Spectral decomposition and discrete series representations on a p-adic group

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Abstract

Let G be a p-adic group. After a short survey on the representation theory of G, I outline my proof of a conjecture of A. Silberger on the infinitesimal character of discrete series representations of G. The conjecture says the following: a cuspidal representation τ of a Levi subgroup L of G corresponds to the infinitesimal character of a discrete series representation of G, if and only if τ is a pole of Harish-Chandra's μ-function of order equal to the parabolic rank of L. The proof uses a spectral decomposition based on a Fourier inversion formula analog to the Plancherel formula. To illustrate the method, the case of the unramified principal series of a semi-simple split group of type B_2 is worked out at the end.

1. Notations: Let F be a non-Archimedean local field. This is a topological field equipped with a discrete valuation $|.|_F$. It will be supposed to be normalized such that the Haar measure on F satisfies the transformation formula $d_F(xy) = |y|_F d_F x$. The topology of F is then defined by the ultrametric distance $d_F(x,y) := |x-y|_F$ and F is complete with respect to this metric. There exists a unique generator of the image of $|.|_F$ which is > 1. It will be denoted by q.

Let $\underline{G}$ be a connected reductive group defined over F and G the group of its F-rational points. So G is a locally compact and totally disconnected group.

The set of equivalence classes of irreducible representations of G will be denoted $\mathcal{E}(G)$. As usual, a representation will often be identified with its equivalence class. The subset of classes of square-integrable representations (i.e. whose matrix coefficients are square integrable functions on G modulo its center) will be denoted $\mathcal{E}_2(G)$. To any representation π in $\mathcal{E}_2(G)$ one can associate the formal degree $\deg(\pi)$, which is defined up to the choice of a Haar measure on G. Sometimes it is necessary to consider the bigger set $\mathfrak{A}(G)$ of equivalence classes of admissible representations of G, which contains the above. (These are no more irreducible but the space of vectors invariant by an open compact subgroup is finite dimensional.)

A subgroup P of G will be called a parabolic subgroup, if it is the group of F-rational points of a parabolic subgroup $\underline{P}$ of $\underline{G}$ defined over F. I will fix a maximal split torus in $\underline{G}$ and let T be the group of its F-rational points. A parabolic subgroup of G will be called semi-standard, if it contains T. There is then a unique Levi factor $\underline{L}$ of $\underline{P}$ which is defined over F and such that the group

L of its F-rational points contains T. The expression "$P = LU$ is a semi-standard parabolic subgroup of G" will then mean that P is a semi-standard parabolic subgroup, U its unipotent radical and L the group of F-rational points of its unique Levi factor which contains T. The functor i_P^G of parabolic induction sends $\mathfrak{A}(L)$ to $\mathfrak{A}(G)$ and will be supposed to be normalized such that it takes unitary representations to unitary representations. (So a representation parabolically induced from an irreducible representation is admissible.)

Let $\mathrm{Rat}(G)$ be the group of rational characters of $\underline{G}$ defined over F and G^1 the intersection of the kernels of the characters of G of the form $|\chi|_F$, $\chi \in \mathrm{Rat}(G)$. An unramified character of G is a homomorphism from the group quotient G/G^1 to $\mathbb{C}^\times$. The group formed by these characters will be designed by $\mathfrak{X}^{\mathrm{ur}}(G)$ and the subgroup formed by the unitary characters by $\mathfrak{X}_0^{\mathrm{ur}}(G)$. The group $\mathfrak{X}^{\mathrm{ur}}(G)$ is an algebraic tori isomorphic to $(\mathbb{C}^\times)^d$ with d equal to the rank of G, $\mathfrak{X}_0^{\mathrm{ur}}(G)$ being isomorphic to $(S^1)^d$. The first group acts on the set $\mathcal{E}(G)$ and the second group on the subset $\mathcal{E}_2(G)$. An orbit with respect to this action will be denoted by $\mathcal{O}$ in the first and $\mathcal{O}_2$ in the second case. So $\mathcal{O}$ is an algebraic variety. The space $\mathcal{O}$ (resp. $\mathcal{O}_2$) is, through the choice of a Haar measure on $\mathfrak{X}^{\mathrm{ur}}(G)$, equipped with a unique measure, such that the action of $\mathfrak{X}^{\mathrm{ur}}(G)$ on $\mathcal{O}$ (resp. of $\mathfrak{X}_0^{\mathrm{ur}}(G)$ on $\mathcal{O}_2$) preserves locally the measures.

