Singular masas of von Neumann algebras: examples from the geometry of spaces of nonpositive curvature[1]

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Abstract

If Γ is a group, then the von Neumann algebra $VN(\Gamma)$ is the convolution algebra of functions $f \in \ell_2(\Gamma)$ which act by convolution on $\ell_2(\Gamma)$ as bounded operators. Maximal abelian $\star$-subalgebras (masas) of von Neumann algebras have been studied from the early days.

If Γ is a torsion free cocompact lattice in a semisimple Lie group G of rank r with no centre and no compact factors then the geometry of the symmetric space $X = G/K$ may be used to define and study masas of $VN(\Gamma)$. These masas are of the form $\mathrm{VN}(\Gamma_0)$, where Γ_0 is the period group of some Γ-periodic maximal flat in X. There is a similar construction if Γ is a lattice in a p-adic Lie group G, acting on its Bruhat-Tits building.

Consider the compact locally symmetric space $M = \Gamma\backslash X$. Assume that T^r is a totally geodesic flat torus in M and let $\Gamma_0 \cong \mathbb{Z}^r$ be the image of the fundamental group $\pi(T^r)$ under the natural monomorphism from $\pi(T^r)$ into $\Gamma = \pi(M)$. Then $\mathrm{VN}(\Gamma_0)$ is a masa of $\mathrm{VN}(\Gamma)$. If in addition $\mathrm{diam}(T^r)$ is less than the length of a shortest closed geodesic in M then $\mathrm{VN}(\Gamma_0)$ is a *singular masa* : its unitary normalizer is as small as possible. This last result is joint work with A. M. Sinclair and R. R. Smith [RSS].

1 Background

Let Γ be an ICC group: each element in Γ other than the identity has infinite conjugacy class. The group von Neumann algebra is the convolution algebra

$$\mathrm{VN}(\Gamma) = \{f \in \ell^2(\Gamma) : g \mapsto f \star g \ \text{ is in } \ B(\ell^2(\Gamma))\}.$$

It is well known that $\mathrm{VN}(\Gamma)$ is a **factor of type** II_1. This means

(a) $\mathrm{VN}(\Gamma)$ is a strongly closed $\star$-subalgebra of $B(\ell^2(\Gamma))$, with trivial centre;

(b) there is a faithful trace on $\mathrm{VN}(\Gamma)$ defined by $\mathrm{tr}(f) := f(1)$.

The group Γ may be embedded as a subgroup of the unitary group of $\mathrm{VN}(\Gamma)$ by identifying an element $\gamma \in \Gamma$ with the corresponding delta function δ_γ. A major result of A. Connes [Co, Corollary 3] implies:

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Theorem. (A. Connes) If Γ_1,Γ_2 are countable amenable ICC groups then $\mathrm{VN}(\Gamma_1) \cong \mathrm{VN}(\Gamma_2)$, the algebra being isomorphic to the hyperfinite II_1 factor.

At the opposite extreme from amenable groups there is the
Rigidity Conjecture (A. Connes) If ICC groups Γ_1,Γ_2 have Property (T) of Kazhdan, then

$$\mathrm{VN}(\Gamma_1) \cong \mathrm{VN}(\Gamma_2) \Rightarrow \Gamma_1 \cong \Gamma_2.$$

Compare this with the

Rigidity Theorem (Mostow-Margulis-Prasad) For $i = 1, 2$, let Γ_i be a lattice in G_i, a connected non-compact simple Lie group with trivial centre, $G_1 \neq \mathrm{PSL}_2(\mathbb{R})$. Then

$$\Gamma_1 \cong \Gamma_2 \Rightarrow G_1 \cong G_2.$$

In Mostow's proof of rigidity ([Mo]: the cocompact, higher rank case), maximal flats of symmetric spaces play an important role. There is some reason to hope that masas of von Neumann algebras might play a similar role for Connes' conjecture.

2 Maximal abelian $\star$-subalgebras

Let $\mathcal{A}$ be a maximal abelian $\star$-subalgebra (masa) of $\mathrm{VN}(\Gamma)$. Say that $\mathcal{A}$ is a **singular masa** if :

$$u \in \mathrm{VN}(\Gamma),\ u \text{ unitary},\ u\mathcal{A}u^* = \mathcal{A} \Rightarrow u \in \mathcal{A}.$$

Singular masas [2] always exist [P1], but are hard to construct explicitly.

