal{B}$ and it is called rational if it is the restriction of a rational mapping on $W$ to $\mathcal{B}$, such that the denominators of the corresponding rational functions do not vanish on $\mathcal{B}$.

For every $b \in \mathcal{B}$ we choose $m$ elements $V_1(b), \ldots, V_m(b)$, in $\mathfrak{g}^*$ depending rationally on $b$. Let $V(b) = \text{span}(V_1(b), \ldots, V_m(b))$ and $\phi^b : \mathbb{R}^m \to V(b)$ defined by $\phi^b(X) = \sum_{i=1}^m x_iV_i(b)$, where $X = (x_1, \ldots, x_m) \in \mathbb{R}^m$.

Let us denote for $(X, b) \in \mathbb{R}^m \times \mathcal{B}$ and for a polarization $b$ at $\phi^b(X)$ in $\mathfrak{g}_b$ the induced representation $\pi_{\phi^b(X), b}$ by $\pi_{(X, b), b}$. Given any Malcev basis $B$ of $\mathfrak{g}_b$ relative to $b$, we can realize the representation in a canonical way on $L^2(\mathbb{R}^r)$ and for every element $u$ in the enveloping $U(\mathfrak{g}_b)$ of $\mathfrak{g}_b$, the operator $d\pi_{\phi^b(X), b}(u)$ becomes a partial differential operator with polynomial coefficients on $\mathbb{R}^r$.

In the following theorem we generalize the theorem 7.7 of [L.M] by replacing the generic points in $\mathfrak{g}^*$ by the generic points of the forms $\phi^b(X), \ X \in \mathbb{R}^m$:

1.2.1 Theorem There exists a Zariski-open subset $O$ in $\mathbb{R}^m \times \mathcal{B}$ such that:

i) For every $(X, b) \in O$ there exists a polarization $b(X, b) = b(\phi^b(X))$ at $\phi^b(X)$ and a Malcev basis $B(X, \phi^b(X))$ of $\mathfrak{g}$ relative to $b(\phi^b(X))$ depending rationally on $(X, b)$. 

ii) For every partial differential operator $D$ on $\mathbb{R}^d$ with polynomial coefficients there exists a rational mapping

$$A : O \to \mathcal{U}(\mathfrak{g}_b), \quad A(X, b) = \sum_{|I| \leq n_d} a_I(X, b)Z^I$$

such that $\pi_{(X, b), \nu}(A(X, b)) = D$.

**Proof.** We use the notations and the proof of [L.M.].

Let $b \in \mathcal{B}$ and $X \in \mathbb{R}^m$; we can construct the indices $j_i(X, b) = j_i(\phi^b(X)) = j_i(\phi^b(X), b); k_i(X, b) = k_i(\phi^b(X)) = k_i(\phi^b(X), b)$ as well as $j_1(X, b)$ and $k_1(X, b)$ corresponding to $(\mathfrak{g}, [\cdot, \cdot])$ as in [L.M]. We put

$$j_1 := \max\{j_1(X, b) : X \in \mathbb{R}^m; b \in \mathcal{B}\},$$

$$k_1 := \max\{k_1(X, b) : X \in \mathbb{R}^m; b \in \mathcal{B}\},$$

and put $\mathcal{B}^1 := \{(X, b) \in \mathbb{R}^m \times \mathcal{B} : j_1(\phi^b(X), b) = j_1 \text{ and } k_1(\phi^b(X), b) = k_1\}$. Then

$$\mathcal{B}^1 = \{(X, b) : \phi^b(X)([Z_{j_1}, Z_{k_1}, b]) \neq 0\}$$

is a Zariski-open in $\mathbb{R}^m \times \mathcal{B}$. Next, for $(X, b) \in \mathcal{B}^1$, we put $(\mathfrak{p}_1(\phi^b(X), b), [\cdot, \cdot]) := \{Y \in \mathfrak{g} : \phi^b(X)([Z_{j_1}, Y, b]) = 0\}$, and

$$Z^1_i(X, b) := Z_i - \frac{\phi^b(X)([Z_{j_1}, Z_{k_1}, b])}{\phi^b(X)([Z_{j_1}, Z_{k_1}, b])}Z_{k_1}, \ i \neq k_1.$$ 

Then $Z^1_i(X, b), i \neq k_1$, form a Jordan-Hölder-basis of $(\mathfrak{p}_1(\phi^b(X), b), [\cdot, \cdot])$.

We identify $(\mathfrak{p}_1(\phi^b(X), b))$ with $\mathfrak{p}_1 := \mathbb{R}^q$, where $q = \dim(\mathfrak{p}_1(\phi^b(X), b))$, we obtain a new r.v.n.$(\mathfrak{p}_1, \mathcal{B}^1)$. Now for $b^1 := (X, b) \in \mathcal{B}^1$, we get $m$ linear forms: $(V_i^1(b^1))_{i=1}^m$ in $\mathbb{R}^q$ given by: $V_i^1(b^1) = V_i^1(X, b) = ((V_i(b), Z_{j_1}^1(X, b)), \ldots, (V_i(b), Z_{k_1}^1(X, b)), (V_i(b), Z_{k_1+1}^1(X, b)), \ldots, (V_i(b), Z_n^1(X, b))).$

We put $V^1(b^1) = \text{span}(V_1^1(b^1), \ldots, V_m^1(b^1))$, and $\phi^{b^1} : \mathbb{R}^m \to V^1(b^1) : \phi^{b^1}(Y) = \sum_{i=1}^m y_iV_i^1(b^1)$.

Applying the same procedure now to $(\mathfrak{p}_1, \mathcal{B}^1)$ instead of $(\mathfrak{g}, \mathcal{B}^1)$, and iterating this process, which stops after a finite number $d$ of steps, we construct indices $j_i(X, b)$ and $k_i(X, b)$ for $i = 1, \ldots, d$, and finally stop at some r.v.n $(\mathfrak{p}_d, \mathcal{B}^d)$ where $\mathcal{B}^d \subset \mathbb{R}^m \times \mathcal{B}^{d-1}$ is Zariski-open. We put $O = \mathcal{B}^d$.

Moreover, it has been shown in [L.M] that for $(X, b^{d-1}) \in O$ the subalgebra $\mathfrak{p}_d(\phi^{b^{d-1}}(X), b^{d-1}) = b(\phi^{b^{d-1}}(X))$ is the Vergne polarization for $\phi^{b^d}(X)$ associated to the basis $\mathcal{Z}$ and there exist rational mappings $Y_i : \mathbb{R}^m \to \mathfrak{g}, \ 1 \leq i \leq d$, such that $\{Y_1(X), \ldots, Y_d(X)\}$ forms a Malcev basis of $\mathfrak{g}$ relative to $b(\phi^{b^{d-1}}(X))$. 

One continues as in the proof of theorem 7.7 in [L.M.].

1.2.2 Proposition Let $g$ be a nilpotent Lie algebra. Let $h$ and $b$ be two subalgebras of $g$. There exists a Malcev-basis $U$ of $g$ relative to $b$, which contains a Malcev-basis of $h$ relative to $h \cap b$.

Proof. We proceed by induction on $\dim(g)$.

Let $g_0$ be an ideal of $g$ with codimension one containing $b$.

i) If $h \subset g_0$, the induction hypothesis gives us a Malcev basis $U_0$ of $g_0$ relative to $b$ which contains a Malcev-basis of $h$ relative to $h \cap b$. Hence we put $U = \{U_0, X\}$, where $X \in g \setminus g_0$.

ii) If $h \not\subset g_0$, we can choose $X \in h$ such that $g = g_0 \oplus \mathbb{R}X$. The induction hypothesis gives us a Malcev-basis $U_0$ of $g_0$ relative to $b$, which contains a Malcev-basis of $h \cap g_0$ relative to $h \cap b$. Hence we put $U = \{U_0, X\}$. ■

1.3 The Bonnet Plancherel Formula

The aim of this section is to describe explicitly the Bonnet Plancherel Formula associated to the disintegration (2). Let $G, H, f, V, \Sigma$ be as in (1.1.b).