2. Plancherel formula: The Plancherel formula for p-adic groups (due to Harish-Chandra (cf. [21])) expresses a smooth, compactly supported and complex valued function f on G by its Fourier transforms. More precisely, for f in $C_c^\infty(G)$ and $\pi \in \mathcal{E}(G)$ one defines an endomorphism $\pi(f)$ of the representation space V_π of π by

$$\pi(f) := \int_G f(g)\pi(g)dg.$$

Let $\Theta_2(G)$ be the set of pairs $(P = LU, \mathcal{O}_2)$ with $P = LU$ a semi-standard Levi subgroup of G and $\mathcal{O}_2$ an orbit in $\mathcal{E}_2(L)$. Two paires are called equivalent $\sim$, if they are conjugated by an element of G.

Let ρ be the action of G on $C_c^\infty(G)$ by right translations. Harish-Chandra defined for every pair $(P = LU, \mathcal{O}_2) \in \Theta_2(G)$ a constant $\gamma(G/L)$ and a function μ on $\mathcal{O}_2$, which extends to a rational function on the $\mathfrak{X}^{\mathrm{ur}}(G)$-orbit $\mathcal{O}$ which contains $\mathcal{O}_2$. He showed that for f in $C_c^\infty(G)$ one has (with a suitable normalization of the measures)

$$f(g) = \sum_{(P=LU,\mathcal{O}_2)\in\Theta_2/\sim} \gamma(G/L)\ |W(L,\mathcal{O}_2)|^{-1} \int_{\mathcal{O}_2} \mathrm{tr}((i_P^G\pi)(\rho(g)f)) \deg(\pi)\mu(\pi)d\pi.$$

(Here $W(L, \mathcal{O}_2)$ denotes a subset of the Weyl group of G relative to T formed by elements which stabilize $\mathcal{O}_2$ and L.)

3. Representation theory of G: (cf. [4]) The functor i_P^G admits a left adjoint functor r_P^G which is called the *Jacquet functor.* A representation $\pi \in \mathcal{E}(G)$ is then called *cuspidal,* when $r_P^G \pi = 0$ for all proper parabolic subgroups P of G. The subset of cuspidal representations will be denoted $\mathcal{E}_c(G)$. Note that to any cuspidal representation π a formal degree $\deg(\pi)$ can be attached.

The classification of cuspidal representations is a deep arithmetical problem which is entirely solved only for GL_N (including SL_N) and the multiplicative group of a central division algebra over F respectively by C. Bushnell and P. Kutzko ([2] and subsequent work treating SL_N) and E.-W. Zink [22]. A conjectural parametrization of this set by N-dimensional irreducible representations of the Weil group of F (which is some distinguished subgroup of the absolut Galois group of F) is the aim of the local Langlands conjectures. For GL_N, they have been proved recently by M. Harris and R. Taylor [5] and, by a more elementary approach, by G. Henniart [9].

Given $\pi \in \mathcal{E}(G)$ there exist a semi-standard parabolic subgroup $P = LU$ and a cuspidal representation $\sigma \in \mathcal{E}_c(L)$ such that π is a subquotient of $i_P^G \sigma$. The G-conjugation class of L and σ is uniquely determined by π. It is called the *cuspidal support* of π.

Remark that any *unitary* cuspidal representation is square integrable and that any orbit $\mathcal{O}$ of a cuspidal representation is formed by cuspidal representations and contains a cuspidal representation that is unitary. On the other hand there exist square integrable representations which are not cuspidal. These are called *special representations.*

Example: Identify $\chi = |.|_F$ with a character of the diagonal subgroup L of SL_2 by the embedding $x \mapsto (x, x^{-1})$. Let B be the Borel subgroup formed by upper triangular matrices. Then the induced representation $i_B^G \chi$ is of length 2 and has a unique subrepresentation which is called the *Steinberg representation.* It is square integrable, but not cuspidal. (Remark that the other subquotient of $i_B^G \chi$ is the unit representation of G.)

