Spectral synthesis for orbits of compact groups in the dual of certain generalized $\mathcal{L}^1$–algebras

Detlev Poguntke

Abstract

Our objects of study are generalized $\mathcal{L}^1$–algebras $\mathcal{L}^1(K,Q)$, where K is a closed normal subgroup of the compact group L, and Q is a commutative Banach algebra whose Gelfand space is a transitive L–space. The main result tells that L–orbits in the dual of $\mathcal{L}^1(K,Q)$ are sets of synthesis, i.e., there is a unique closed two–sided ideal in $\mathcal{L}^1(K,Q)$ whose hull coincides with a given L–orbit. Also the empty set is a set of synthesis, which means that each proper closed two–sided ideal is contained in the kernel of an irreducible involutive representation. To this end, L–fixed projections in $\mathcal{L}^1(K,Q)$ are constructed. Such projections are also useful in other circumstances.

Introduction

The notion of sets of synthesis (or Wiener sets) is best known in the case of $\mathcal{L}^1(G)$, G a locally compact abelian group. In this case the (Gelfand) structure space $\mathcal{L}^1(G)^\wedge$ of the commutative Banach algebra $\mathcal{L}^1(G)$ can be identified with the Pontryagin dual $G^\wedge$. With each closed ideal $\mathcal{I}$ in $\mathcal{L}^1(G)$, one can associate a closed subset of $G^\wedge$, namely the hull $h(\mathcal{I}) := \{\chi \in G^\wedge \mid \ker_{\mathcal{L}^1(G)} \chi \supset \mathcal{I}\}$. A closed subset A of $G^\wedge$ is called a *set of synthesis* if there is only one closed ideal $\mathcal{I}$ in $\mathcal{L}^1(G)$ with $h(\mathcal{I}) = A$. In this case $\mathcal{I}$ is necessarily equal to the kernel $k(A) := \bigcap_{\chi\in A} \ker_{\mathcal{L}^1(G)} \chi$ of A. For some results on sets of synthesis in the case of abelian groups compare [13, 14].

Usually, the kernel is defined in the equivalent way: $k(A) = \{f \in \mathcal{L}^1(G) \mid \widehat{f} = 0 \text{ on } A\}$. We have chosen the above formulation, because then all the introduced notions generalize immediately to arbitrary Banach algebras, as soon as one agrees on the structure space to be considered. In the present article we study algebras of the following type:

Let K be as closed normal subgroup of a compact group L, and let Q be a symmetric semi–simple involutive commutative Banach algebra. Symmetry means in the commutative case that $\omega(q^*) = \overline{\omega(q)}$ for all $q \in Q$ and all ω in the Gelfand space $Q^\wedge$, i.e., for all multiplicative linear functionals $\omega : Q \to \mathbb{C}$. Suppose that L acts strongly continuously (from the right) on Q with the usual properties:

$$(\lambda q + r)^\ell = \lambda q^\ell + r^\ell, (qr)^\ell = q^\ell r^\ell, (q^*)^\ell = (q^\ell)^*, q^{\ell m} = (q^\ell)^m,$$

$\|q^\ell\| = \|q\|$, and $L \ni \ell \longmapsto q^\ell \in Q$ is continuous. Then L acts on $Q^\wedge$, $(\ell\omega)(q) = \omega(q^\ell)$, and we suppose that this action is *transitive. These assumptions are retained throughout the article.*

Now one can form our object of study, the generalized $\mathcal{L}^1$–algebra $\mathcal{L}^1(K,Q)$, compare [7], multiplication and involution being given by

$$\begin{aligned}(f \star g)(a) &= \int_K f(a\,b)^{b^{-1}} g(b^{-1}) d\,b \\ f^*(a) &= f(a^{-1})^{*a}\end{aligned}$$

for $a \in K$, $f, g \in \mathcal{L}^1(K,Q)$. $\mathcal{L}^1(K,Q)$ carries a natural L–action,

$$f^\ell(a) = f(\ell\,a\,\ell^{-1})^\ell$$

satisfying the usual properties (as written above for the pair (L,Q)).

As structure space of $\mathcal{L}^1(K,Q)$ we take the collection $\mathrm{Priv}_*\,\mathcal{L}^1(K,Q)$ of kernels of all irreducible involutive representations of $\mathcal{L}^1(K,Q)$ in Hilbert spaces equipped with the Jacobson topology. This space carries a natural L–action. Our main goal is to show that L–orbits are sets of synthesis. En passant, we also prove that the empty set is a set of synthesis (sometimes called Wiener property, see [8, 10]), i.e., each proper closed ideal in $\mathcal{L}^1(K,Q)$ is contained in the kernel of an involutive irreducible representation, that there exist operators of finite rank in the image of irreducible representations, and that $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ coincides with $\mathrm{Priv}\,\mathcal{L}^1(K,Q)$, the collection of primitive ideals in $\mathcal{L}^1(K,Q)$.

The proofs are more or less exercises in representation theory of compact groups, based on the existence of the Haar measure, particularly on the following easy, but useful lemma, whose proof is omitted.

Lemma 0.1. *Let $\iota : E \longrightarrow F$ be a bounded linear dense injection of Banach spaces. Suppose that a compact group G acts continuously on E and F by linear isometries, and that ι intertwines the action. If either all G–isotypical components in E or in F are finite–dimensional then ι induces an isomorphism of each of the components and, as a consequence, an isomorphism from the collection $E^{(G)}$ of G–finite vectors onto $F^{(G)}$. Moreover, $E^{(G)}$ resp. $F^{(G)}$ is dense in E resp. F.*

1 The C^*–hull and the irreducible involutive representations of $\mathcal{L}^1(K,Q)$

Let us fix a base point $\omega \in Q^\wedge$. Then $Q^\wedge$ can be identified with the space L/L_ω of cosets, where, of course, L_ω denotes the stabilizer of ω; for $x \in L$ we denote by $[x] = x\,L_\omega$ the corresponding coset. The Gelfand transform can be identified with an injective map $\mathcal{G} : Q \longrightarrow \mathcal{C}(L/L_\omega)$; it is L–equivariant if $\ell \in L$ acts on $\varphi \in \mathcal{C}(L/L_\omega)$ via $\varphi^\ell([x]) = \varphi([\ell x])$. The map $\mathcal{G}$ induces an injective morphism of involutive Banach algebras.

$$\mathcal{L}^1(K, Q) \to \mathcal{L}^1(K, \mathcal{C}(L/L_\omega)). \tag{1.1}$$

Each involutive representation π of $\mathcal{L}^1(K, Q)$ is given by a covariance pair (π', π''), π' being a continuous unitary representation of K, π'' an involutive representation of Q:

$$\pi(f)\xi = \int_K \pi'(a)\pi''(f(a))\xi da \tag{1.2}$$

for $f \in \mathcal{L}^1(K, Q)$.