For $\sigma \in \Sigma, g \in G_\sigma$ we define the operator: $q_{g,\sigma} : \mathcal{H}_\sigma^\infty \rightarrow \mathbb{C}$ by

$$\langle q_{g,\sigma}, \xi \rangle = \int_{H/B(g_\sigma) \cap H} \xi(hg) \chi_f(h) dh.$$  

It has already been shown in [Fuj.1] that the integral on the right is well defined (here it suffices to use that for $g \in G_\sigma, \chi_{g\sigma}(h) = \chi_f(h)$ for all $h \in H$), the operator $q_{g,\sigma}$ is continuous and for all $h \in H$, $\pi_\sigma(h)q_{g,\sigma} = \chi_f(h)q_{g,\sigma}$.

Let $\varphi$ be in $\mathcal{S}(G)$. For $\sigma \in \mathcal{V}$, the operator $\pi_\sigma(\varphi)$ is a kernel-operator, whose kernel $K_{\pi_\sigma(\varphi)}$ is given by

$$K_{\pi_\sigma(\varphi)}(x, y) = \int_{B(\sigma)} \varphi(xby^{-1}) \chi_\sigma(b) db, \quad x, y \in G.$$  

Furthermore, for any Malcev-basis $\mathcal{Y} = \{Y_1, \ldots, Y_d\}$ of $g$ relative to $b(\sigma)$, the function

$$(s, t) \mapsto K_{\pi_\sigma(\varphi)}(\prod_{i=1}^d \exp(s_iY_i), \prod_{i=1}^d \exp(t_iY_i))$$

is a Schwartz-function on $\mathbb{R}^d \times \mathbb{R}^d$ (see [C.G.]).
Let us recall some results from [L.M.].

Let $Z_1, \ldots, Z_n \in \mathfrak{g}$ be a basis of $\mathfrak{g}$, and put

$$L := \sum_{j=1}^{n} Z_j^2 \in \mathfrak{u}(\mathfrak{g}),$$

where $\mathfrak{u}(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$. Let $N \in \mathbb{N}$. Since $(1 - L)^N$ is hypoelliptic for every $N \in \mathbb{N}^*$, there exists a local fundamental solution $E_N \in \mathcal{D}'(U)$ of $(1 - L)^N$ on a neighbourhood $U$ of $e \in G$, i.e.

$$(1 - L)^N E_N = \delta_e \text{ in } U.$$

Since $(1 - L)^N$ is hypoelliptic, we have that $E_N$ is $C\infty$ on $G \setminus \{e\}$ and for $d \in \mathbb{N}$, if $N$ is big enough $E_N$ is in $C^d(G)$. Hence $E_N$ is in $L^1(G) \cap L^2(G)$ and is even of class $C^d$ in $L^1(G)$.

We recall that the $N$th Sobolev $L^1$-norm on $G$ is defined by

$$\|f\|_{N,1} = \sum_{|\alpha| \leq N} \|Z_\alpha \ast f\|_1 + \sum_{|\alpha| \leq N} \|f \ast Z_\alpha\|_1,$$

where $Z_\alpha = Z_1^{\alpha_1} \ast \cdots \ast Z_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

1.3.1 Proposition  There exists $N \in \mathbb{N}$ such that for almost all $\sigma \in \Sigma$ and all $g \in G_\sigma$ the distribution $q_{g, \sigma}$ is an element of $H_{\sigma}^N$ and

$$\int_{G/B(\sigma)} \int_{H/H \cap B(g\sigma)} |K_{\sigma}(\varphi)(u, hg)| \, dh \, du \leq C_{\sigma, g} \|\varphi\|_{N,1} \varphi \in \mathcal{S}(G),$$

for some constant $C_{\sigma, g}$.

Proof. Let $\sigma \in \Sigma$ and $g \in G$, such that $g\sigma \in \mathcal{V}$. Let $B_1 = \langle Y_1, \ldots, Y_d \rangle$ be a Malcev-basis of $\mathfrak{g}$ relative to $\mathfrak{b}(g\sigma)$ such that $B'_1 = \langle Y_{i_1}, \ldots, Y_{i_r} \rangle$ is a Malcev-basis of $\mathfrak{h}$ relative to $\mathfrak{b} \cap \mathfrak{b}(g\sigma)$ (according to 1.2.2). Then we have

$$\langle q_{g, \sigma}, \xi \rangle = \int_{\mathbb{R}^r} \xi(s) \exp(t_1 Y_{i_1}) \cdots \exp(t_r Y_{i_r}) g e^{-i\sum_{j=1}^{r} t_j Y_{i_j}} dt_1 \cdots dt_r$$

$$= \int_{\mathbb{R}^r} (\xi \circ E_{B'_1})(t_1, \ldots, t_r)(\chi_f \circ E_{B'_1})(t_1, \ldots, t_r) dt_1 \cdots dt_r$$

where $\xi(g') = \xi(g'g)$.

Let $P_\sigma$ be a function on $G/B(\sigma)$ such that $S \mapsto P_\sigma(E'_{B_1}(S))$ is a polynomial on $\mathbb{R}^d$ of degree $\leq 2r$ such that

$$c_{g, \sigma} = \int_{\mathbb{R}^r} \frac{1}{|P_\sigma(E'_{B_1}(T))|} dT < \infty.$$
Then we get
\[ |\langle q_{g,\sigma}, \xi \rangle| \leq c_{g,\sigma} \| (P_{\sigma} \cdot \xi)_{g} \|_{\infty}. \]

Let now \( B_2 = B_2(q\sigma) \) be the Malcev-basis of \( g \) relative to \( b(q\sigma) \) obtained by theorem (1.2.1) applied to the affine subspace \( V \) and the one point set \( B \).

Let \( \tilde{Q}_{g}(S) = P_{\sigma}(E_{B_2}^{e}(S)g) = P_{\sigma}(E_{B_1}^{e}(E_{B_1}^{e-1} \circ E_{B_2}^{e})(S)g), \) \( S \in \mathbb{R}^{d} \), whose coefficients depend on \( g, \sigma \) and whose degree is bounded by an integer \( M_{1} \) independent of \( g, \sigma \) (by 1.1.a.1).

Moreover we can see that for some constant \( c'_{g,\sigma} \) big enough the polynomial \( c'_{g,\sigma} F = c'_{g,\sigma}(1 + \|T\|^{2})^{M_{1}}, T \in \mathbb{R}^{d} \), dominates the function \( \tilde{Q}_{g} \) on \( \mathbb{R}^{d} \).

Hence
\[ \| \tilde{Q}_{g,\sigma} \|_{\infty} \leq c'_{g,\sigma} \| F_{\xi_{g}} \cdot E_{B_2} \|_{\infty} \leq c'_{g,\sigma} \| D(F_{\xi_{g}} \cdot E_{B_2}) \|^{2} \]

for some fixed partial differential operator with constant coefficients on \( \mathbb{R}^{d} \). Now by theorem (1.2.1) we have that for almost all \( \sigma \in V \) there exists \( a(\sigma) \in \mathcal{U}(g) \) such that : \( d\pi_{\sigma}(a(\sigma)) = D \circ \) multiplication by \( F \). Moreover the degree of \( d\pi_{\sigma}(a(\sigma)) \) is bounded by a constant \( N \) independent of \( \sigma \). Thus for almost all \( \sigma \in \Sigma \) and all \( g \in G \)
\[ |\langle q_{g,\sigma}, \xi \rangle| \leq c''_{g,\sigma} \| \xi \|_{N} \]

for some big enough constant \( c''_{g,\sigma} \).