Classification scheme: The Langlands classification (cf. [17]) gives a description of the set $\mathcal{E}(G)$ up to the knowledge of the tempered representations of its Levi subgroups. Tempered representations can be constructed by parabolic induction from square integrable representations. For GL_N a representation parabolically induced from a square integrable representation is irreducible, but for other groups this may fail and it is not known yet how to describe the different components.

The next step below is to construct all square integrable representations from the cuspidal ones. This is known for GL_N by the work of Bernstein and Zelevinsky [22], for unramified principal series representations by Kazhdan and Lusztig [10] and for split classical groups (under some assumption on the reducibility points) by the results of C. Moeglin and Moeglin-Tadic ([12] and [13]). There are also several results of A. Silberger in [18] and [19].

4. A conjecture of Silberger: A. Silberger conjectured also the following result:

Theorem: (cf. [8] corollaire **8.7**) *Let $P = LU$ be a parabolic subgroup of G and τ an irreducible cuspidal representation of L. Then $i_P^G\tau$ has a subquotient in $\mathcal{E}_2(G)$ precisely when the following two conditions hold:*

i) the restriction of τ to the center of G is a unitary representation;

ii) τ is a pole of μ of order $rk_{ss}(G) - rk_{ss}(L)$. (Here μ denotes Harish-Chandra's μ-function as defined in **2.***).*

Let us make the second condition more precise. For this, I will first explain the notion of an affine rootal hyperplane. Fix a maximal split torus T_L in the center of L and let $\Sigma(P)$ be the set of roots of T_L in $Lie(U)$. Define $a_L = \mathrm{Hom}(\mathrm{Rat}(L), \mathbb{R})$ and let a_L^* be the dual space. It contains $\Sigma(P)$. There is a natural map $H_L : L \to a_L$. One defines a surjection from the complexified vector space $a_{L,\mathbb{C}}^*$ to $\mathfrak{X}^{\mathrm{ur}}(L)$, by sending λ to the character χ_λ with $\chi_\lambda(l) := q^{-\langle H_L(l), \lambda\rangle}$ (recall that q is the unique generator > 1 of the image of $|.|_F$). The restriction of this map to a_L^* is injective and so $\Re(\chi_\lambda) := \Re(\lambda)$ is well defined. An affine rootal hyperplane in $a_{L,\mathbb{C}}^*$ is then by definition an affine hyperplane defined by a coroot $\alpha^\vee$, $\alpha \in \Sigma(P)$.

Let $\mathcal{O}$ be the orbit of τ. An affine rootal hyperplane in $\mathcal{O}$ is then by definition the image of an affine rootal hyperplane in $a_{L,\mathbb{C}}^*$ by the composed map $a_{L,\mathbb{C}}^* \to \mathfrak{X}^{\mathrm{ur}}(L) \to \mathcal{O}$, the second arrow being given by the action of $\mathfrak{X}^{\mathrm{ur}}(L)$ on $\mathcal{O}$.

It is known since Harish-Chandra that the poles and zeroes of μ lie on finitely many affine rootal hyperplanes in $\mathcal{O}$. Let $\mathcal{S}_0$ be the set of affine zero hyperplanes of μ and $\mathcal{S}_1$ the set of affine polar hyperplanes. The affine zero hyperplanes are of order 2 and the polar ones are of order 1. So the order of the pole of μ in τ is $|\{S \in \mathcal{S}_1 \,|\tau \in S\}| - 2|\{S \in \mathcal{S}_0 \,|\tau \in S\}|$.

Remark: By a conjecture of Langlands [11], which has been proved by F. Shahidi [16] in the case of G quasi-split and τ generic, the function μ on $\mathcal{O}$ can be expressed as product and quotient of L-functions attached to τ.