As π'' extends to a representation of $\mathcal{C}(L/L_\omega)$, so does π, i.e., the C^*–hull of $\mathcal{L}^1(K, Q)$ is the C^*–transformation algebra $C^*(K, \mathcal{C}(L/L_\omega)$. We obtain three dense continuous inclusions

$$\begin{array}{ccccc} \mathcal{L}^1(K, Q) & \longrightarrow & \mathcal{L}^1(K, \mathcal{C}(L/L_\omega)) & \longrightarrow & C^*(K, \mathcal{C}(L/L_\omega)) \\ & & & \uparrow & \\ & & & \mathcal{C}(K \times L/L_\omega) & \end{array} \tag{1.3}$$

where $\mathcal{C}(K \times L/L_\omega)$ is equipped with the uniform norm. The compact group $G := L \ltimes (K \times K)$ with multiplication law

$$(\ell, k_1, k_2)(t, a_1, a_2) = (\ell\, t, t^{-1} k_1 t\, a_1, f^{-1} k_2 t\, a_2) \tag{1.4}$$

acts on all these four spaces, on $\mathcal{L}^1(K, Q)$ via

$$\sigma(\ell, k_1, k_2) f = \left(\varepsilon_{k_1} \star f \star \varepsilon_{k_2^{-1}}\right)^{\ell^{-1}}, \tag{1.5}$$

where

$$(\varepsilon_k \star f)(a) = f(k^{-1}a), (f \star \varepsilon_k)(a) = f(a\, k^{-1})^k \text{ for } a, k \in K, f \in \mathcal{L}^1(K, Q), \tag{1.6}$$

on $\mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$ via

$$(\sigma(\ell, k_1, k_2)\varphi)(a, [x]) = \varphi\left(k_1^{-1}\ell^{-1} a\, \ell\, k_2, \left[k_2^{-1}\ell^{-1} x\right]\right) \tag{1.7}$$

for $\varphi \in \mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$, $a \in K$, $[x] \in L/L_\omega$.

The action on $\mathcal{C}(K \times L/L_\omega)$ is obtained by restriction, the action on $C^*(K, \mathcal{C}(L/L_\omega))$ by functoriality, but it is not explicitly needed. By its very construction all three inclusions are G–invariant.

Sometimes the isomorphic copy $G' = (K \times K) \rtimes L$ of G is also useful, where the multiplication is given by $(k_1, k_2, \ell)(a_1, a_2, t) = (k_1 \ell\, a_1 \ell^{-1}, k_2 \ell\, a_2 \ell^{-1}, \ell\, t)$. Via the canonical isomorphism

$$\delta : G' \longrightarrow G, \delta(k_1, k_2, \ell) = (\ell, \ell^{-1} k_1 \ell, \ell^{-1} k_2 \ell) \tag{1.8}$$

one obtains representations σ' of G' in the above four spaces, for instance

$$\sigma'(k_1, k_2, \ell)\varphi(a, [x]) = \varphi\left(\ell^{-1}k_1^{-1}a\, k_2\ell, \left[\ell^{-1}k_2^{-1}a\right]\right) \tag{1.9}$$

for $\varphi \in \mathcal{C}(K \times L/L_\omega)$.

If we restrict σ' to the subgroup $H' := K \rtimes L = (K \times \{e\}) \rtimes L \leq G'$ we just get the left regular representation of H' in $\mathcal{C}(H'/L_\omega = K \times L/L_\omega)$, whose H'–isotypical components are finite–dimensional. We conclude that the isotypical components of σ in $\mathcal{C}(K \times L/L_\omega)$ are finite–dimensional as well. The Lemma in the introduction tells us:

Proposition 1.10. *All the G–isotypical components in the four spaces $\mathcal{L}^1(K, Q)$, $\mathcal{L}^1(K, \mathcal{C}(L/L_\omega)$, $\mathcal{C}(K \times L/L_\omega)$, $C^*(K, \mathcal{C}(L/L_\omega)$ are finite–dimensional. In fact, they coincide as well as the collections of G–finite vectors, which are dense in the respective spaces.*

For later use we define here the group $H := L \ltimes K = L \ltimes (K \times \{e\}) \leq G$ which is isomorphic to H' via

$$\gamma : H' \longrightarrow H\ ,\ \gamma(k, \ell) = (\ell, \ell^{-1}k\,\ell). \tag{1.11}$$

The group L acts on involutive representations π of $\mathcal{L}^1(K, Q)$ (or of $C^*(K, \mathcal{C}(L/L_\omega))$ via

$$(\ell\,\pi)(f) = \pi(f^\ell) = \pi(\sigma(\ell)^{-1}(f)). \tag{1.12}$$

For a continuous irreducible unitary representation α of $K_\omega = L_\omega \cap K$ in V_α we define an irreducible involutive representation π_α of $C^*(K, \mathcal{C}(L/L_\omega)$ in the Hilbert space

$$\begin{aligned}(1.13)\quad \mathfrak{H}_\alpha = \mathcal{L}^2_{K_\omega}(K, V_\alpha) := \{\xi : K \longrightarrow V_\alpha \mid \xi &\text{ is measurable},\\ \xi(k\,a) = \alpha(a)^{-1}\xi(k) \text{ for } a \in K_\omega, k \in K, &\text{ and } \int_{K/K_\omega} \|\xi(k)\|^2 dk < \infty\}\end{aligned}$$

by the covariance pair $(\pi'_\alpha, \pi''_\alpha)$

$$\begin{aligned}(\pi'_\alpha(k)\xi)(a) &= \xi(k^{-1}a)\\ (\pi''_\alpha(\psi)\xi)(a) &= \psi([a])\xi(a).\end{aligned}$$

In particular, for $\varphi \in \mathcal{C}(K \times L/L_\omega)$ and $\xi \in \mathfrak{H}_\alpha$ the vector $\pi_a(\varphi)\xi$ is given by

$$(\pi_\alpha(\varphi)\xi)(b) = \int_K \varphi(y, [y^{-1}b])\xi(y^{-1}b)dy. \tag{1.14}$$

If π is any irreducible representation of $C^*(K, \mathcal{C}(L/L_\omega)$ given by the covariance pair (π', π'') then π'' is supported by a K–orbit in L/L_ω as $K\backslash L/L_\omega$ is Hausdorff.