For the second statement, we remark that, since \( K_{\pi_{\sigma}(\varphi)} \) is a Schwartz-function on \( G \times G \) modulo \( B(\sigma) \times B(\sigma) \) by Howe’s result (see [C.G.]), the function
\[ G \ni u \mapsto \eta_{v}(u) = K_{\pi_{\sigma}(\varphi)}(u,v), \ v \in G, \]
is in \( \mathcal{S}(G/B(\sigma), \chi_{\sigma}) \) and so by the arguments for the first statement
\[
\int_{G/B(\sigma)} \int_{H/H \cap B(\sigma)} |K_{\pi_{\sigma}(\varphi)}(hg,v)| d\hat{h} dv = \int_{H/H \cap B(\sigma)} \left( \int_{G/B(\sigma)} |K_{\pi_{\sigma}(\varphi)}(hg,v)| dv \right) d\hat{h}
\]
\[ \leq \int_{H/H \cap B(\sigma)} \left( \int_{G/B(\sigma)} |\pi_{\sigma}(a'(g\sigma))\eta_{v}(hg)| dv \right) d\hat{h} \]
for some element \( a'(g\sigma) \) in the enveloping algebra of \( g \), whose degree is bounded by a constant \( N \) which does not depend on \( g\sigma \) according to (1.2.1). Since
\[
\int_{G/B(\sigma)} |\pi_{\sigma}(a'(g\sigma))\eta_{v}(hg)| dv = \int_{G/B(\sigma)} a'(g\sigma) * \varphi(hgv^{-1}) \chi_{\sigma}(b) db d\hat{v}
\]
\[ \leq \int_{G} |a'(g\sigma) * \varphi(v)| dv = \int_{G} |a'(g\sigma) * \varphi(v)| dv \leq c_{g,\sigma} \| \varphi \|_{N,1}, \]
(for some constant $c_{g\sigma}$ depending on $a'(g\sigma)$) for all $h \in H$, it follows that
\[
\int_{G/B} \int_{H/H \cap B(g\sigma)} |K_{\pi_s(\varphi)}(v, hg)| dq d\hat{v} \leq c_{g\sigma} \|\varphi\|_{N,1} \int_{H/H \cap B(g\sigma)} \frac{1}{P(\varphi)} dh \leq C_{g\sigma} \|\varphi\|_{N,1}
\]
(for some new constant $C_{g\sigma}$).

This gives us the one dimensional operators:
\[
Q_{g,\sigma} = P_{q_{g,\sigma},q_{g,\sigma}} : \mathcal{H}_{\sigma}^N \to \mathcal{H}_{\sigma}^{-N}, \quad Q_{g,\sigma}(\xi) = \langle \xi, q_{g,\sigma}\rangle_{q_{g,\sigma}}, \quad \xi \in \mathcal{H}_{\sigma}^N.
\]

In particular for $\varphi \in S(G)$, $\pi_s(\varphi) \circ Q_{g,\sigma} = P_{\pi_s(\varphi)q_{g,\sigma},q_{g,\sigma}}$ (see [G.H.L.S.]).

For $\sigma \in \Sigma$, we define the operator $U_\sigma : \mathcal{H}_{\sigma}^N \to \mathcal{H}_{\sigma}^{-N}$ as the integral of these operators:
\[
U_\sigma = \int_{\Gamma_\sigma} Q_{g,\sigma} d\lambda_\sigma(g).
\]

We have the following:

1.3.2 Proposition \textit{For almost all $\sigma \in \Sigma$ we have: $U_\sigma : \mathcal{H}_{\sigma}^N \to \mathcal{H}_{\sigma}^{-N}$ is trace class.}

Proof. Let $\sigma \in \Sigma, s \in G_\sigma$. We recall that the rank one operator $Q_{s,\sigma}$ has a trace which is given by:
\[
\text{tr}(Q_{s,\sigma}) = \text{tr}(A_{\sigma}^{-N} \circ Q_{s,\sigma} \circ A_{\sigma}^{-N}) = \langle \pi_s(E_N)q_{s,\sigma}, \pi_s(E_N)q_{s,\sigma}\rangle,
\]
where $A_{\sigma}^{-N} = \pi_s(E_N)$ (see [G.H.L.S.]).

On the other hand for $\psi \in \mathcal{H}_{\sigma}^\infty$ we have:
\[
\langle \pi_s(E_N)q_{s,\sigma}, \psi \rangle = \langle q_{s,\sigma}, \pi_s(E_N)^*\psi \rangle = \int_{H/B(s \cdot \sigma) \cap H} \overline{\pi_s(E_N)^*}\psi(h \chi_f(h)) dh \]
\[
= \int_{H/B(s \cdot \sigma) \cap H} \int_{G/B(\sigma)} K_{\pi_s(E_N)}(hs, u) \psi(u) du \chi_f(h) dh
\]
\[
= \int_{H/B(s \cdot \sigma) \cap H} \int_{G/B(\sigma)} K_{\pi_s(E_N)}(u, hs) \psi(u) du \chi_f(h) dh.
\]
As $N$ is increasing, the function $E_N$ becomes smoother and smoother and the kernel function

$$(u, h) \mapsto K_{π_σ(E_N)}(u, hs)$$

is decreasing more and more rapidly at infinity, and so for $N$ big enough, this function is in $L^1(G/B(σ), σ) \otimes L^1(H/B(s \cdot σ) \cap H, f)$ for almost all $σ ∈ V$ (see 1.3.1). Hence, using Fubini, we can deduce that

$$\langle π_σ(E_N)q_{s,σ}, ψ \rangle = \int_{G/B(σ)} \int_{H/B(s \cdot σ) \cap H} K_{π_σ(E_N)}(u, hs) d\nu(u) d\sigma $$

$$= \langle \eta_{s,σ}, ψ \rangle$$

where $η_{s,σ}(u) = \int_{H/B(s \cdot σ) \cap H} K_{π_σ(E_N)}(u, hs) \chi_f(h^{-1}) d\sigma$ is in $L^2(G/B(s \cdot σ), s \cdot σ)$.

Hence

$$\text{tr}(Q_{s,σ}) = \langle \eta_{s,σ}, \eta_{s,σ} \rangle$$

$$= \int_{G/B(σ)} \int_{H/B(s \cdot σ) \cap H} K_{π_σ(E_N)}(g, h′s) \chi_f(h^{-1}) d\sigma d\sigma'$$

$$= \int_{G/B(σ)} \int_{H/B(s \cdot σ) \cap H} K_{π_σ(E_N)}(g, h's) \chi_f(h^{-1}) d\sigma d\sigma'$$

$$= \int_{G/B(σ)} \int_{H/B(s \cdot σ) \cap H} \int_{B(s \cdot σ)} E_N(gbs^{-1}h^{-1}) \chi_σ(b) d\chi_f(h^{-1}) d\sigma' d\sigma d\sigma'$$

Now for $q ∈ C_0(G)$, it has been shown in [B.L.2] that

$$\int_{H/B(s \cdot σ) \cap H} \int_{B(s \cdot σ)} q(hb^{-1}) \chi_σ(b) d\chi_f(h^{-1}) d\sigma'$$

$$= \int_{B(s \cdot σ)/B(s \cdot σ) \cap H} \int_{H} q(hb^{-1}) \chi_σ(b) d\chi_f(h^{-1}) dσ' d\sigma ddb$$

We obtain:

$$\text{tr}(Q_{s,σ}) = \langle T_{s,σ}(P_{H,f}(E_N)), T_{s,σ}(P_{H,f}(E_N)) \rangle_{H_σ,σ} = \|T_{s,σ}(P_{H,f}(E_N))\|^2_{H_σ,σ}$$