5. Strategy of proof: Let $\mathcal{O}$ be the orbit of τ in $\mathcal{E}_c(L)$. All its elements can be realized as representations in a same vector space which will be denoted E. For σ in an open set of $\mathcal{O}$ and $P' = LU'$ a second parabolic subgroup with Levi factor L, one has an operator $J_{P|P'}(\sigma) : i_{P'}^G E \to i_P^G E$ which intertwines the representations $i_{P'}^G \tau$ and $i_P^G \tau$. In an open cone of $\mathcal{O}$ it is defined by the converging integral

$$(J_{P|P'}(\sigma)v)(g) := \int_{U\cap U'\backslash U} v(ug)du,$$

where v is considered as an element of the space $i_{P'}^G E$ equipped with the representation $i_{P'}^G \sigma$. It is a rational function in σ and the composed operator $J_{P|\overline{P}}(\sigma)J_{\overline{P}|P}(\sigma)$ is scalar and equals the inverse of the μ-function. For w in the Weyl group W of G with respect to T one defines an operator $\lambda(w)$ which induces an isomorphism between the representations $i_P^G\sigma$ and $i_{wP}^G w\sigma$.

The following lemma was crucial for the proof of a matrix Paley-Wiener theorem in [7]:

Lemma: (cf. [7] **0.2**) *Let f be in $C_c^\infty(G)$. Identify $(i_P^G\sigma)(f)$ to an element of $i_P^G E \otimes i_P^G E^\vee$. There exists a polynomial map $\xi_f : \mathcal{O} \to i_{\overline{P}}^G E \otimes i_P^G E^\vee$ with image in a finite dimensional space such that*

$$(i_P^G\sigma)(f) = \sum_{w\in W(L,\mathcal{O})} (J_{P|\overline{wP}}(\sigma)\lambda(w) \otimes J_{P|wP}(\sigma)\lambda(w))\xi_f(w^{-1}\sigma).$$

as rational funtions in σ. (Here $W(L,\mathcal{O})$ has the same meaning than in **2.***.)*

Remark that the poles of $J_{P|\overline{P}}$ are on the affine rootal hyperplanes in $\mathcal{S}_0$ and that the poles of $\mu J_{P|\overline{P}}$ are on the affine rootal hyperplanes in $\mathcal{S}_1$.

Let $C_c^\infty(G)_{\mathcal{O}}$ be the subspace of $C_c^\infty(G)$ formed by the functions f such that $(i_{P'}^G\sigma')(f) = 0$ for all $\sigma' \in \mathcal{O}'$ with $(P', \mathcal{O}') \not\sim (P, \mathcal{O})$.

Proposition: *Let f be in $C_c^\infty(G)_{\mathcal{O}}$. Identify $\xi_f(\sigma)$ with an element in* $\mathrm{Hom}(i_{\overline{P}}^G E$, $i_P^G E)$. *For $g \in G$ one has*

$$(*) \qquad f(g) = \int_{\Re(\sigma)=r\gg_P 0} \gamma(G/L)\deg(\sigma)\,\mathrm{tr}((i_P^G\sigma)(g^{-1})J_{P|\overline{P}}(\sigma)\xi_f(\sigma))\mu(\sigma)d\Im(\sigma).$$

(The symbol $\int_{\Re(\sigma)=r\gg_P 0}$ means that one fixes r in a_L^* such that $\langle r, \alpha^\vee\rangle \gg 0$ for all $\alpha \in \Sigma(P)$ and that one integrates on the compact set $\chi_r\,\mathcal{O}_2$. Here $\mathcal{O}_2$ is the subset formed by the unitary representations in $\mathcal{O}$ and the integral is taken with respect to the fixed measure on $\mathcal{O}_2$.)

The strategy of proof of the theorem in **4.** is then to compare the expression (*) in the above proposition with the one given by the Plancherel formula in **2.**. This is done after a contour shift to the unitary axis.

Example: Suppose G semi-simple and L maximal. Then $\mathfrak{X}^{\mathrm{ur}}(L) \simeq \mathbb{C}^\times$ and the theory of complex functions in one variable applies: the integral (*) is a sum of residues and an integral over the unitary orbit $\mathcal{O}_2$. The residues correspond by the remark after the above lemma to the poles of μ.

In the Plancherel formula for $f \in C_c^\infty(G)_{\mathcal{O}}$, there appear only terms corresponding to the equivalence class of pairs $(P' = L'U', \mathcal{O}_2)$ with either $P' = P$ or with $P' = G$ and $\mathcal{O}_2$ equal to the set formed by a single square integrable representation of G. The cuspidal support of this representations is necessarily contained in the G-orbit of $\mathcal{O}$. The first term is an integral over $\mathcal{O}_2$ and the other terms are discrete.