For a suitable $\ell \in L$, $(\ell\pi)''$ is supported by the K–orbit through the origin, i.e., $\ell\pi$ may be considered as a representation of $C^*(K, \mathcal{C}(K/K_\omega))$. Such algebras, actually in much higher generality, were studied in [5]. In particular, we know that $C^*(K, \mathcal{C}(K/K_\omega))$ is liminal, and that $\ell\pi$ is equivalent to one of the above π_α. Therefore, we have

1.15. *Each irreducible involutive representation of $C^*(K, \mathcal{C}(L/L_\omega))$ is equivalent to one of the collection $\ell\pi_\alpha, \ell \in L, \alpha \in K_\omega^\wedge$. Moreover, $(\ell\pi_\alpha)(C^*(K, \mathcal{C}(L/L_\omega))$ is equal to algebra $\mathcal{K}(\mathfrak{H}_\alpha)$ of compact operators on $\mathfrak{H}_\alpha$.*

Next, we investigate, which of those representations are equivalent. If $\ell_1\pi_\alpha \sim \ell_2\pi_\beta$ then $\pi_\beta \sim \ell_2^{-1}\ell_1\pi_\alpha$. If follows that their restrictions to $\mathcal{C}(L/L_\omega)$ (i.e., their second components considered as covariance pairs) must be carried by the same K–orbit which means $\ell_2^{-1}\ell_1 \in L_\omega K$. Write $\ell_2^{-1}\ell_1 = \ell_0 k$ with $k \in K$ and $\ell_0 \in L_\omega$. As $k\pi_\alpha \sim \pi_\alpha$ because $k\pi_\alpha(f) = \pi_\alpha(f^k) = \pi_\alpha(\varepsilon_{k^{-1}} \star f \star \varepsilon_k) = \pi'_\alpha(k^{-1})\pi_\alpha(f)\pi'_\alpha(k)$, compare (1.6), we find that $\ell_0\pi_\alpha \sim \pi_\beta$. Further, it is easy so see:

1.16. *For $\ell_0 \in L_\omega$ one has $\ell_0\pi_\alpha \sim \pi_\beta$ if and only if $\ell_0\alpha \sim \beta$ (as representations of K_ω, $(\ell_0\alpha)(b) = \alpha(\ell_0^{-1} b\, \ell_0)$ for $b \in K_\omega$).*

Also for later use we write down an intertwining operator explicitly. If $\mathcal{U} : V_\alpha \to V_\beta$ is a unitary operator with $\mathcal{U}\alpha(\ell_0^{-1} b\, \ell_0) = \beta(b)\mathcal{U}$ for all $b \in K_\omega$ then define $\mathcal{U}' : \mathfrak{H}_\alpha \to \mathfrak{H}_\beta$ by

$$(\mathcal{U}'\xi)(k) = \mathcal{U}(\xi(\ell_0^{-1} k\, \ell_0)), k \in K. \tag{1.17}$$

These arguments work also the other way around, and we conclude:

1.18. *$\ell_1\pi_\alpha \sim \ell_2\pi_\beta$ means that $\ell_2^{-1}\ell_1$ can be written in the form $\ell_2^{-1}\ell_1 = \ell_0 k$ with $k \in K$ and $\ell_0 \in L_\omega$ satisfying $\ell_0\alpha \sim \beta$.*

In view of this observation we choose an indexed set of representatives of the L_ω–orbits in $K_\omega^\wedge$, i.e., we take a collection α_j, $j \in J$, of concrete continuous irreducible unitary representations of K_ω in V_j with the following properties.

1.19. *Each continuous irreducible unitary representation of K_ω is equivalent to $\ell\alpha_j$ for some $j \in J$, $\ell \in L_\omega$. If $\ell\alpha_j \sim \ell'\alpha_{j'}$, for $\ell, \ell' \in L_\omega$ and $j, j' \in J$ then $j = j'$.*

With those representations α_j we construct as above the representations $\pi_j := \pi_{\alpha_j}$ of $C^*(K, \mathcal{C}(L/L_\omega))$ or of $\mathcal{L}^1(K, Q)$ in $\mathfrak{H}_j := \mathfrak{H}_{\alpha_j}$. Our discussion shows:

Proposition 1.20. *Each of the irreducible involutive representations of $\mathcal{L}^1(K, Q)$ is equivalent to one of the form $\ell\pi_j$, $\ell \in L$, $j \in J$. For $\ell_1, \ell_2 \in L$ and $i, j \in J$ the condition $\ell_1\pi_i \sim \ell_2\pi_j$ is equivalent to $i = j$ and $\ell_2^{-1}\ell_1 \in L^j K$, where L^j denotes the stabilizer of α_j in L_ω. (L^j is of finite index in L_ω.) In other words, the set $\mathcal{L}^1(K, Q)^\wedge$ of equivalence classes is a disjoint union of the L–orbits $L\pi_j$, $j \in J$, and the L–stabilizer of π_j is $L^j K$.*

Remark 1.21. *This description can be used to write down all the members of* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$*, however, it is not clear at present that inequivalent representations yield different kernels. But they do as we shall see later.*

Remark 1.22. *The description of* $\mathcal{L}^1(K,Q)^\wedge$ *given in (1.20) is the one we are going to use in the sequel. A little more canonical is the following one, also suggested by the above discussion. The group* $\mathfrak{U}_\omega := L_\omega K$ *acts from the left on* $K_\omega^\wedge$*: For* $\ell_0 k$*,* $\ell_0 \in L_\omega$*,* $k \in K$ *and* $\alpha \in K_\omega^\wedge$ *the element* $\alpha' = \ell_0 k \cdot \alpha \in K_\omega^\wedge$ *is given by* $\alpha'(v) = \alpha(\ell_0^{-1} v \ell_0)$*. And* $\mathfrak{U}_\omega$ *acts also from the left on* L *by right translations:* $u \cdot x = xu^{-1}$ *for* $x \in L$*,* $u \in \mathfrak{U}_\omega$*. Thus,* $\mathfrak{U}_\omega$ *acts on* $L \times K_\omega^\wedge$*. By (1.15), there is a surjection* $L \times K_\omega^\wedge \longrightarrow \mathcal{L}^1(K,Q)^\wedge$*, and by (1.16) the fibers of this map are exactly the* $\mathfrak{U}_\omega$*–orbits. Moreover, the* L*–action on* $\mathcal{L}^1(K,Q)^\wedge$ *corresponds to translation on* $L \times K_\omega^\wedge$ *in the first variable. Clearly, the space of* $\mathfrak{U}_\omega$*–orbits in* $L \times K_\omega^\wedge$ *can be identified with the disjoint union* $\bigcup_{j\in J} L/L^j K$ *in an obvious manner, respecting the* L*–action.*

2 The kernel operators for the representations $\ell\pi_j$, a surjectivity theorem

Many questions in harmonic analysis depend on an appropriate description of the image of the Fourier transform; this principle applies also to non–commutative situations. We shall write down the kernel functions which give the operators $(\ell\pi_j)(\varphi)$ of the previous section, and shall prove a surjectivity theorem describing the image $(\ell\pi_j)(\varphi)$, $\ell \in L$, $\varphi \in \mathcal{C}(K \times L/L_\omega)$. Using G–equivariance we shall obtain a result for $\mathcal{C}(K \times L/L_\omega)^{(G)} = \mathcal{L}^1(K,Q)^{(G)}$.