If $\mathcal{A} = \mathrm{VN}(\Gamma_0)$, where Γ_0 is a subgroup of Γ, then
$\mathrm{VN}(\Gamma_0)$ embeds as a subalgebra of $\mathrm{VN}(\Gamma)$ via $f \mapsto \overline{f}$, where

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let $\Gamma_1 < \Gamma_0 < \Gamma$, with Γ_0 abelian. Define the commutant $\mathrm{VN}(\Gamma_1)'$ to be the centralizer of $\mathrm{VN}(\Gamma_1)$ in $\mathrm{VN}(\Gamma_0)$. Suppose that, for all $x \notin \Gamma_0$, the set*

$$A_x = \{x_1^{-1} x x_1 : x_1 \in \Gamma_1\}$$

is infinite. Then $\mathrm{VN}(\Gamma_1)' = \mathrm{VN}(\Gamma_0)$. In particular, $\mathrm{VN}(\Gamma_0)$ is a masa of $\mathrm{VN}(\Gamma)$.

(This result is contained in [Di], in the case $\Gamma_1 = \Gamma_0$.)

[2] If the unitary normalizer of $\mathcal{A}$ generates $\mathrm{VN}(\Gamma)$ then $\mathcal{A}$ is a *Cartan* masa. $\mathrm{VN}(\Gamma)$ may not contain a Cartan masa: e.g. $\Gamma = \mathbb{F}_2$. S. Popa [P2] has recently used Cartan masas to construct isomorphism invariants for certain II_1 factors.

Proof. Let $f \in \mathrm{VN}(\Gamma_1)'$ and $x \notin \Gamma_0$.
Then $\delta_{x_1^{-1}} * f * \delta_{x_1} = f$ (for all $x_1 \in \Gamma_1$)
$\Rightarrow f$ is constant on A_x
$\Rightarrow f = 0$ on A_x (since $f \in \ell^2(\Gamma)$ and $\# A_x = \infty$)
$\Rightarrow f(x) = 0$ (for all $x \notin \Gamma_0$)
$\Rightarrow f \in \mathrm{VN}(\Gamma_0)$. □

There is a **conditional expectation** $\mathbb{E}_{\mathcal{A}} : \mathrm{VN}(\Gamma) \to \mathcal{A}$ onto any masa $\mathcal{A}$, which extends to an orthogonal projection on $\ell^2(\Gamma)$. If $\mathcal{A} = \mathrm{VN}(\Gamma_0)$, where Γ_0 is an abelian subgroup of Γ and if $f \in \mathrm{VN}(\Gamma)$, then

$$\mathbb{E}_{\mathcal{A}} f(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition.[SS] Say that $\mathcal{A}$ is a **strongly singular masa** of $\mathrm{VN}(\Gamma)$ if

$$\|\mathbb{E}_{u\mathcal{A}u^*} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} \geq \|u - \mathbb{E}_{\mathcal{A}}(u)\|_2$$

for all unitaries $u \in \mathrm{VN}(\Gamma)$. [Here $\| \cdot \|_{\infty,2}$ means: operator norm on domain, ℓ^2 norm on range.]

This condition implies that any unitary $u \in \mathrm{VN}(\Gamma)$ which normalizes $\mathcal{A}$ necessarily lies in $\mathcal{A}$. Therefore $\mathcal{A}$ is a singular masa.

2.1 Construction of masas

Let Γ be an ICC group and let Γ_0 be an abelian subgroup. Here is a condition that ensures that $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$.

(SS) If $x_1, \dots, x_m, y_1, \dots, y_n \in \Gamma$ and

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 y_j \,, \tag{2.1}$$

then some $x_i \in \Gamma_0$.

Theorem. *Condition* (SS) *implies that* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

The proof of this result is contained in [RSS]. It can be used to construct strongly singular masas of $\mathrm{VN}(\Gamma)$, for certain geometrically defined groups Γ, acting on spaces of nonpositive curvature.