It is then rather easy to show that the two integrals and the discrete terms in both formulas correspond to each other, proving the theorem in this simple case. (Remark that the theorem **4.** was already known in this case.)

Unfortunately, in the case of a Levi subgroup with corank bigger than one, poles of the intertwining operators do appear and it is not evident at all, that and how they cancel. The proof of the theorem **4.** follows then the following steps:

i) *Formulation of a convenient multi-dimensional residue theorem:* this is achieved by a generalization of the residue theory for root systems due to E. P. van den Ban and H. Schlichtkrull [3] to our situation (see also the paper [6] of G. Heckman and E. Opdam). Let $\mathcal{S}$ be the union of the sets $\mathcal{S}_0$ and $\mathcal{S}_1$. Define $\mathcal{A}(\mathcal{S})$ as the set of affine subspaces of $\mathcal{O}$ which are connected components of finite intersections of elements in $\mathcal{S}$. The subset of spaces in $\mathcal{A}(\mathcal{S})$, where μ can have a non trivial residue will be denoted $\mathcal{A}_\mu(\mathcal{S})$. I also fix a set $[\mathcal{A}(\mathcal{S})]$ of representatives of conjugation classes in $\mathcal{A}(\mathcal{S})$. To an affine hyperplane A in $\mathcal{A}(\mathcal{S})$ one attaches a semi-standard Levi subgroup L_A of G. The origine of A will be denoted $r(A)$ and ϵ_A will be an element in some positive Weyl chamber of $a_{L_A}^*$.

With $\Delta_{\mathcal{O}}$ some set of positive roots associated to $\mathcal{O}$, $W_{\Delta_{\mathcal{O}}}$, $W^{L_A}(L, \mathcal{O})$ and $W_{L_A}^+(L, \mathcal{O})$ some sets of Weyl group elements and $\mathcal{P}_{\mathcal{S}}(L_A)$ some set of generalized parabolic subgroups with Levi component L_A, the residue formula applied to the integral (*) gives then

$$\sum_{\Omega\subseteq\Delta_{\mathcal{O}}}\ \sum_{A\in[\mathcal{A}(\mathcal{S})],L_A=L_\Omega} |W_{\Delta_{\mathcal{O},L_A}}|^{-1}\ |\mathcal{P}_{\mathcal{S}}(L_A)|^{-1}\gamma(G/L)\int_{\Re(\sigma)=r(A)+\epsilon_A}\deg(\sigma)|\operatorname{Stab}(A)|^{-1}$$
$$\sum_{w'\in W^{L_A}(L,\mathcal{O})}\ \sum_{w\in W^+_{L_A}(L,\mathcal{O})} \operatorname{Res}^P_{w'A}(\operatorname{tr}((i^G_P\sigma)(g^{-1})J^{-1}_{\overline{P}|P}(ww'\sigma)\xi_f(ww'\sigma)))d_A\Im(\sigma).$$

Here Res^P_A is an operator from the space of rational functions on $\mathcal{O}$, which are regular outside the hyperplanes in $\mathcal{S}$, to some space of rational functions on A. It turns out to be uniquely determined by r and P. It is a sum of composed residue operators relative to affine hyperplanes in $\mathcal{S}$ containing A.

ii) *Identification of the continuous part:* This is done with help of an induction hypothesis. One sees then that one can replace ϵ_A by zero in the above formula.

iii) *Elimination of the undesirable poles with help of test functions.* These already appeared in [7] at a crucial step, although they played a different role there.

With this one gets the following result:

The induced representation $i^G_P\tau$ has a subquotient in $\mathcal{E}_2(G)$ if and only if the restriction of τ to the center of G is unitary, $A\in\mathcal{A}_\mu(\mathcal{S})$, $L_A=G$, and

(**) $$\sum_{w\in W(L,\mathcal{O})}(\operatorname{Res}_{wA}\mu)(w\sigma)\neq 0.$$

But a theorem of E. Opdam (cf. [15] theorem 3.29) shows, that the condition (**) is always satisfied, finishing the proof of the theorem in **4.**. Remark that Opdam proved in [15] a spectral decomposition for affine Hecke algebras, which applies in particular to Iwahori-Hecke algebras and through it for example to the unramified principal series of a p-adic group.