Given j, ℓ, φ as above, we recall, (1.7), (1.14), that $(\ell\pi_j)(\varphi)$ in $\mathcal{B}(\mathfrak{H}_j)$ is given by

$$\begin{aligned}
[(\ell\pi_j)(\varphi)\xi](b) &= [\pi_j(\varphi^\ell = \sigma(\ell^{-1})\varphi)\xi](b) = \int_K \varphi(\ell\, y\, \ell^{-1}, [\ell y^{-1} b])\xi(y^{-1}b)dy \\
&= \int_K \varphi(\ell\, bc^{-1}\ell^{-1}, [\ell\, c])\xi(c)dc = \int_{K/K_\omega} (R_j\varphi)(\ell, b, c)\xi(c)dc,
\end{aligned}$$

where $R_j\varphi : L \times K \times K \to \mathcal{B}(V_j)$ is defined by

$$(R_j\varphi)(\ell, b, c) = \int_{K_\omega} \varphi(\ell\, b\, s^{-1}c^{-1}\ell^{-1}, [\ell\, c])\alpha_j(s)^{-1}ds. \tag{2.1}$$

Clearly, the functions $R_j\varphi$ are continuous, but they share also three covariance properties, which are most easily expressed by viewing $R_j\varphi$ as a function on $G = L \ltimes (K \times K)$. To this end, we choose intertwining operators between $\ell\alpha_j$ and α_j, $\ell \in L^j$:

2.2. *Let* $X_j(\ell)$*,* $\ell \in L^j$*, be a unitary operator on* V_j *satisfying* $\alpha_j(\ell^{-1} r\, \ell) = X_j(\ell)^* \alpha_j(r) X_j(\ell)$ *for all* $r \in K_\omega$*.*

For later use we remark:

2.3. *For each $\ell_0 \in L^j$ there exists a function c defined in a neighborhood of ℓ_0 in L^j with values in $\mathbb{T}$ such that $c(\ell_0) = 1$ and $\ell \longmapsto c(\ell)X_j(\ell)$ is continuous on this neighborhood.*

To see (2.3) define $F(\ell)$, $\ell \in L^j$, by $F(\ell) = \int_{K_\omega} \alpha_j(x)X_j(\ell_0)(\ell\,\alpha_j)(x)^{-1}dx \in \mathcal{B}(V_j)$. Clearly, F is continuous, and $F(\ell_0) = X_j(\ell_0)$. For ℓ sufficiently close to ℓ_0 the operator $F(\ell)$ is invertible (or, equivalently, different from 0 as it is an intertwining operator). Then $F'(\ell) = \|F(\ell)\|^{-1}F(\ell)$ is unitary, F' is still continuous, and $F'(\ell) = c(\ell)X_j(\ell)$ for suitable numbers $c(\ell) \in \mathbb{T}$.

$\square$

2.4. *The $\mathcal{B}(V_j)$–valued function $R_j\varphi$ on G satisfies: $(R_j\varphi)(t\,k,b,c) = (R_j\varphi)(t,k\,b,k\,c)$ for all $(t,b,c) \in G$, all $k \in K$ or, equivalently,*

(i) $$(R_j\varphi)(g\,x) = R_j\varphi(g)$$

for all $g \in G$, all $x \in \Delta := \{(k^{-1},k,k) \mid k \in K\}$. Observe that Δ is a subgroup of G.

(ii) $$(R_j\varphi)(g(\ell,1,1)) = X_j(\ell)^* R_j\varphi(g)X_j(\ell)$$

for all $g \in G$, $\ell \in L^j$.

(iii) $$(R_j\varphi)(g(1,u,v)) = \alpha_j(u)^{-1}(R_j\varphi)(g)\alpha_j(v)$$

for all $g \in G$ and $u,v \in K_\omega$.

Denote the space of all continuous $\mathcal{B}(V_j)$–valued functions on G satisfying (i), (ii), (iii) by $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$.

All these properties are easy to check as well as

2.5. *For all $x,g \in G$ one has $R_j(\sigma(g)\varphi)(x) = (\tau_j(g)R_j\varphi)(x)$, where $\tau_j(g)$ on $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ is just left translation, i.e., $(\tau_j(g)\Phi)(x) = \Phi(g^{-1}x)$.*

Property (i) of (2.4) implies that the members Φ of $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ are completely determined by their restrictions $\rho(\Phi)$ to $H = L \ltimes K = L \ltimes (K \times \{1\}) \leq G$, in other words,

2.6. *ρ is an isomorphism from $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ onto the space $\mathcal{C}_{L^j,K_\omega}(H,\mathcal{B}(V_j))$ of all continuous functions $\Lambda : H \to \mathcal{B}(V_j)$ satisfying*

(a) $\Lambda(h(\ell,1)) = X_j(\ell)^{-1}\Lambda(h)X_j(\ell)$ *for all $h \in H$, $\ell \in L^j$.*

(b) $\Lambda(h(1,u)) = \alpha_j(u)^{-1}\Lambda(h)$ *for all $h \in H$, $u \in K_\omega$.*

To transform this space into a space of functions on $H' = K \rtimes L$ along the canonical isomorphism $\gamma : H' \to H$, see (1.11), we first note:

2.7. *The subgroup $M^j := K_\omega \rtimes L^j$ of H' has a canonical continuous (irreducible) representation β_j in the space $\mathcal{B}(V_j)$ given by*

$$\beta_j(x, \ell_0)^{-1}(A) = X_j(\ell_0)^* \alpha_j(x)^{-1} A X_j(\ell_0)$$

for $A \in \mathcal{B}(V_j)$.

2.8. *γ induces an isomorphism $\widetilde{\gamma}, \widetilde{\gamma}(\Lambda)(k, \ell) = \Lambda(\ell, \ell^{-1} k\, \ell)$, from $\mathcal{C}_{L^j, K_\omega}(H, \mathcal{B}(V_j))$ onto the space $\mathcal{C}_{M^j}(H', \mathcal{B}(V_j))$ of all continuous functions $\Psi : H' \to \mathcal{B}(V_j)$ satisfying*

$$\Psi(h(x, \ell_0)) = \beta_j(x, \ell_0)^{-1} \Psi(h)$$

for all $h \in H'$, $(x, \ell_0) \in M^j$.

These easy, but a little confusing changes of viewpoints simplify the proof of the following surjectivity theorem. In fact, they are not absolutely necessary, but they shed some light on the structure of the elements in $\mathcal{C}_{\Delta, L^j, K_\omega, K_\omega}(G, \mathcal{B}(V_j))$.

Theorem 2.9. *Given $j \in J$ and $\Phi \in \mathcal{C}_{\Delta, L^j, K_\omega, K_\omega}(G, \mathcal{B}(V_j))$ there exists $\varphi \in \mathcal{C}(K \times L/L_\omega)$ satisfying $R_j \varphi = \Phi$ and $R_i \varphi = 0$ for all $i \in J$, $i \neq j$.*

Proof. By what we have seen above, it is enough to find φ with $(\widetilde{\gamma} \circ \rho)(\Phi) = (\widetilde{\gamma} \circ \rho)(R_j \varphi)$ and $(\widetilde{\gamma} \circ \rho)(R_i \varphi) = 0$ for $i \in J$, $i \neq j$. For short we put $\Psi = (\widetilde{\gamma} \circ \rho)(\Phi)$. We choose an orthonormal basis $v_1, \ldots, v_n$ of V_j, and identify, for fixed $h = (k, \ell) \in H'$, the operator $\Psi(h)$ with an $n \times n$–matrix,

$$\Psi(h) v_t = \sum_{s=1}^{n} \Psi(h)_{st} v_s$$

for $1 \leq t \leq n$.