On the other hand one has by (3)

$$\int_{G/B(σ)} \int_{H/B(s \cdot σ) \cap H} \|T_{s,σ}(P_{H,f}(E_N))\|^2_{H_σ,σ} dσ = \int_V \langle T_φ(P_{H,f}(E_N)), T_φ(P_{H,f}(E_N)) \rangle_{H_σ,σ} dφ = \|P_{H,f}(E_N)\|^2_{H_σ,σ} \text{ by (4)}. $$
Hence for almost all $\sigma \in \Sigma$

$$\|U_\sigma\|_1 = \int_{\Gamma_\sigma} \text{tr}(Q_{\sigma}) d\lambda_\sigma(\dot{g}) < \infty$$

and the integral

$$U_\sigma = \int_{\Gamma_\sigma} Q_{\sigma} d\lambda_\sigma(\dot{g})$$

converges in the space of the trace-class operators.

\[\blacksquare\]

1.3.4. **Theorem** There exists $N \in \mathbb{N}$, such that for every $\varphi \in \mathcal{S}(G)$ and for almost all $\sigma \in \Sigma$, we have that the operator $\pi_\sigma(\varphi) \circ U_\sigma : \mathcal{H}_\sigma^N \to \mathcal{H}_\sigma^N$ is trace class and

$$< S_{H,f}, \varphi > = \int_{\Sigma} \text{tr}(\pi_\sigma(\varphi) \circ U_\sigma) d\nu(\sigma).$$

**Proof.** Let $\sigma \in \Sigma, s \in G_\sigma$ and $\varphi \in \mathcal{S}(G)$. An argument similar to (*) permits us to write $\pi_\sigma(\varphi)q_{s,\sigma}(u) = \varphi_{s,\sigma}(u) = \int_{H/B(s,\sigma) \cap H} K_{\pi_\sigma(\varphi)}(u,h)^{\chi_f(h)} dh$, for all $u \in G$.

Then

$$\langle \pi_\sigma(\varphi)q_{s,\sigma}, q_{s,\sigma} \rangle = \int_{H/B(s,\sigma) \cap H} \varphi_{s,\sigma}(hs)^{\chi_f(h)} dh$$

$$= \int_{H/B(s,\sigma) \cap H} K_{\pi_\sigma(\varphi)}(hs,h')^{\chi_f(h')^{-1}} dh' \chi_f(h) dh$$

$$= \int_{H/B(s,\sigma) \cap H} K_{\pi_\sigma(\varphi)}(h,h')^{\chi_f(h'h')^{-1}} dh' dh.$$  

We recall that $\pi_\sigma(\varphi) \circ U_\sigma = \pi_\sigma(\varphi) \circ \int_{H/B(s,\sigma) \cap H} P_{\pi_\sigma(\varphi)q_{s,\sigma}, q_{s,\sigma}} d\lambda_\sigma(\dot{s}) = \int_{\Gamma_\sigma} P_{\pi_\sigma(\varphi)q_{s,\sigma}, q_{s,\sigma}} d\lambda_\sigma(\dot{s}).$

Hence we deduce that

$$\text{tr}(\pi_\sigma(\varphi) \circ U_\sigma) = \int_{\Gamma_\sigma} \int_{H/B(s,\sigma) \cap H} \int_{H/B(s,\sigma) \cap H} K_{\pi_\sigma(\varphi)}(h,h')^{\chi_f(h'h')^{-1}} dh' dh d\lambda_\sigma(\dot{s}).$$

(\*)

Now we recall that, from [B.L.2] one has

$$\langle S_{H,f}, \varphi \rangle = \int_V \langle S_\phi, \varphi \rangle d\mu(\phi)$$

$$= \int_{\Sigma} \int_{\Gamma_\sigma} \int_{H/B(s,\sigma)} T_{s,\sigma}(P_{H,f}(\varphi))(h)^{\chi_f(h)} dhd\lambda_\sigma(\dot{s}) d\nu(\sigma)$$

(by (3) and (5)).
On the other hand
\[
\int_{\Gamma} \int_{H/H \cap B(s, \sigma)} T_{s, \sigma}(P_{H, f}(\varphi))(h) \chi_f(h) dh d\lambda_\sigma(s)
\]
\[
= \int_{\Gamma} \int_{H/H \cap B(s, \sigma)} \int_{B(s, \sigma)/B(s, \sigma) \cap H} P_{H, f}(\varphi)(b) \chi_f(b) db \chi_f(h) dh d\lambda_\sigma(s)
\]
\[
= \int_{\Gamma} \int_{H/H \cap B(s, \sigma)} \int_{B(s, \sigma)/B(s, \sigma) \cap H} \varphi(h b h') \chi_f(h') dh' \chi_f(b) db \chi_f(h) dh d\lambda_\sigma(s)
\]
\[
= \int_{\Gamma} \int_{H/H \cap B(s, \sigma)} \int_{B(s, \sigma)/B(s, \sigma) \cap H} \varphi(h b b h') \chi_f(h') dh' \chi_f(b) db \chi_f(h) dh d\lambda_\sigma(s).
\]
Then by (**) *(***)
\[
\int_{\Gamma} \int_{H/H \cap B(s, \sigma)} T_{s, \sigma}(P_{H, f}(\varphi))(h) \chi_f(h) dh d\lambda_\sigma(s)
\]
\[
= \int_{\Gamma} \int_{H/H \cap B(s, \sigma)} \int_{H/H \cap B(s, \sigma) \cap H} \int_{B(s, \sigma)} \varphi(h b b h') \chi_f(h) dh' dh d\lambda_\sigma(s)
\]
\[
= \int_{\Gamma} \int_{H/H \cap B(s, \sigma)} \int_{H/H \cap B(s, \sigma) \cap H} \int_{B(s, \sigma)} K_{\pi_\sigma(\varphi)}(h, h') \chi_f(h) dh' dh d\lambda_\sigma(s)
\]
\[
= \text{tr}(\pi_\sigma(\varphi) \circ U_\sigma).
\]
Whence
\[
\langle S_{H, f}, \varphi \rangle = \int \text{tr}(\pi_\sigma(\varphi) \circ U_\sigma) d\nu(\sigma).
\]

2. The Bonnet Plancherel formula for a class of completely solvable Lie group

In this part we take, as mentioned in the introduction, the semi-direct product
\[G = N H;\] where \(N = \exp(\mathfrak{n})\) is nilpotent and normal in \(G\), and \(H = \exp(\mathfrak{h})\) is
abelian and acts semi-simply on \(N\) with real eigenvalues. Let \(\chi = \chi_f\) be a unitary character of \(H\) (where \(f \in \mathfrak{g}^*\)). We consider the representation \(\tau_f = \text{Ind}_{H}^{G} \chi_f\) and
we assume that \(\tau_f\) has finite multiplicity.

Let us recall some results given in the paper [Cu.2].

2.1 Generalities and main results

2.1.1 \(C^\infty\) vectors

Let \(G\) be an exponential solvable Lie group and \(K\) a closed subgroup of \(G\). Fix a
choice of right Haar measures \(dg, dk\) on \(G\) and \(K\). We write \(\Delta_G, \Delta_K\) for the modular
functions of \( G, K \) (respectively). If \( \chi \) is a unitary character of \( K \), the induced representation \( \pi_{\chi} = Ind_{K}^{G} \chi \) acts in the space \( C_{K}(G, K, \chi) = \{ f \in C_{K}(G) : f(kg) = \chi(k)f(g) \ \forall k \in K, g \in G; f \ \text{compactly supported mod} \ K \} \), by the formula

\[
\pi_{\chi}(g)f(x) = f(xg)q(g)^{1/2}.
\]

Here \( q = q_{K,G} : G \to \mathbb{R}_{+}^{*} \) is a smooth function on \( G \) satisfying \( q(e) = 1, q(kg) = \Delta_{K,G}(k)q(g) \).