The identities with the terms in the Plancherel formula contain also informations on the formal degree and on the position of the discrete series representations in the induced representation. Opdam was for example able to deduce from his identities some invariance properties of the formal degree on L-packets of discrete series representations in his context.

The method employed here may be considered as a local analog of the spectral decomposition and the theory of the residual spectrum due to Langlands [11] in the field of automorphic forms.

Appendix: The case of B_2

Let now $\underline{G}$ be a semi-simple split group of type B_2 defined over F. Fix a minimal semi-standard parabolic subgroup $P = TU$ of G. Then $a_0^* := a_T^* \simeq \mathbb{R}^2$. The set $\Sigma(P)$ of roots of T in Lie(U) can be written in the form $\{\alpha, \beta, \alpha{+}2\beta, \alpha{+}\beta\}$, where β is the short root. Let $\Sigma^\vee(P)$ be the set of roots dual to the roots in $\Sigma(P)$. One has $\Sigma^\vee(P) = \{\alpha^\vee, \beta^\vee, \alpha^\vee + \beta^\vee, 2\alpha^\vee + \beta^\vee\}$. The set $\{\alpha^\vee, \beta^\vee\}$ is a base of a_0 and the dual bases of a_0^* will be denoted $\{\omega_\alpha, \omega_\beta\}$. Observe that $\langle \alpha^\vee, \beta\rangle = -1$ and $\langle \beta^\vee, \alpha\rangle = -2$.

Let τ be the trivial representation of T. The orbit $\mathcal{O}$ of τ with respect to $\mathfrak{X}^{\mathrm{ur}}(T)$ is isomorphic to $(\mathbb{C}^\times)^2$. Define $\tau_\lambda := \tau \otimes \chi_\lambda$. The μ-function on $\mathcal{O}$ is given by

$$\mu(\tau_{x\omega_\alpha + y\omega_\beta}) = C\frac{(1-q^x)(1-q^{-x})(1-q^y)(1-q^{-y})(1-q^{x+y})(1-q^{-x-y})}{(1-q^{1+x})(1-q^{1-x})(1-q^{1+y})(1-q^{1-y})(1-q^{1+x+y})(1-q^{1-x-y})} \times \frac{(1-q^{2x+y})(1-q^{-2x-y})}{(1-q^{1+2x+y})(1-q^{1-2x-y})},$$

where C is a constant > 0.

The affine hyperplanes of $\mathcal{O}$ which are polar for μ are the images of the affine hyperplanes in a_0^* of the form $\langle \gamma^\vee, \lambda\rangle = c$ with $c = -1$ or $c = 1$, $\gamma \in \Sigma(P)$. The zero affine hyperplanes are the images of the affine hyperplanes $\langle \gamma^\vee, \lambda\rangle = 0$ in a_0^*, $\gamma \in \Sigma(P)$. So they correspond to the lines generated respectively by the vectors $\overrightarrow{0\omega_\alpha}$, $\overrightarrow{0\omega_\beta}$, $\overrightarrow{0\alpha}$ and $\overrightarrow{0\beta}$.