Let G be a semisimple Lie group of rank r with no centre and no compact factors. Let Γ be a torsion free cocompact lattice in G. Then Γ acts freely on the symmetric space $X = G/K$ and the quotient manifold $M = \Gamma\backslash X$ is a compact locally symmetric space, with fundamental group $\pi(M) = \Gamma$.

Let $T^r \subset M$ be a totally geodesic embedding of a flat r-torus in M. The inclusion $i : T^r \to M$ induces an injective homomorphism $i_* : \pi(T^r) \to \pi(M)$. (Reason: no geodesic loop in M can be null-homotopic.)

Let $\Gamma_0 = i_*\pi(T^r) \cong \mathbb{Z}^r < \Gamma$. Under these assumptions, the following results hold.

Theorem A. $\mathrm{VN}(\Gamma_0)$ *is a masa of* $\mathrm{VN}(\Gamma)$.

Theorem B. [RSS] *Let σ be the length of a shortest closed geodesic in M. If* $\mathrm{diam}(T^r) < \sigma$ *then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

3 Proofs

Theorem A is a consequence of a stronger result. Recall that a geodesic L in X is *regular* if it lies in only one maximal flat. See the appendix below for further details. A regular geodesic in $M = \Gamma\backslash X$ is, by definition, the image of a regular geodesic in X under the canonical projection. It follows from [Mo, §11] that T^r contains a closed regular geodesic.

Theorem A′. Let $x_1 \in \Gamma_0$ be the class of a *regular* closed geodesic c in T^r, and

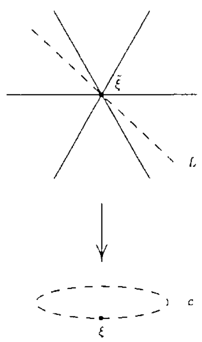
$$\Gamma_1 = \langle x_1 \rangle \cong \mathbb{Z} < \Gamma_0.$$

Then $\mathrm{VN}(\Gamma_1)' = \mathrm{VN}(\Gamma_0)$.

Consequently $\mathrm{VN}(\Gamma_0)$ is the unique masa containing $\mathrm{VN}(\Gamma_1)$.

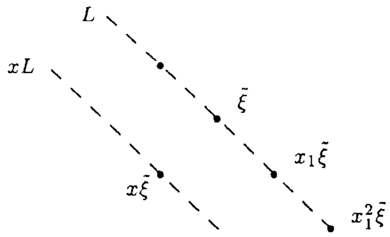
Proof of Theorem A′. (Using Lemma 2.1.)

Lift c to a geodesic L in X through $\tilde{\xi}$, where $p(\tilde{\xi}) = \xi$.



Regularity of the geodesic c means that L lies in a *unique* maximal flat F_0 and $p(F_0) = T^r$.

Now x_1 acts on L by translation.



Suppose that $A_x = \{x_1^{-n} x x_1^n : n \in \mathbb{Z}\}$ is finite, and let

$$\delta = \sup\{d(\eta, x_1^{-n} x x_1^n \eta) : \eta \in [\tilde{\xi}, x_1\tilde{\xi}], n \in \mathbb{Z}\}.$$

Then

$$d(x_1^n \eta, x x_1^n \eta) \leq \delta \qquad (\eta \in [\tilde{\xi}, x_1\tilde{\xi}],\ n \in \mathbb{Z}).$$

Therefore

$$d(\zeta, x\zeta) \leq \delta \text{ for all } \zeta \in L.$$

In other words, L is a parallel translate of xL. This implies that L and xL lie in a common maximal flat, namely F_0. In particular $x\tilde{\xi} \in F_0$. It follows that $p[\tilde{\xi}, x\tilde{\xi}]$ is a closed geodesic in T^r. Consequently $x \in \Gamma_0$. $\square$

Rather than proving Theorem B in complete generality, we prove a special case of it, which contains all the essential ideas of the general proof [RSS].

Corollary. *Let $\Gamma = \pi(M_g)$, the fundamental group of a compact Riemann surface M_g of genus $g \geq 2$. Let c be a closed geodesic of minimal length σ in M_g. Let $\gamma_0 = [c] \in \Gamma$, and let $\Gamma_0 \cong \mathbb{Z}$ be the subgroup of Γ generated by γ_0. Then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of the* II_1 *factor* $\mathrm{VN}(\Gamma)$.