The space \( K \setminus G \) carries a relatively invariant measure \( d\gamma \) with modulus \( q^{-1} \) which satisfies:

\[
\int_{K \setminus G} f(\gamma g) d\gamma = \int_{K \setminus G} f(\gamma) q(\gamma^{-1}) d\gamma
\]

where \( f \in C_c(K \setminus G) \).

The Hilbert space \( H_{\pi_{\chi}} = L^{2}(G, K, \chi) \) is the completion of \( C_{K}^{\infty}(G, K, \chi) \) under the norm \( \|f\| = (\int_{K \setminus G} |f(\gamma)|^{2} d\gamma)^{1/2} \).

Now let \( \pi \) be a unitary representation of \( G \) on a Hilbert space \( H_{\pi} \), we denote by \( H_{\pi}^{\infty} \) the Fréchet space of smooth vectors of \( \pi \). Its anti-dual space is denoted by \( H_{\pi}^{\infty} \). It is well known that \( \pi(D(G))H_{\pi}^{\infty} \subset H_{\pi}^{\infty} \) where \( D(G) = C_{c}^{\infty}(G) \).

2.1.2 Algebraic structure

Let \( \mathfrak{g} = \mathfrak{n} + \mathfrak{h} \) where \( \mathfrak{n} \) is nilpotent, \( [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \) and where \( \mathfrak{h} \) is an abelian subalgebra of \( \mathfrak{g} \) such that \( ad(\mathfrak{h}) \) consists of semi-simple endomorphisms with real eigenvalues.

In [Cu.2] it has been shown that if \( \tau_{f} \) is of finite multiplicity then the Lie algebra \( \mathfrak{g} \) has a basis \( \mathcal{B} = \{ C_{1}, \cdots, C_{a}, V_{1}, \cdots, V_{\nu}, X_{1}, \cdots, X_{u}, Y_{1}, \cdots, Y_{u}, A_{1}, \cdots, A_{u}, B_{1}, \cdots, B_{\nu} \} \) such that

\[ \mathfrak{n} = \text{vect} < C_{1}, \cdots, C_{a}, V_{1}, \cdots, V_{\nu}, X_{1}, \cdots, X_{u}, Y_{1}, \cdots, Y_{u} > \]

and \( \mathfrak{h} = \text{vect} < A_{1}, \cdots, A_{u}, B_{1}, \cdots, B_{\nu} > \). Furthermore we have:

i) \( [X_{h}, Y_{h'}] = 0 \) if and only if \( h \neq h' \) and \( [X_{h}, Y_{h}] \) is central in \( \mathfrak{n} \) for \( 1 \leq h \leq u \).

ii) For every \( h, h' \) \( [X_{h}, X_{h'}] = [Y_{h}, Y_{h'}] = 0 \).

iii) \( \text{cent}(\mathfrak{g}) = \text{vect} < C_{1}, \cdots, C_{a} > \), and \( \text{cent}(\mathfrak{n}) = \text{vect} < C_{1}, \cdots, C_{a}, V_{1}, \cdots, V_{\nu} > \).

iv) \( [A_{h}, X_{h}] = -X_{h}; [A_{h}, Y_{h}] = Y_{h}; [A_{h}, X_{h'}] = [A_{h}, Y_{h'}] = 0 \) for \( h \neq h' \).

v) \( [B_{k}, X_{h}] = \alpha_{k,h}X_{h}, \ \alpha_{k,h} \in \mathbb{R}; [B_{k}, Y_{h}] = 0; [A_{h}, V_{k}] = 0; [B_{k}, V_{k}] = V_{k} \) and \( [B_{k}, V_{k'}] = 0 \) for \( k \neq k' \).

(see Theorem 1.8 in [Cu.2]), we have simplified here the notations of Currey).
2.1.3 Plancherel formula

Let $\tau$ be the monomial representation: $\tau = \tau_f = \text{Ind}_H^G \chi_f$. To decompose $\tau$ means to describe the spectrum of $\tau$, the multiplicities and the equivalence class of the Plancherel measure in terms of the coadjoint orbit picture.

In the case of a completely solvable Lie group, it has been shown in [Li.1] that the spectral decomposition formula is given by $\tau = \int_{(f+\mathfrak{h}^\perp)/H} \pi_\theta d\nu(\theta)$ where $\nu$ is a pushforward of a finite measure on $(f+\mathfrak{h}^\perp)$ which is equivalent to Lebesgue measure.

In the case with which we are concerned where $G = NH$ and $\tau_f$ has finite multiplicity, it has been shown in [Cu.2] that the set of generic $H$–orbits in the decomposition of $\tau_f$ admits a natural algebraic cross-section $\Sigma$ and the measure $\nu$ is given as an explicit measure on $\Sigma$.

Furthermore we can choose $f|n = 0$.

The cross-section in $f + \mathfrak{h}^\perp$ is $f + \Sigma$ and is given as follows:

Fixing a choice of signs $\theta = (\epsilon, \delta) = (\epsilon_1, \ldots, \epsilon_u, \delta_1, \ldots, \delta_v) \in \{1, -1\}^d$; $d = u + v$, one has $\Sigma = \bigcup_{\theta \in \{1, -1\}^d} \Sigma_\theta$ where $\Sigma_\theta = \{ l \in \Omega \cap \mathfrak{h}^\perp; \ l(Y_k) = \epsilon_k, \ 1 \leq k \leq u \text{ and } l(V_i) = \delta_i, \ 1 \leq i \leq v \}$. Here $\Omega = \Omega_0 \cap \Omega_1$, where $\Omega_0$ is the set of $G$–orbits having maximal dimension in $\mathfrak{g}^*$ and $\Omega_1$ consists with $H$–orbits of maximal dimension. The irreducible representations which correspond to $G$–orbits $G \cdot l$, $l \in \Omega \cap (f+\mathfrak{h}^\perp)$, are sufficient to decompose $\tau_f$.

There exists a dense open subset $D_\theta$ of $\mathbb{R}^a \times \mathbb{R}^u$ such that

$$\Sigma_\theta = \{ \sum_{h=1}^a \xi_h C_h^* + \sum_{i=1}^\nu \delta_i V_i^* + \sum_{k=1}^u \epsilon_k Y_k^* + \sum_{k=1}^u \mu_k X_k^*; \ (\xi, \mu) \in D_\theta \} \quad (7)$$

(see [Cu.2], we have made a small change of notations).

Let $F$ be a function on $f + \mathfrak{h}^\perp$. One has

$$\int_{f+\Sigma} F(l) dl = \sum_{\theta \in \{1, -1\}^d} \int_{\mathbb{R}^a \times \mathbb{R}^u} F(f + a \sum_{h=1}^a \xi_h C_h^* + \sum_{i=1}^\nu \delta_i V_i^* + \sum_{k=1}^u \epsilon_k Y_k^* + \sum_{k=1}^u \mu_k X_k^*) \ d\xi d\mu.$$

Now for $l \in \Sigma$, an $H$–covariant generalized vector for $\pi_l$ is defined formally by; for $\psi \in \mathcal{H}_l^{\infty}$

$$\beta_l(\psi) = \int_H \overline{\psi(h)q_{B,G}^{1/2}q_{H,G}^{-1/2} \chi_f(h)} dh,$$

(see 2.1 in [Cu.2]).