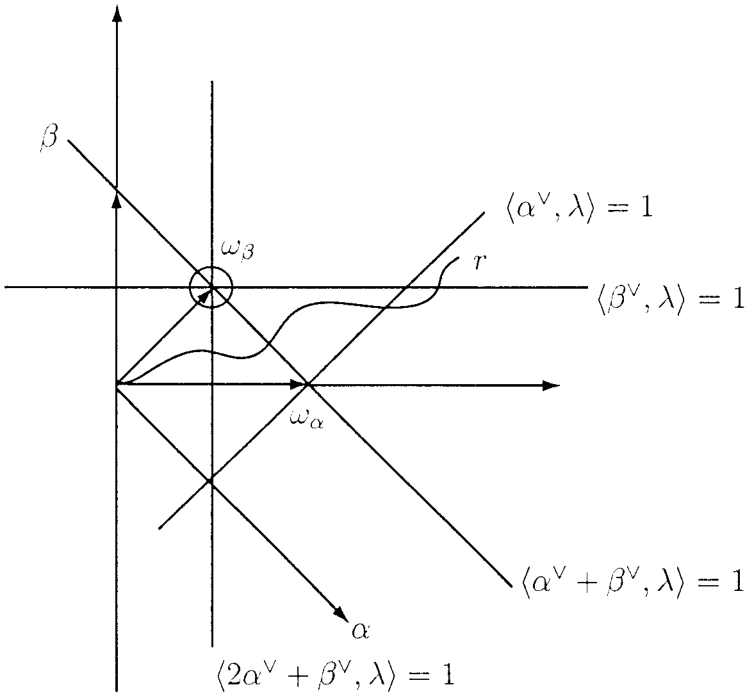
Fix $r = r_\alpha\omega_\alpha + r_\beta\omega_\beta \in a_0^*$ with $r_\alpha \gg 0$ and $r_\beta \gg 0$. To calculate for $f \in C_c^\infty(G)_{\mathcal{O}}$ the integral

$$(\#) \qquad \int_{\Re(\sigma)=r} \mathrm{tr}((i_P^G\sigma)(g^{-1})J_{P|\overline{P}}(\sigma)\xi_f(\sigma))\mu(\sigma)d\Im(\sigma),$$

one first moves r to 0 following the circuit below. Each intersection of this circuit with a polar hyperplane H gives a residue, which is a rational function in σ with $\Re(\sigma) \in H$. For each intersection point r_H, one has to move r_H on H near to the origine $r(H)$ of this affine hyperplane (which is the point with minimal distance to the origine of a_0^*). Each intersection point with an affine polar or zero hyperplane $H' \neq H$ of this segment on H can give rise to another non trivial residue. It turns out that there is only one case where a zero affine hyperplane gives a non trivial residue: this happens for the intersection of the polar affine hyperplane

$\langle \beta^\vee, \lambda\rangle = 1$ with the zero affine hyperplane $\langle \alpha^\vee, \lambda\rangle = 0$. The intersection point is ω_β.

Remark that the point r_H can be moved to the origine of the hyperplan H, if the origine is a regular point. (One verifies that only the affine polar hyperplane $\langle \alpha^\vee + \beta^\vee, \lambda\rangle = 1$ has an origine, which is not regular.)



To simplify the notations let $g = 1$. (For $g \neq 1$ one gets the analog result.) Then one sees that $(\#)$ is with $\epsilon > 0$ equal to

(1) $\quad \operatorname{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+\frac{3}{2}\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+\frac{3}{2}\omega_\alpha})) \operatorname{Res}_{x=\frac{3}{2}} \operatorname{Res}_{y=1}(\mu(\tau_{x\omega_\alpha+y\frac{\beta}{2}}))$

(2) $\quad + \operatorname{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+(1+\frac{\pi i}{\log q})\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+(1+\frac{\pi i}{\log q})\omega_\alpha})) \operatorname{Res}_{x=1+\frac{\pi i}{\log q}} \operatorname{Res}_{y=1}(\mu(\tau_{x\omega_\alpha+y\frac{\beta}{2}}))$

(3) $\quad + \operatorname{Res}_{x=\frac{1}{2}}(\operatorname{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+x\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+x\omega_\alpha})) \operatorname{Res}_{y=1}(\mu(\tau_{y\frac{\beta}{2}+x\omega_\alpha}))_{|x=\frac{1}{2}}$

(4) $\quad + \dfrac{\log q}{2\pi}\displaystyle\int_{x=0}^{\frac{2\pi}{\log q}} \operatorname{tr}(J_{P|\overline{P}}(\tau_{\frac{\beta}{2}+ix\omega_\alpha})\xi_f(\tau_{\frac{\beta}{2}+ix\omega_\alpha})) \operatorname{Res}_{y=1}(\mu(\tau_{ix\omega_\alpha+y\frac{\beta}{2}}))dx$