In terms of this basis we construct explicitly a function $\widetilde{\psi}$ on K_ω such that $\int_{K_\omega} \widetilde{\psi}(u) \alpha_j(u) du = \Psi(h)$, namely

$$\widetilde{\psi}(h; u) := n \sum_{s,r=1}^{n} \langle \alpha_j(u)^{-1} v_s, v_r \rangle \Psi(h)_{sr} \quad \text{for} \quad u \in K_\omega. \tag{2.10}$$

The orthogonality relations, see e.g. [4, p. 278], readily imply

$$\Psi(h) = \int_{K_\omega} \widetilde{\psi}(h; u) \alpha_j(u) du. \tag{2.11}$$

Indeed, as $\langle \alpha_j(u)^{-1} v_s, v_r \rangle$ is the entry in the matrix corresponding to $\alpha_j(u)^{-1}$ at position r, s, the sum $\sum_{s=1}^{n} \langle \alpha_j(u)^{-1} v_s, v_r \rangle \Psi(h)_{sr}$ is nothing but the entry in the matrix corresponding to $\alpha_j(u)^{-1} \Psi(h)$ at position r, r.

But $\alpha_j(u)^{-1}\Psi(h) = \Psi(h(u,1))$, hence

$$\widetilde{\psi}(h;u) = n \operatorname{Tr} \Psi(h(u,1)). \tag{2.12}$$

In particular, the function $K_\omega \ni u \longmapsto \widetilde{\psi}(h(u^{-1},1);u)$ is constant for all $h \in H'$. From (2.8) and (2.12) it follows that $\widetilde{\psi}(h(1,\ell);1) = \widetilde{\psi}(h;1)$ for all $h \in H'$, all $\ell \in L^j$. Therefore,

$$\varphi(k,[t]) := \sum_{\ell \in L_\omega/L^j} \widetilde{\psi}(k, t\,\ell; 1) \tag{2.13}$$

is a function on $K \times L/L_\omega$, and we claim that this φ has the desired properties.

At a point $(b,t) \in H' = K \rtimes L$ one finds for an $i \in J$:

$$\begin{aligned} [(\widetilde{\gamma} \circ \rho)(R_i\varphi)]\,(b,t) &= \int_{K_\omega} \varphi(btst^{-1},[t])\alpha_i(s)ds \\ &= \sum_{\ell \in L_\omega/L^j} \int_{K_\omega} ds\,\widetilde{\psi}((b,t)(s,\ell);1)\alpha_i(s)ds. \end{aligned}$$

We introduce artificially another integration

$$\widetilde{\psi}((b,t)(s,\ell);1) = \int_{K_\omega} du\,\widetilde{\psi}((b,t)(s,\ell)(u^{-1},1);u).$$

As $(s,\ell)(u^{-1},1) = (s\,\ell\,u^{-1}\ell^{-1},\ell)$, with the new integration variable $s' = s\,\ell u^{-1}\ell^{-1} \in K_\omega$ one gets

$$[(\widetilde{\gamma} \circ \rho)(R_i\varphi)](b,t) = \sum_{\ell \in L_\omega/L^j} \int_{K_\omega} ds \int_{K_\omega} du\,\widetilde{\psi}((b,t)(s,\ell);u)\alpha_i(s)\alpha_i(\ell\,u\,\ell^{-1}).$$

If $i \neq j$, then the representation $\ell^{-1}a_i$, $\ell \in L_\omega$, is not equivalent to α_j, hence by the construction of $\widetilde{\psi}$, (2.10), and the orthogonality relations the integral $\int_{K_\omega} du\,\widetilde{\psi}(h;u)\,(\ell^{-1}\alpha_i)\,(u)$ vanishes for all $h \in H'$. Therefore, $(\widetilde{\gamma} \circ \rho)(R_i\varphi)$ is equal to zero.

If $i = j$, for the same reason the above integral over u vanishes if ℓ is outside L^j. Thus, we obtain

$$[(\widetilde{\gamma} \circ \rho)(R_j\varphi)](b,t) = \int_{K_\omega} ds \int_{K_\omega} du\ \widetilde{\psi}((b,t)(s,1);u)\alpha_j(s)\alpha_j(u).$$

The integration over u can be carried out using (2.11),

$$[(\widetilde{\gamma} \circ \rho)(R_j\varphi)](b,t) = \int_{K_\omega} ds\,\alpha_j(s)\Psi((b,t)(s,1)).$$

But the integrand is constant, hence $[(\widetilde{\gamma}\circ\rho)(R_j\varphi)](b,t) = \Psi(b,t) = [(\widetilde{\gamma}\circ\rho)(\Phi)](b,t)$ as desired. □

Corollary 2.14. *Given $j \in J$, for each G–finite kernel function $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ there exists a G–finite function $f \in \mathcal{L}^1(K,Q)$ such that the operator $(\ell\pi_j)(f) = \pi_j(f^\ell)$ is given by the kernel $\Phi(\ell,-,-)$, and that $(\ell\pi_i)(f) = 0$ for all $\ell \in L$, $i \in J$, $i \neq j$.*

Proof. First of all, by (2.9) there exists an $\varphi \in \mathcal{C}(K \times L/L_\omega)$, considered as a subspace of $\mathcal{L}^1(K,\mathcal{C}(L/L_\omega))$, with the corresponding properties as stated in the Corollary. As the map $\Pi_{i\in J}R_i : \mathcal{C}(K \times L/L_\omega) \to \Pi_{i\in J}\mathcal{C}_{\Delta,L^i,K_\omega,K_\omega}(G,\mathcal{B}(V_i))$ is injective ("uniqueness of the Fourier transform") and G–equivariant, the function φ has necessarily to be G–finite, whence it is "contained" in $\mathcal{L}^1(K,Q)^{(G)}$, see (1.10). $\square$

Corollary 2.15. *If $i,j \in J$ and $\ell_1,\ell_2 \in L$ have the property that $\ell_1\pi_i$ is not equivalent to $\ell_2\pi_j$ then there exists a (G–finite) $f \in \mathcal{L}^1(K,Q)$ such that $(\ell_1\pi_i)(f) = 0$, but $(\ell_2\pi_j)(f) \neq 0$. In view of (1.20), see also (1.21), this means that the canonical map from the set of equivalence classes of irreducible involutive representations of $\mathcal{L}^1(K,Q)$ onto* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *is a bijection. Thus, a parametrization of* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *is obtained.*

Proof. If $i \neq j$, then, in view of Corollary 0.1, one only has to note the evident fact that there is a G–finite vector $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ with $\Phi(\ell_2,-,-) \not\equiv 0$.