2.1.3.1. Theorem [Cu.2] The integral (8) is absolutely convergent for every $\psi \in \mathcal{H}_l^{\infty}$ and $\beta_l$ is continuous on $\mathcal{H}_l^{N}$ for a certain integer $N$ (see [Cu.2] proof of theorem 2.2).
The distribution-theoretic Plancherel formula which is equivalent to the disintegration of $\tau_f$ is

$$\langle \tau_f(\omega)\alpha_\tau, \alpha_\tau \rangle = \int_{f+\Sigma} \langle \pi(\omega)\beta, \beta \rangle |R(l)| dl$$

where $R(l) = ((2\pi)^n l([X_1, Y_1])l([X_2, Y_2])\cdots l([X_n, Y_n]))^{-1}$ with $n = \dim(\mathfrak{n})$ and $\alpha_\tau$ is the generalized cyclic vector for $\tau$: $\alpha_\tau(\xi) = \xi(1)$ for $\xi \in \mathcal{H}_\infty^\pi$ (cf. [Cu.2] Theorem 3.2).

Of course the reference [Cu.2] contains more information than is conveyed here.

### 2.2 The Bonnet Plancherel formula

The aim of this section is to describe explicitly the Bonnet Plancherel Formula associated to the disintegration of $\tau_f$. Let $G, H, f$ (and so on) be as above. We recall that the distribution $S_{H,X_l}$, defined on $D(G)$ by: $\langle S_{H,X_l}, \varphi \rangle = \int_H \varphi(h)\chi_f(h)\Delta^{1/2}_{G,H}(h) dh$, is positive.

By the theorem of P. Bonnet [Bon.], there exist positive nuclear operators $U_{\varphi} : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty$, such that

$$\langle S_{H,X_l}, \varphi \rangle = \int_G \text{tr}(\pi(\varphi)U_{\varphi}(\pi))d\mu(\pi), \quad \varphi \in D(G).$$

We shall show that the operators $U_{\varphi}$ are finite sum of rank one operators. The first step is a determination of a cross-section for $G$-orbits in $G.(f + \Sigma)$.

Let $l = f + l_0 \in f + \Sigma$. By (2.1.3) there exists $\theta = (\epsilon, \delta) = (\epsilon_1, \cdots, \epsilon_u, \delta_1, \cdots, \delta_v) \in \{-1, 1\}^d$ such that $l_0 \in \Sigma_\theta$: $l_0 = \sum_{h=1}^u \xi_h C_h^* + \sum_{i=1}^v \delta_i V_i^* + \sum_{k=1}^u \epsilon_k Y_k^* + \sum_{k=1}^u \mu_k X_k^*$; the $G$-orbit of $l$ consists of elements $l'$ of the form:

$$l' = \sum_{h=1}^u \xi_h C_h^* + \sum_{i=1}^v \delta_i w_i V_i^* + \sum_{k=1}^u y_k Y_k^* + \sum_{k=1}^v x_k X_k^* + \sum_{k=1}^u P_k(w_i, x_k, y_k)A_k^* + \sum_{i=1}^v b_i B_i^*$$

where $w_i \in [0, +\infty]$ \quad $1 \leq i \leq \nu$, \quad $x_k, y_k, b_i \in \mathbb{R}$ and $P_k$ are polynomials in $x_k, y_k$ and rationals in $w_i$, \quad $1 \leq k \leq u$.

It has been shown in [Cu.2] that

$$O_l = G.l \cap (f + \Sigma) = \bigcup_{\epsilon' \in \{-1, 1\}^u} \{ \sum_{h=1}^u \xi_h C_h^* + \sum_{i=1}^v \delta_i V_i^* + \sum_{k=1}^u \epsilon_k Y_k^* + \sum_{k=1}^u \epsilon_k \epsilon'_k \mu_k X_k^* \}\{ \sum_{i=1}^v \delta_i \epsilon_i w_i V_i^* \}.$$ (9)
We give a cross-section for $G$- orbits in $G.(f + \Sigma)$ as the set

$$
\Gamma = \left\{ f + \sum_{h=1}^{a} \xi_h G_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} Y_k^* + \sum_{k=1}^{u} \mu_k X_k^*, \quad (\xi_h, \mu_k) \in \mathbb{R}^a \times \mathbb{R}^u, \right. \\
\text{and} \quad \delta = (\delta_1, \cdots, \delta_\nu) \in \{-1, 1\}^\nu \right\} \\
= \bigcup_{\delta \in \{-1, 1\}^\nu} \Gamma_\delta.
$$

We see that our cross-section $\Gamma$ for $G$- orbits in $G.(f + \Sigma)$ is contained in $f + \Sigma$.

Furthermore we decompose the Lebesgue measure on $f + \Sigma$ into integral of measures on $\Omega, l \in \Gamma$: Given a function $F$ on $(f + \Sigma)$ we write:

$$
\int_{f+\Sigma} F(l) dl = \int_{f+\Sigma} F(\phi) d\mu_\sigma(\phi) d\nu(\sigma) \quad (10)
$$

$$
= \sum_{\delta \in \{-1, 1\}^\nu} \int_{\mathbb{R}^a \times \mathbb{R}^u} \sum_{\epsilon \in \{-1, 1\}^u} F(f + \sum_{h=1}^{a} \xi_h G_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} \epsilon_k Y_k^* + \sum_{k=1}^{u} \epsilon_k \mu_k X_k^*) \, d\xi. \\
$$

On the other hand recall that for all $\omega \in D(G)$ we have by (Cu.2):

$$
\langle \tau_f(\omega) \alpha_\tau, \alpha_\tau \rangle = \int_{f+\Sigma} \langle \tau_i(\omega) \beta_i, \beta_i \rangle |R(l)| dl,
$$

where $R(l) = ((2\pi)^u \prod_{k=1}^{u} l([X_k,Y_k]))^{-1}$.

**Remarks**

i) From the construction of vectors $X_k, Y_k$ one can verify that $l([X_k,Y_k]) \neq 0$ for all $l \in \Omega$.

ii) Since for all $1 \leq k \leq u, \quad [X_k,Y_k] \in cent(n)$ then for every $\sigma \in \Gamma$ by (9) we have $R(\sigma) = R(l) \quad \forall l \in G \cdot \sigma \cap (f + \Sigma)$. Thus we can write $R(l) = R(f, \delta, \xi)$ as a function uniquely depending on $f, \delta = (\delta_1, \cdots, \delta_\nu)$ and $\xi = (\xi_1, \cdots, \xi_a)$.

Let us write $\pi(\xi,\delta,\epsilon,\mu)$ for the irreducible representation associated to the element

$$
l = l(\xi, \delta, \epsilon, \mu) = f + \sum_{h=1}^{a} \xi_h G_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} \epsilon_k Y_k^* + \sum_{k=1}^{u} \mu_k X_k^* \quad \text{in} \quad g^*.
$$

We deduce that:

$$
\langle \tau_f(\omega) \alpha_\tau, \alpha_\tau \rangle = \sum_{\delta \in \{-1, 1\}^\nu} \int_{\mathbb{R}^a \times \mathbb{R}^u} \sum_{\epsilon \in \{-1, 1\}^u} \langle \pi(\xi,\delta,\epsilon,\mu)(\omega) \beta_{(\xi,\delta,\epsilon,\mu)}, \beta_{(\xi,\delta,\epsilon,\mu)} \rangle |R(f, \delta, \xi)| d\xi. \\
$$

(11)

Let now $\sigma = f + \sum_{h=1}^{a} \xi_h G_h^* + \sum_{i=1}^{\nu} \delta_i V_i^* + \sum_{k=1}^{u} Y_k^* + \sum_{k=1}^{u} \mu_k X_k^* \in \Gamma \subset (f + \Sigma)$.
For every $l \in G \cdot \sigma \cap (f + \Sigma)$ there exists by (9) an $\epsilon \in \{-1, 1\}^u$ such that:

$$l = f + \sum_{h=1}^{a} \xi_h C_h^* + \sum_{i=1}^{\nu'} \delta_i V_i^* + \sum_{k=1}^{u} \epsilon_k Y_k^* + \sum_{k=1}^{u} \epsilon_k \mu_k Y_k^*.$$ 

Put for $1 \leq k \leq u$:

$$a_k(\sigma) = \langle \sigma, [X_k, Y_k] \rangle.$$ 

Since $[X_k, Y_k] \in \text{cent}(n)$, we have that $a_k(\sigma) = a_k(l)$. Then by the obvious remark (i) one has $a_k(\sigma) \neq 0$.