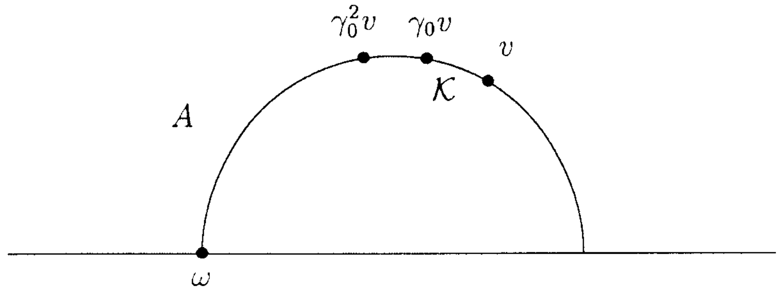
Proof. This uses condition (SS). The universal covering of M_g is the Poincaré upper half-plane

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

The boundary of $\mathfrak{H}$ is $\partial\mathfrak{H} = \mathbb{R} \cup \{\infty\}$. Also Γ acts isometrically on $\mathfrak{H}$.

The minimal closed geodesic c lifts to a geodesic A in $\mathfrak{H}$. Fix $v \in A$, and let $\mathcal{K} = [v, \gamma_0 v]$. Then

$$A = \bigcup_{n \in \mathbb{Z}} \gamma_0^n \mathcal{K} = \Gamma_0 \mathcal{K}. \tag{3.1}$$



Suppose that $x_1, \dots, x_m, y_1, \dots, y_n \in \Gamma$ and

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 y_j \,. \tag{3.2}$$

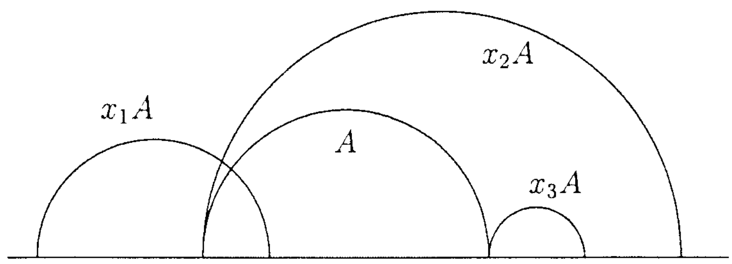
Let $\delta = \max\{d(y_j\kappa, \kappa); 1 \le j \le n,\ \kappa \in \mathcal{K}\}$. Then

$$\begin{aligned} y_j\mathcal{K} \underset{\delta}{\subset} \mathcal{K} \ &\Rightarrow\ \Gamma_0 y_j \mathcal{K} \underset{\delta}{\subset} \Gamma_0\mathcal{K} = A \\ &\Rightarrow\ x_i\Gamma_0 y_j\mathcal{K} \underset{\delta}{\subset} x_i A \end{aligned}$$

[Here the notation $P \underset{\delta}{\subset} Q$ means that $d(p, Q) \le \delta$, for all $p \in P$.]
Hence

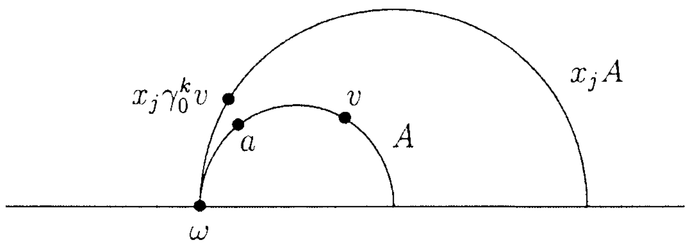
$$A = \Gamma_0\mathcal{K} \underset{\delta}{\subset} x_1A \cup x_2A \cup \cdots \cup x_mA \,. \tag{3.3}$$

This implies that each boundary point of A is a boundary point of some x_jA.