If $i = j$, then $\ell_2^{-1}\ell_1 \notin L^jK$ by (1.20). Choose $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))^{(G)}$ such that $\Phi(\ell_2,-,-) \not\equiv 0$. Next choose a representative function μ in $\mathcal{R}(G/L^jK \ltimes (K \times K))$, i.e., a G–finite function which is constant on $L^jK \ltimes (K \times K)$–cosets, such that $\mu(\ell_2,1,,1) = 1$, but $\mu(\ell_1,1,1) = 0$. This can be done because $(\ell_1,1,1)(L^jK \ltimes (K \times K)) \neq (\ell_2,1,1)(L^jK \ltimes (K \times K))$. Then $\mu\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))^{(G)}$ satisfies $(\mu\Phi)(\ell_2,-,-) \neq 0$, but $(\mu\Phi)(\ell_1,-,-) = 0$. By (2.14) there exists $f \in \mathcal{L}^1(K,Q)$ such that the operator $(\ell\pi_j)(f)$ is represented by the kernel function $(\mu\Phi)(\ell,-,-)$. $\square$

Corollary 2.16. *The L–orbits in* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *are open and closed with respect to the Jacobson topology.*

Proof. Any L–orbit is of the form $\Omega_j := \{\ker \ell\pi_j \mid \ell \in L\}$ for a certain $j \in J$. To show that Ω_j is open, take the kernel of $\Omega'_j := \{\ker \ell\pi_i \mid i \in J,\, i \neq j',\, \ell \in L\}$, i.e., $\mathcal{I}'_j := \{f \in \mathcal{L}^1(K,Q) \mid (\ell\pi_i)(f) = 0 \text{ for all } \ell \in L,\, i \in J,\, i \neq j\}$.

Then, by definition, the hull of $\mathcal{I}'_j$ is closed, and $h(\mathcal{I}'_j)$ contains Ω'_j. On the other hand, for any $\ell \in L$, by (2.14) there exists $f \in \mathcal{I}'_j$ such that $(\ell\pi_j)(f) \neq 0$, which implies that $h(\mathcal{I}'_j) = \Omega'_j$, whence Ω_j is open.

To show that Ω_j is closed, take the kernel of Ω_j, i.e., $\mathcal{I}_j := \{f \in \mathcal{L}^1(K,Q) \mid (\ell\pi_j)(f) = 0 \text{ for all } \ell \in L\}$. Clearly, the closed set $h(\mathcal{I}_j)$ contains Ω_j. On the other hand, if $\ell \in L$, $i \in J$, $i \neq j$, then by (2.14) there exists $f \in \mathcal{L}^1(K,Q)$ such that $(\ell\pi_i)(f) \neq 0$, but $(\ell'\pi_j)(f) = 0$ for all $\ell' \in L$. This means that $f \in \mathcal{I}_j$, and $\ker \ell\pi_i \notin h(\mathcal{I}_j)$, whence $\Omega_j = h(\mathcal{I}_j)$ is closed. $\square$

Remark 2.17. *We did not claim anything on the internal (Jacobson) topology of the various L–orbits in* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$*. Presumably, if one assumes that Q is regular, i.e., the Gelfand topology coincides with the Jacobson topology on the structure space $Q^\wedge$, then those orbits carry their natural topology, which would imply that* $\mathrm{Priv}_*\mathcal{L}^1(K,Q)$ *is homeomorphic to* $\mathrm{Priv}_*C^*(K,\mathcal{C}(L/L_\omega))$*, i.e.,* $\mathcal{L}^1(K,Q)$ *is $*$–regular in the sense of [1], where originally this class of groups/algebras was denoted by* $[\Psi]$*. But I must admit that I did not study this circle of questions seriously. Certainly G–finite functions are too algebraic in nature in order to separate arbitrary closed sets in* $\mathrm{Priv}_*C^*(K,\mathcal{C}(L/L_\omega))$ *from points.*

The consideration in this section can also be used to "compute" the C^*–hull of $\mathcal{L}^1(K,Q)$, which is the same as the C^*–hull of $\mathcal{L}^1(K,\mathcal{C}(L/L_\omega))$. Given $j \in J$ and chosen intertwining operators $X_j(\ell)$, $\ell \in L^j$, as in (2.2), we define unitary operators $Y_j(\ell)$ on $\mathfrak{H}_j$, compare (1.17), by

$$(Y_j(\ell)\xi)(k) = X_j(\ell)\xi(\ell^{-1}k\ell) \tag{2.18}$$

for $\ell \in L^j$, $k \in K$, $\xi \in \mathfrak{H}_j$ satisfying

$$(\ell\pi_j)(f) = \pi_j(f^\ell) = Y_j(\ell)^*\pi_j(f)Y_j(\ell)$$

for $f \in \mathcal{L}^1(K,Q)$ or in $C^*(K,\mathcal{C}(L/L_\omega))$.

Using these operators we define a space of continuous functions from L into the algebra $\mathcal{K}(\mathfrak{H}_j)$ of compact operators on $\mathfrak{H}_j$ as follows.

2.19. $\mathcal{C}_{L^jK}(L,\mathcal{K}(\mathfrak{H}_j))$ *consists of all continuous functions* $T : L \to \mathcal{K}(\mathfrak{H}_j)$ *satisfying*

$$\begin{aligned} T(\ell\,\ell') &= Y_j(\ell')^*T(\ell)Y_j(\ell') \text{ for } \ell' \in L^j, \text{ and} \\ T(\ell\,k) &= \pi_j'(k)^*T(\ell)\pi_j'(k) \text{ for } k \in K. \end{aligned}$$

Observe that each $Y_j(\ell')$ normalizes $\pi_j'(K)$. Each $\Phi \in \mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G,\mathcal{B}(V_j))$ yields an element $T_\Phi \in \mathcal{C}_{L^jK}(L,\mathcal{K}(\mathfrak{H}_j))$ via

$$\left(T_\Phi(\ell)\xi\right)(a) = \int_{K/K_\omega} \Phi(\ell,a,b)\xi(b)db \tag{2.20}$$

for $a \in K$, $\xi \in \mathfrak{H}_j = \mathcal{L}^2_{K_\omega}(K,V_j)$.

As a matter of fact, if $\Phi = R_j\varphi$, $\varphi \in \mathcal{C}(K \times L/L_\omega)$, one has

$$T_{R_j\varphi}(\ell) = (\ell\pi_j)(\varphi). \tag{2.21}$$

There is an action ν_j of the group $G = L \ltimes (K \times K)$ on $\mathcal{C}_{L^jK}(L,\mathcal{K}(\mathfrak{H}_j))$ given by

$$(\nu_j(\ell,k_1,k_2)T)(\ell') = \pi_j'(\ell'^{-1}\ell\,k_1\ell^{-1}\ell')T(\ell^{-1}\ell')\pi_j'(\ell'^{-1}\ell\,k_2^{-1}\ell^{-1}\ell') \tag{2.22}$$

for $(\ell, k_1, k_2) \in G$, $\ell' \in L$.