(5) $\quad + \dfrac{\log q}{2\pi}\displaystyle\int_{y=0}^{\frac{2\pi}{\log q}} \operatorname{tr}(J_{P|\overline{P}}(\tau_{\frac{\alpha}{2}+iy\omega_\beta})\xi_f(\tau_{\frac{\alpha}{2}+iy\omega_\beta})) \operatorname{Res}_{x=1}(\mu(\tau_{x\frac{\alpha}{2}+iy\omega_\beta}))dy$

(6)ϵ $\quad + \dfrac{\log q}{2\pi}\displaystyle\int_{z=0}^{\frac{2\pi}{\log q}} \operatorname{tr}(J_{P|\overline{P}}(\tau_{(iz+\epsilon)\frac{\alpha}{2}+\omega_\beta})\xi_f(\tau_{(iz+\epsilon)\frac{\alpha}{2}+\omega_\beta})) \operatorname{Res}_{x=1}(\mu(\tau_{(iz+\epsilon)\frac{\alpha}{2}+x\omega_\beta}))dz$

(7) $\quad + \dfrac{\log q}{2\pi}\displaystyle\int_{t=0}^{\frac{2\pi}{\log q}} \operatorname{tr}(J_{P|\overline{P}}(\tau_{it\beta+\frac{\omega_\alpha}{2}})\xi_f(\tau_{it\beta+\frac{\omega_\alpha}{2}})) \operatorname{Res}_{y=1}(\mu(\tau_{y\frac{\omega_\alpha}{2}+it\beta}))dt$

(8) $\quad + \displaystyle\int_{\Re(\sigma)=0} \operatorname{tr}(J_{P|\overline{P}}(\sigma)\xi_f(\sigma))\mu(\sigma)d\sigma.$

One observes that

$$(6)_\epsilon - (6)_{-\epsilon} = \mathrm{Res}_{z=0}(\mathrm{tr}(J_{P|\overline{P}}(\tau_{z\frac{\alpha}{2}+\omega_\beta})\xi_f(\tau_{z\frac{\alpha}{2}+\omega_\beta}))) \,\mathrm{Res}_{x=1}(\mu(\tau_{z\frac{\alpha}{2}+x\omega_\beta}))_{|z=0}$$

and verfies that

$$0 = (3) + \frac{1}{2}\,\mathrm{Res}_{z=0}(\mathrm{tr}(J_{P|\overline{P}}(\tau_{z\frac{\alpha}{2}+\omega_\beta})\xi_f(\tau_{z\frac{\alpha}{2}+\omega_\beta}))) \,\mathrm{Res}_{x=1}(\mu(\tau_{z\frac{\alpha}{2}+x\omega_\beta}))_{|z=0}.$$

So (3) cancels after replacing $(6)_\epsilon$ by $(6')_\epsilon = \frac{1}{2}((6)_\epsilon + (6)_{-\epsilon})$. According to our general results (cf. step ii) in **5.**), one verifies directly that the integrant in $((6)_\epsilon + (6)_{-\epsilon})$ is a regular function for $\epsilon = 0$, i.e. $(6')_\epsilon = (6')_0 =: (6')$.

With this one sees easily, that (8) corresponds to the term in the Plancherel formula coming from the unitary orbit of the unit representation of $L = T$, that $(6') + (5)$ corresponds to the term coming from the orbit of the Steinberg representation of the Levi subgroup L_α and that $(4) + (7)$ corresponds to the term coming from the Steinberg representation of the Levi subgroup L_β. The term (1) corresponds to the one in the Plancherel formula coming from the square-integrable representation of G which is the unique subrepresentation of $i_P^G\tau_{\omega_\alpha+\omega_\beta}$. The term (2) comes from the unique square-integrable representation of G, which is a subrepresentation of $i_P^G\tau_{\omega_\beta+(\frac{1}{2}+\frac{\pi i}{\log q})\omega_\alpha}$.

Remark that these results on the discrete series of G of type B_2 were already known to P. Sally and M. Tadic [20] (see also [1] for a complete Plancherel formula in this setting). However, in general it is much more difficult to find explicitly the subquotients of an induced representation which are square-integrable (see for example the case of a group of type G_2 studied in the appendix **A.** to [8] which is the local analog to the case studied in the appendix III to [14]).

The material of this article together with all the proofs will appear in the Journal de l'Institut de Mathématiques de Jussieu.

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