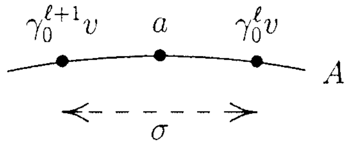


Now $\omega = \gamma_0^\infty v$ is a boundary point of some x_jA. We show that this implies $x_j \in \Gamma_0$. Choose $k \in \mathbb{Z}$, $a \in A$ such that

$$d(x_j\gamma_0^k v, a) < \frac{\sigma}{2} \,.$$



Choose $\ell \in \mathbb{Z}$ such that $d(a, \gamma_0^\ell v) \le \frac{\sigma}{2}$:



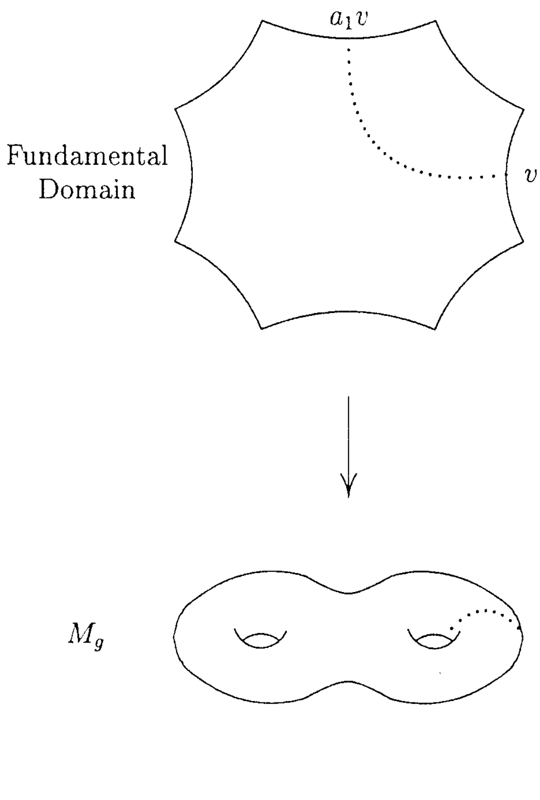
Then $d(\gamma_0^{-\ell} x_j \gamma_0^k v, v) = d(x_j\gamma_0^k v, \gamma_0^\ell v) \le d(x_j\gamma_0^k v, a) + d(a, \gamma_0^\ell v) < \sigma$.
This implies $\gamma_0^{-\ell} x_j \gamma_0^k = 1$. For otherwise $[v, \gamma_0^{-\ell} x_j \gamma_0^k v]$ projects to a closed geodesic in M_g of length $< \sigma$.

Therefore $x_j = \gamma_0^{\ell-k} \in \Gamma_0$. $\square$

In the usual presentation of $\pi(M_g)$,

$$\Gamma = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

we can take $\gamma_0 \in \{a_i^{\pm 1}, b_j^{\pm 1}\}$.



3.1 The ICC Property

Recall that $VN(\Gamma)$ is a II_1 factor if and only if the group Γ is *ICC*. If Γ were a lattice in a semisimple Lie group then the argument of [GHJ, Lemma 3.3.1] (which uses the Borel density theorem) proves that Γ is *ICC*. However not all the groups of interest to us are embedded in a natural way as subgroups of linear groups. We therefore show how to use a geometric argument to verify the *ICC* property of Γ. This argument applies much more generally; in particular to the group actions on buildings which we consider later.

Proposition. A group Γ of isometries of $\mathfrak{H}$ which acts cocompactly on $\mathfrak{H}$ is ICC.

Proof. By assumption, $\Gamma\mathcal{K} = \mathfrak{H}$ where $\mathcal{K} \subset \mathfrak{H}$ is compact.

Let $x \in \Gamma - \{1\}$. Suppose that $C = \{y^{-1}xy : y \in \Gamma\}$ is finite.

Let $\delta = \max\{d(\kappa, y^{-1}xy\kappa) : \kappa \in \mathcal{K}, y \in \Gamma\}$. Then

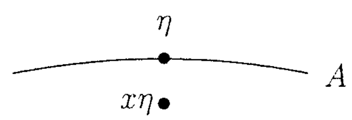
$$d(y\kappa, xy\kappa) = d(\kappa, y^{-1}xy\kappa) \leq \delta, \qquad y \in \Gamma, \kappa \in \mathcal{K}.$$

Therefore, for all $\xi \in \mathfrak{H}$

$$d(\xi, x\xi) \leq \delta. \tag{3.4}$$

Choose $\eta \in \mathfrak{H}$ such that $x\eta \neq \eta$

Choose a geodesic A in $\mathfrak{H}$ with $\eta \in A$, $x\eta \notin A$.