Let $g_l = \prod_{k=1}^{u} \exp(y_k Y_k) \prod_{k=1}^{u} \exp(x_k X_k) \prod_{h=1}^{\nu'} \exp(v_h V_h) \in N$, where $x_k = \frac{1 - \epsilon_k}{\alpha_k(l)}$, $y_k = \frac{\epsilon_k - 1}{\alpha_k(l)} \mu_k$, and $v_h = -\delta_h \sum_{k=1}^{u} \frac{1 - \epsilon_k}{\alpha_k(l)} \alpha_{h,k} \mu_k$.

**2.2.1 Lemma.** We have that:

$$l = g_l \cdot \sigma$$

**Proof.** We recall that

$$g = \text{vect} < C_1, \cdots, C_u, V_1, \cdots, V_\nu, X_1, \cdots, X_u, Y_1, \cdots, Y_\nu, A_1, \cdots, A_u, B_1, \cdots, B_\nu > .$$

According to the expressions of $\sigma$, $l$ and since the vectors $C_h$ and $V_i$ are central in $n$ we have $g_l \cdot \sigma(C_h) = l(C_h), \ 1 \leq h \leq u$, and $g_l \cdot \sigma(V_i) = l(V_i), \ 1 \leq i \leq \nu$.

Fix $s \in \{1, \cdots, \nu\}$, we have by (2.1.2.v) and the fact that $f_{\alpha} = 0$

$$g_l \cdot \sigma(B_s) = \sigma(Ad(\prod_{h=1}^{\nu'} \exp(-v_h V_h) \prod_{k=1}^{u} \exp(-x_k X_k))(B_s))$$

$$= \sigma(Ad(\prod_{h=1}^{\nu'} \exp(-v_h V_h))(B_s + \sum_{k=1}^{u} x_k \alpha_{s,k} X_k))$$

$$= \sigma(B_s + v_s V_s + \sum_{k=1}^{u} x_k \alpha_{s,k} X_k)$$

$$= \sigma(B_s) + \delta_s v_s + \sum_{k=1}^{u} x_k \alpha_{s,k} \mu_k$$

$$= \sigma(B_s) - \sum_{k=1}^{u} x_k \alpha_{s,k} \mu_k + \sum_{k=1}^{u} x_k \alpha_{s,k} \mu_k$$

$$= \sigma(B_s) = l(B_s) = f(B_s).$$
For $1 \leq i \leq u$, we have by (2.1.2.v), (2.1.2.iv), (2.1.2.ii) and by the fact that $f_{ij} = 0$:

$$g_l \cdot \sigma(A_i) = \sigma(Ad(\prod_{k=1}^{u} \exp(-x_k X_k))(A_i + y_i Y_i))$$
$$= \sigma(A_i + y_i Y_i - x_i X_i - x_i y_i [X_i, Y_i])$$
$$= \sigma(A_i) + y_i - x_i \mu_i - x_i y_i \sigma(a)$$
$$= \sigma(A_i) + \frac{\epsilon_i - 1}{a_i(\sigma)} \mu_i + \frac{\epsilon_i - 1}{a_i(\sigma)} \mu_i + \frac{(\epsilon_i - 1)^2}{a_i(\sigma)} \mu_i$$
$$= \sigma(A_i) + \frac{\mu_i}{a_i(\sigma)} (2\epsilon_i - 2 + (\epsilon_i)^2 - 2\epsilon_i)$$
$$= \sigma(A_i) + \epsilon_i \mu_i$$
$$= l(A_i),$$

and

$$g_l \cdot \sigma(X_i) = \sigma(Ad(\prod_{k=1}^{u} \exp(-x_k X_k))(X_i - y_i [X_i, X_i]))$$
$$= \sigma(X_i + y_i [X_i, Y_i])$$
$$= \sigma(X_i) + (\epsilon_i) \mu_i$$
$$= \epsilon_i \mu_i$$
$$= l(X_i)$$

Thus $g_l \cdot \sigma = l$.

We turn now to Bonnet’s operators. First we define for every $l \in G \cdot \sigma \cap (f + \Sigma)$ an operator $\beta_l' : \mathcal{H}_l^\infty \to \mathbb{C}$ by

$$\beta_l'(\psi) = \int_{H} \frac{\psi(g_l^{-1} h)}{q_{H, G}^{1/2} q_{H, G}^{-1/2} \chi_f(h)} dh$$

and a function $\psi_{g_l}$ by $\psi_{g_l}(g') = \psi(g_l^{-1} g')$, $g' \in G$. We can see that $\psi_{g_l}$ is an element of $\mathcal{H}_l^\infty$. Indeed, the covariance condition is satisfied.

Let $B(l)$ be the Vergne polarization associated to $l$ and to our Jordan-Hölder basis of $g$. For $g' \in G, b \in B(l)$ we have $\psi_{g_l}(bg') = \psi(g_l^{-1} bg') = \psi(g_l^{-1} bg_l g_l^{-1} g').$ Since
$l = g_l \cdot \sigma$, we have that then $B(l) = g_l B(\sigma) g_l^{-1}$ and $b' = g_l^{-1} b g_l \in B(\sigma)$. Hence

$$
\psi_{g_l}(b') = \psi(b' g_l^{-1}) \\
= \chi_{\sigma}(b') \psi(g_l^{-1}) \quad (\psi \in H^\infty_{\sigma}) \\
= \chi_{\sigma}(b') \psi_{g_l}(g').
$$

Evidently $\psi_{g_l}$ is $C^\infty$ function. We obtain $\beta'_l(\psi) = \beta_l(\psi_{g_l})$ where $\beta_l$ is as in (8).

Then using (2.1.3.1) we have that (12) converges for all $\psi \in H^\infty_{\sigma}$ and $\beta'_l \in H^\infty_{\sigma}$.

Let $\sigma \in \Gamma, l \in G \cdot \sigma \cap (f + \Sigma)$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_u) \in \{-1, 1\}^u$ such that $\epsilon_k = l(Y_k)$. Since $l$ depends only on $\epsilon$ we put $\beta'_l = \beta'_\epsilon$ and we define the operator $U_{\sigma} : H^\infty_{\sigma} \rightarrow H^\infty_{\sigma}$ by:

$$
U_{\sigma} = \sum_{\epsilon \in \{-1, 1\}^u} P_{\beta'_\epsilon, \beta'_l}.
$$

Here $P_{\beta'_\epsilon, \beta'_l} : H^\infty_{\sigma} \rightarrow H^-_{\sigma}$ is a rank one operator defined by $P_{\beta'_\epsilon, \beta'_l}(\psi) = \langle \psi, \beta'_\epsilon \rangle \beta'_l$.

We have the following:

**2.2.2 Theorem** Let $G = \exp(g)$ be the semi direct product; where $N = \exp(n)$ is nilpotent and normal in $G$, and $H = \exp(h)$ is abelian and acts semi-simply on $N$ with real eigenvalues. Let $f$ be a linear functional of $g$ such that $f([h, h]) = \{0\}$ and $\chi_f$ the corresponding unitary character of $H$. Let $\tau_f = Ind^G_H \chi_f$ and assume that $\tau_f$ has finite multiplicity. Let $\Sigma \subset g^*$ be the cross-section for the $H$-orbit in $\Omega \cap h^\perp$ given in [Cu.2]. Then there exists a cross-section $\Gamma$ for the $G$-orbit in $G \cdot (f + \Sigma)$, a measure $\nu$ on $\Gamma$, such that for every $\omega \in D(G)$ we have:

$$
\langle \tau_f(\omega) \rho_\tau, \rho_\tau \rangle = \int_{\Gamma} \text{tr}(\pi_{\sigma}(\omega) \circ U_{\sigma}) d\nu(\sigma)
$$

where $U_{\sigma}, \sigma \in \Gamma$, is defined in (13).