The map $\Phi \longmapsto T_\Phi$ of (2.20) is G-equivariant, i.e.,

$$T_{\tau_j(g)\Phi} = \nu_j(g) T_\Phi. \tag{2.23}$$

Furthermore, it is a matter of routine to check that this map is in fact dense (and injective) if both spaces are equipped with the uniform norm.

By means of (2.9) we conclude that the map

$$\mathcal{C}(K \times L/L_\omega) \ni \varphi \longmapsto T_{R_j\varphi} \in \mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j)) \tag{2.24}$$

has a dense image. Moreover, this map is multiplicative, if the multiplication in $\mathcal{C}_{L^jK}\ (L, \mathcal{K}(\mathfrak{H}_j))$ is defined pointwise, and if $\mathcal{C}(K \times L/L_\omega)$ is considered as a subalgebra of the crossed product $\mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$.

Finally, we define the space (C^*–algebra) $\mathcal{D}$ consisting of all functions ψ on $J \times L$ such that the value $\psi(j, \ell)$ is contained in $\mathcal{K}(\mathfrak{H}_j)$, in fact $\psi(j, -) \in \mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j))$, and that ψ vanishes at infinity, i.e., for all $\varepsilon > 0$ there exists a finite subset $J_\varepsilon \subset J$ such that

$$\|\psi(j, \ell)\|_{\mathcal{K}(\mathfrak{H}_j)} < \varepsilon \text{ for all } \ell \in L, j \in J \smallsetminus J_\varepsilon. \tag{2.25}$$

On $\mathcal{D}$, a norm is defined by

$$\|\psi\| = \sup_{(j,\ell) \in J \times L} \|\psi(j, \ell)\|_{\mathcal{K}(\mathfrak{H}_j)}.$$

The operations on $\mathcal{D}$ are defined pointwise, in particular the multiplication.

The previous discussion and (2.9) imply:

Proposition 2.26. *The map $\mathcal{C}(K \times L/L_\omega) \ni \varphi \longmapsto \psi \in \mathcal{D}, \psi(j, \ell) = T_{R_j\varphi}(\ell)$, extends to an isomorphism from $C^*(K, \mathcal{C}(L/L_\omega))$ onto $\mathcal{D}$.* □

Using once more the Lemma of the introduction, resp. its consequence (1.10) we obtain:

Corollary 2.27. *The map $\mathcal{L}^1(K, Q) \ni f \longmapsto \psi \in \mathcal{D}$, $\psi(j, \ell) = (\ell\pi_j)(f)$, yields an isomorphism from $\mathcal{L}^1(K, Q)^{(G)}$ onto $\mathcal{D}^{(G)}$, where the action of G on the various pieces $\mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j))$ of $\mathcal{D}$ is given in (2.22).* □

3 Invariant projectors of finite rank, applications to the ideal theory in $\mathcal{L}^1(K, Q)$

The first purpose of this section is to show that each $(\ell\pi_j)(\mathcal{L}^1(K, Q))$ contains projectors of finite rank, in fact realized by L–invariant functions. Further, we show that $\mathrm{Priv}_*\mathcal{L}^1(K,Q) = \mathrm{Priv}\mathcal{L}^1(K,Q)$, that $\mathcal{L}^1(K,Q)$ has the Wiener property, and that L–orbits are sets of synthesis.

A basic ingredient of the proofs is the fact that a certain subgroup of the unitary group $\mathfrak{U}(\mathfrak{H}_j)$, $j \in J$, is actually compact.

It is easy to check that for any $j \in J$ the subset

$$\mathfrak{U}_j := \{t\, Y_j(\ell)\pi'_j(k) \mid t \in \mathbb{T}, \ell \in L^j, k \in K\} \tag{3.1}$$

of the unitary group $\mathfrak{U}(\mathfrak{H}_j)$ is a subgroup, in fact, the $Y_j(\ell)$ normalize $\pi'_j(K)$, as we observed earlier.

Lemma 3.2. $\mathfrak{U}_j$ *is compact w.r.t. the strong operator topology.*

Proof. Let a sequence (or net) $t_n Y_j(\ell_n)\pi_j(k_n)$ be given. W.l.o.g. we may assume that (ℓ_n) converges to ℓ_0, and that (k_n) converges to k_0. Choosing, as in (2.3), a function c on a neighborhood of ℓ_0, with values in $\mathbb{T}$, one can arrange that $\ell \longmapsto c(\ell)Y_j(\ell)$ is (locally) continuous w.r.t. the strong operator topology.

As $t_n Y_j(\ell_n)\pi_j(k_n) = (c(\ell_n)^{-1}t_n)c(\ell_n)Y_j(\ell_n)\pi_j(k_n)$, passing once more to a subnet we can get that $(c(\ell_n)^{-1}t_n)$ converges in $\mathbb{T}$ to t_0. But then the subnet converges to $t_0 Y_j(\ell_0)\pi_j(k_0)$. □

As a further application of (2.14) (and of the previous lemma which shows that $\mathfrak{H}_j$ decomposes into an orthogonal sum of finite–dimensional $\mathfrak{U}_j$–invariant subspaces) we obtain the following theorem.

Theorem 3.3. *Given $j \in J$ for each $\mathfrak{U}_j$–invariant finite–dimensional subspace $\mathfrak{F}$ of $\mathfrak{H}_j$ there exists an L–fixed G–finite vector $\mathfrak{p} = \mathfrak{p}_{j,\mathfrak{F}}$ in $\mathcal{L}^1(K,Q)$ such that $(\ell\pi_j)(\mathfrak{p}) = \pi_j(\mathfrak{p})$ is the orthogonal projection on $\mathfrak{F}$, and $(\ell\pi_i)(\mathfrak{p}) = 0$ for all $\ell \in L$, $i \in J$, $i \neq j$. Moreover, $\mathfrak{p}^* = \mathfrak{p}$ and $\mathfrak{p} * \mathfrak{p} = \mathfrak{p}$.*

Proof. Denote by $p : \mathfrak{H}_j \to \mathfrak{H}_j$ the orthogonal projection onto $\mathfrak{F}$. From the $\mathfrak{U}_j$–invariance of $\mathfrak{F}$ it follows that the constant map $L \ni \ell \longmapsto p \in \mathcal{K}(\mathfrak{H}_j)$ is contained in $\mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j))$. Hence the function ψ_p on $J \times L$ given by

$$\psi_p(i,\ell) = \delta_{ij}\, p \tag{3.4}$$

is contained in the C^*–algebra $\mathcal{D}$. Actually, ψ_p is a G–finite vector. The details of the proof of this statement are left to the reader. We just remark the following crucial fact. Define a representation of K in $\mathfrak{F}$ by restricting π'_j, i.e.,

$$\widetilde{\pi}_j(k) = \pi'_j(k)\Big|_{\mathfrak{F}}, k \in K. \tag{3.5}$$

Then there is a subgroup $L_{\mathfrak{F}}$ of L of finite index such that $\ell\,\widetilde{\pi}_j$ is unitarily equivalent to $\widetilde{\pi}_j$ for all $\ell \in L_{\mathfrak{F}}$.