Now it follows from (3.4) that $A \underset{\delta}{\subset} xA$. This implies that $A = xA$. In particular $x\eta \in A$, a contradiction. $\square$

3.2 A Free Group Analogue

If X is a finite connected graph with fundamental group $\Gamma = \pi(X)$ then Γ is a finitely generated free group. Also Γ acts freely and cocompactly on the universal covering tree $\tilde{X}$ with boundary $\partial\tilde{X}$. Let $\Gamma_0 \cong \mathbb{Z}$ be the subgroup of Γ generated by one of the free generators of Γ.

This setup is a combinatorial analogue of the Corollary above, where the fundamental group Γ of a Riemann surface acts on the Poincaré upper half plane. Exactly the same proof shows that Γ, Γ_0, satisfy condition **(SS)**.

In the figure below, $\Gamma = \mathbb{Z} \star \mathbb{Z} = \langle a, b\rangle$, the free group on two generators and $\Gamma_0 = \langle a \rangle \cong \mathbb{Z}$. Thus $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$.

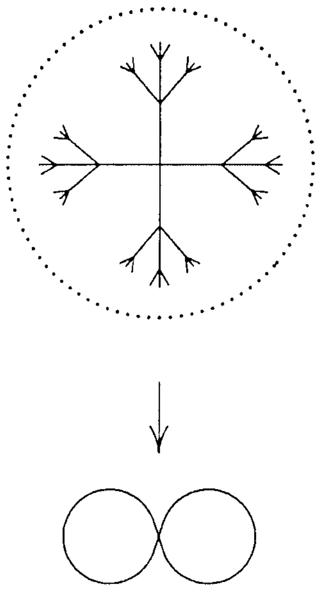
3.3 Euclidean Buildings

More generally, suppose Γ acts freely and transitively on the vertex set of a euclidean building Δ and Γ_0 is an abelian subgroup which acts transitively on the vertex set of an apartment (flat). Then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$. [The proof is essentially the same as that of Theorem B.]

There exist many examples where $\Gamma < \mathrm{PGL}_3(\mathbb{K})$, $\mathbb{K}$ a nonarchimedean local field [CMSZ].

$\tilde{X}$ and $\partial\tilde{X}$

X



Example: $\mathbb{K} = \mathbf{F}_4((X))$, the field of Laurent series with coefficients in the field $\mathbf{F}_4$ with four elements. Let Γ be the torsion free group with generators $x_i, 0 \leq i \leq 20$, and relations (written modulo 21):

$$\begin{cases} x_j x_{j+7} x_{j+14} = x_j x_{j+14} x_{j+7} = 1 & 0 \leq j \leq 6, \\ x_j x_{j+3} x_{j-6} = 1 & 0 \leq j \leq 20. \end{cases}$$

For each j, $0 \leq j \leq 6$,

$$\Gamma_0 = \langle x_j, x_{j+7}, x_{j+14} \rangle \cong \mathbb{Z}^2$$

satisfies the hypotheses.

3.4 A Borel subgroup

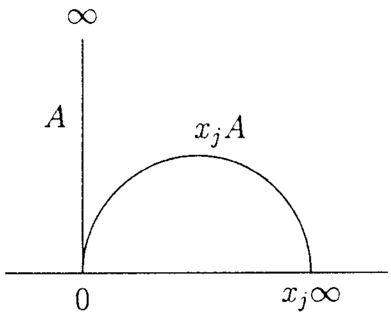
The geometric methods outlined above apply also to other situations. Here is an example.

Proposition. Let Γ be the upper triangular subgroup of $\mathrm{PSL}_n(\mathbb{Q})$, $n \geq 2$, and let Γ_0 be the diagonal subgroup of Γ. Then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$.