**Proof.** Let $\omega \in D(G)$. We have $\pi_{\sigma}(\omega) \circ U_{\sigma} = \sum_{\epsilon \in \{-1, 1\}^u} P_{\pi_{\sigma}(\omega) \circ \beta'_\epsilon, \beta'_l}$. Hence

$$
\text{tr}(\pi_{\sigma}(\omega) \circ U_{\sigma}) = \sum_{\epsilon \in \{-1, 1\}^u} \langle \pi_{\sigma}(\omega) \beta'_\epsilon, \beta'_l \rangle.
$$

On the other hand, for all $\psi \in H^\infty_{\sigma}$, we have:

$$
\langle \pi_{\sigma}(\omega) \beta'_l, \psi \rangle = \langle \beta'_l, \pi_{\sigma}(\omega^*) \psi \rangle = \beta_l((\pi_{\sigma}(\omega^*) \psi)_{g_l}); \text{ where } l = g_l \cdot \sigma.
$$
Since for all \( x \in G \)

\[
(\pi_\sigma(\omega^*)\psi)_{\eta_1}(x) = \pi_\sigma(\omega^*)\psi(g_1^{-1}x)
\]

\[
= \int_G \omega^*(y)(\pi_\sigma(y)\psi)(g_1^{-1}x)dy
\]

\[
= \int_G \omega^*(y)\psi(g_1^{-1}xy)q(y)^{1/2}dy
\]

\[
= \pi_l(\omega^*)\psi_{\eta_1}(x),
\]

it follows that \((\pi_\sigma(\omega^*)\psi)_{\eta_1} = \pi_l(\omega^*)\psi_{\eta_1}\).

Thus

\[
\langle (\pi_\sigma(\omega^*)\beta')_{\eta_1}, \psi_{\eta_1} \rangle_{\mathcal{H}_t} = \langle \pi_\sigma(\omega^*)\psi \rangle_{\mathcal{H}_t} = \langle \beta', \pi_l(\omega^*)\psi \rangle_{\mathcal{H}_t} = \langle \pi_l(\omega)\beta', \psi_{\eta_1} \rangle_{\mathcal{H}_t}.
\]

Hence \(\pi_l(\omega)\beta_l = (\pi_\sigma(\omega)\beta'_l)_{\eta_1}\) and \((\pi_\sigma(\omega)\beta'_l, \beta'_l) = (\pi_\sigma(\omega)\beta'_l, \beta_l) = (\pi_\epsilon(\omega)\beta_\epsilon, \beta_\epsilon)\).

We deduce that

\[
\text{tr}(\pi_\sigma(\omega)U_\sigma) = \sum_{\epsilon \in \{-1, 1\}^n} \langle \pi_\epsilon(\omega)\beta_\epsilon, \beta_\epsilon \rangle.
\]

The formulas (10) and (11) permit us to conclude, the measure \(\nu\) is given on each \(\Gamma_\delta\) by: \(|R(f, \delta, \xi)|d\xi d\mu\). ■

2.3 Exemple ([Cu.2])

Let \(g = \text{vect} < B, A, X, Y, Z >\) with non vanishing brackets

\[
\]

Here \(\mathfrak{h} = \text{vect} < A, B >\) and \(\mathfrak{n} = \text{vect} < X, Y, Z >\).

For \(l \in \mathfrak{g}^*\) we write \(l = (\lambda, \gamma, \mu, \alpha, \theta)\) where \(\lambda = l(Z); \gamma = l(Y); \mu = l(X); \alpha = l(A); \theta = l(B); \Omega_0 = \{l \in \mathfrak{g}^*; \lambda \neq 0\}\) and \(\Omega_1 = \{l \in \mathfrak{g}^*; \gamma \neq 0\}\) and the set \(\Omega\) of generic linear functionals is \(\Omega = \Omega_0 \cap \Omega_1\).

The cross-section for \(H\)-orbits in \(\mathfrak{h}^\perp \cap \Omega\) is given in [Cu.2] as:

\[
\Sigma = \{(\delta, \epsilon, \mu, 0, 0); \ \mu \in \mathbb{R}; \ (\epsilon, \delta) \in \{-1, 1\}^2\} = \cup \Sigma_\theta.
\]

Now the cross-section for \(G\)-orbits in \(G \cdot \Sigma\) is: \(\Gamma = \cup_{\delta \in \{-1, 1\}} \Gamma_\delta\) where
\[ \Gamma = \{ (\delta, 1, \mu, 0, 0) ; \mu \in \mathbb{R}, \delta \in \{-1, 1\} \}. \]

Let \( \sigma \in \Gamma \); there exits \( \delta \in \{-1, 1\} \) such that \( \sigma = (\delta, 1, \mu, 0, 0) \). The theorem (2.2.2) says that the Bonnet Plancherel measure is given on each \( \Gamma_\delta \) by \((2\pi)^{-3}d\mu\).

For \( l \in G \cdot \sigma \) \( \exists \epsilon = l(Y) \) such that \( l = g_l \cdot \sigma \), here we have: \( V = Z \); and since \([B, X] = X\) then for \( \epsilon = -1 \)
\[ g_l = \exp(-2\mu Y)\exp(\frac{2}{\delta}X)\exp(-2\mu \frac{\delta}{\delta^2}Z). \]

The operator \( \beta_l \) is given in [Cu.2]:
\[ \beta_l(\psi) = \int_{\mathbb{R}^2} \psi(\exp(sB)\exp(tA))e^{s}e^{(t-s)\frac{\epsilon}{2}} dsdt. \]

Thus the formula for the operator \( \beta'_l \) is:
\[ \beta'_l(\psi) = \beta'_1(\psi) = \int_{\mathbb{R}^2} \psi(g_l^{-1}\exp(sB)\exp(tA))e^{s}e^{(t-s)\frac{\epsilon}{2}} dsdt = \beta_1(\psi_{g_l}). \]

Then Bonnet’s operator \( U_\sigma \) is given by
\[ U_\sigma = \sum_{\epsilon_1 \in \{-1, 1\}} P_{\beta'_1, \beta'_1} \] where \( P_{\beta'_1, \beta'_1}(\psi) = \langle \psi, \beta'_1 \rangle \beta'_1 \).

Furthermore for \( \epsilon = 1, \beta'_1 = \beta_\sigma \), then
\[ U_\sigma = P_{\beta'_1, \beta'_1} + P_{\beta_1, \beta_1}. \]

Now for \( \omega \in D(G) \) we have: \( \pi_\sigma(\omega) \circ U_\sigma = P_{\pi(\delta, 1, \mu)(\omega)\beta'_1, \beta'_1} + P_{\pi(\delta, 1, \mu)(\omega)\beta_1, \beta_1}. \)

Then:
\[ \operatorname{tr}(\pi_\sigma(\omega) \circ U_\sigma) = \langle \pi(\delta, 1, \mu)(\omega)\beta'_1, \beta'_1 \rangle + \langle \pi(\delta, 1, \mu)(\omega) \circ \beta_1, \beta_1 \rangle \]
By theorem (2.2.2) we have the Bonnet Plancherel formula:
\[ \langle \tau_f(\omega)\alpha_\tau, \alpha_\tau \rangle = (2\pi)^{-3} \sum_{\delta \in \{-1, 1\}} \int_{\mathbb{R}} \operatorname{tr}(\pi(\delta, 1, \mu)(\omega)U(\delta, 1, \mu))d\mu. \]

References


The Bonnet Plancherel formula for monomial representations . . .


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