We illustrate the proof in the case $n = 2$. Here Γ acts on the Poincaré upper half plane $\mathfrak{H}$.

$$\Gamma = \{g \in \mathrm{PSL}_2(\mathbb{Q}) : g\infty = \infty\} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

$$\Gamma_0 = \{g \in \Gamma : g0 = 0\} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}.$$



Note that Γ_0 stabilizes the geodesic

$$\begin{aligned} A &= \mathbb{R}^+ i \\ &= \Gamma_0\mathcal{K}, \qquad \text{where } \mathcal{K} = [i, 2i]. \end{aligned}$$

In order to show that condition (SS) holds, proceed as in the proof of the Corollary in Section 3. As in (3.3), suppose that

$$A \underset{\delta}{\subset} x_1A \cup x_2A \cup \cdots \cup x_mA,$$

for some $x_1, \ldots, x_m \in \Gamma$, and $\delta > 0$. Then 0 is a boundary point of some x_jA. Now since $x_j \in \Gamma$, $x_j\infty = \infty$. Therefore $x_jA = A$ and $x_j0 = 0$. It follows that $x_j \in \Gamma_0$. □

4 Appendix: Symmetric Spaces

We conclude with a quick summary of some essential facts about symmetric and locally symmetric spaces [BH].

Let G be a semisimple Lie group with no centre and no compact factors.

The corresponding *symmetric space* is $X = G/K$ where K is a maximal compact subgroup.

The *rank* r of X is the dimension of a maximal *flat* in X. That is, the maximal dimension of an isometrically embedded euclidean space in X.

A geodesic L in X is *regular* if it lies in only one maximal flat; it is called *singular* if it is not regular.

Let F be a maximal flat in X and let $x \in F$. Let S_x denote the union of all the singular geodesics through x. A connected component of $F - S_x$ is called a *Weyl chamber* with origin x.

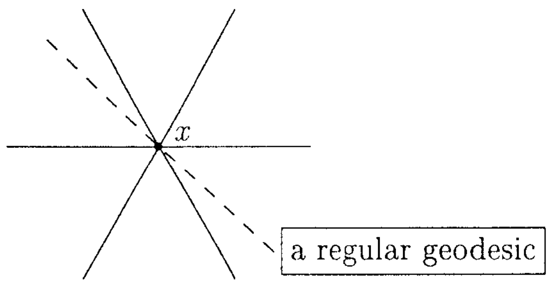
Example Consider a rank 2 example.

$$\begin{aligned} G &= \mathrm{SL}_3(\mathbb{R}) \\ X &= \{x \in \mathrm{SL}_3(\mathbb{R}) : x \ \text{ is positive definite}\} \end{aligned}$$

G acts transitively on X by $x \mapsto gxg^t$ and the stabilizer of I is $\mathrm{SO}_3(\mathbb{R})$. Therefore

$$X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$$

A maximal flat F is 2-dimensional. There are six Weyl chambers in F with a given origin $x \in F$.



A flat through I has the form $\exp \mathfrak{a}$, where $\mathfrak{a}$ is a linear subspace of

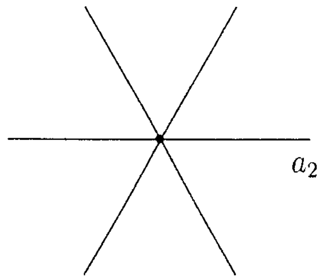
$$S_n(\mathbb{R}) = \{x \in \mathrm{M}_n(\mathbb{R}) : x = x^t, \, \mathrm{trace}(x) = 0\} \qquad \text{(the tangent space at } I\text{)}$$

such that $xy = yx$ for all $x, y \in \mathfrak{a}$.

The geodesic $t \mapsto \exp tx$ through I in X is regular if and only if the eigenvalues of $x \in S_n(\mathbb{R})$ are all distinct. To see why this is so, consider

$$\mathfrak{a}_0 = \{\mathrm{diag}(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 0\}.$$

Parametrize elements of $\mathfrak{a}_0$ by points on a plane through the origin in $\mathbb{R}^3$, as in the figure below.



If a_1, a_2, a_3 are all distinct then a matrix in $S_n(\mathbb{R})$ which commutes with $a = \operatorname{diag}(a_1, a_2, a_3)$ is necessarily diagonal and so lies in $\mathfrak{a}_0$. Thus the geodesic $t \mapsto \exp ta$ lies in a unique maximal flat $\exp \mathfrak{a}_0$.

4.1 Locally Symmetric Spaces

Let Γ be a torsion free cocompact lattice in G.

$M = \Gamma \backslash X$ is a compact *locally symmetric space* of nonpositive curvature.

$X = G/K$ is the universal covering space of M and the fundamental group of M is $\pi(M) = \Gamma$.

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