Corollary (2.27) delivers a (unique) element $\mathfrak{p}$ in $\mathcal{L}^1(K,Q)^{(G)}$ with $(\ell\pi_i)(\mathfrak{p}) = \psi_p(i,\ell)$ for all $i \in J$, $\ell \in L$. □

Retaining the previous notations, we consider the corner $\psi_p \mathcal{D} \psi_p$ in $\mathcal{D}$, which clearly can be identified with

$$\mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j)) \cap \mathcal{C}(L, \mathcal{B}(\mathfrak{F})) =: \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F})), \tag{3.6}$$

i.e., with the space of all continuous functions $L \to \mathcal{B}(\mathfrak{F})$ satisfying the transformation properties w.r.t. L^jK analogous to (2.19); of course, π'_j has to be replaced by $\widetilde{\pi}_j$.

All the irreducible representations of $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$ are given by evaluation at points $\ell \in L$, in particular, they live in $\mathfrak{F}$, and $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))^\wedge$ can be identified with L/L^jK.

The map

$$\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p} \ni f \longmapsto (\ell \longmapsto (\ell\pi_j)(f)) \in \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F})) = \psi_p \mathcal{D} \psi_p \tag{3.7}$$

yields a dense embedding of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ into the C^*–algebra $\psi_P \mathcal{D} \psi_p$. Further, this map is L–equivariant if we let L act on $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$ by left translations. In particular, the L–finite vectors in the two spaces coincide. We are going to show that the map of (3.7) is actually the C^*–completion of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$. The proof of this rests on the fact that

$$\mathcal{L}^1(K, Q) \ \text{ is symmetric} \tag{3.8}$$

as a "compact extension" of the symmetric algebra Q by [9, Theorem 1]. Here a few words on symmetric Banach algebras are in order, for more information see [2, 11, 14]. (There are also more recent contributions, for instance by Pták, but we do not need these results.) In [11] such algebras are called completely symmetric, in [2] hermitean. An involutive Banach algebra $\mathcal{A}$ is called symmetric if for all $a \in \mathcal{A}$ the spectrum of a^*a is contained in $[0, \infty)$. By the theorem of Shirali–Ford, see [2, p. 226], this is equivalent to the fact that all hermitean elements, i.e., $a^* = a$, have a real spectrum. The latter property is evidently conserved by adding a unit to the algebra. This observation is here important because we wish to use some results of [11], which were there only formulated for algebras with unit. Also we note that closed involutive subalgebras of symmetric algebras are symmetric (the spectra of hermitean elements in the subalgebra can be "computed" by means of [2, Prop. 14, p. 25]. In particular, the subalgebra $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ of $\mathcal{L}^1(K, Q)$ is symmetric.

Proposition 3.9. *The C^*–hull of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ is $\psi_p \mathcal{D} \psi_p \cong \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$.*

Proof. We have to show that each irreducible involutive representation θ of $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ in some Hilbert space $\mathfrak{K}$ can be lifted along the map of (3.7). By the symmetry of $\mathcal{L}^1(K, Q)$ the representation θ can be extended to an irreducible involutive representation μ of $\mathcal{L}^1(K, Q)$ in some Hilbert space $\mathfrak{H} \supset \mathfrak{K}$, compare [11, Thm. 1, p. 311, Ch. V, § 23], i.e., restricting μ to $\mathfrak{p} * \mathcal{L}^1(K, Q) * \mathfrak{p}$ and $\mathfrak{K}$ yields the original θ. By (1.20) there exist $i \in J$ and $\ell \in L$ such that $\mathfrak{H} = \mathfrak{H}_i$ and $\mu = \ell\pi_i$. Since $(\ell\pi_i)(\mathfrak{p} * \mathcal{L}^1(K,Q) * \mathfrak{p}) = 0$ for $i \neq j$, we have to have $i = j$. Moreover, $\mathfrak{K}$ is contained in $\mathfrak{F} = (\ell\pi_j)(\mathfrak{p})$. As $(\ell\pi_j)(\mathcal{L}^1(K,Q))$ is dense in $\mathcal{K}(\mathfrak{H}_j)$, we conclude that $(\ell\pi_j)(\mathfrak{p} * \mathcal{L}^1(K,Q) * \mathfrak{p}) = p(\ell\pi_j)(\mathcal{L}^1(K,Q))p$, which may be considered as a subspace of $\mathcal{B}(\mathfrak{F})$, has in fact to coincide with $\mathcal{B}(\mathfrak{F})$. It follows that $\mathfrak{K} = \mathfrak{F}$, and $\theta = \ell\pi_j\big|_{\mathfrak{p}*\mathcal{L}^1(K,Q)*\mathfrak{p},\mathfrak{F}}$ which clearly implies what we had to show. $\square$

The construction of the element $\mathfrak{p}$ can be made more explicit in terms of kernel functions. Actually, the following considerations will be a little more general giving the kernel functions for all elements in $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$ (still j and $\mathfrak{F}$ are fixed).

For $k \in K$ define $B(k) : \mathfrak{F} \to V_j$ by $B(k)\xi = \xi(k)$. This map is well–defined as $\mathfrak{F}$ consists of continuous functions, because $\mathfrak{F}$ is finite–dimensional and invariant under $\pi'_j(K)$. The latter invariance and the transformation property of the members of $\mathfrak{H}_j = \mathcal{L}^2_{K_\omega}(K, V_j)$ yield

$$B(ku) = \alpha_j(u)^{-1}B(k), B(ak) = B(k)\widetilde{\pi}_j(a)^{-1} \tag{3.10}$$

for all $a, k \in K$, $u \in K_\omega$.

In particular, the operator $B(k)$ depends continuously on $k \in K$.

Moreover, the invariance of $\mathfrak{F}$ under $Y_j(\ell), \ell \in L^j$, leads to

$$B(\ell^{-1}k\,\ell) = X_j(\ell)^*B(k)Y_j(\ell), \ell \in L^j. \tag{3.11}$$

For the definition of $Y_j(\ell)$ see (2.18).

With the family $B(k)$ of operators we also have their adjoints $B(k)^* : V_j \to \mathfrak{F}$, and it is easy to check that the projection $p : \mathfrak{H}_j \to \mathfrak{F}$ may be written as

$$p\,\xi = \int_{K/K_\omega} B(k)^*\xi(k)dk. \tag{3.12}$$

This means that p is given by the kernel function

$$F(a,b) = B(a)B(b)^*, a, b \in K. \tag{3.13}$$

The above transformation laws for B imply

$$F(ka,kb) = F(a,b), F(\ell^{-1}a\ell, \ell^{-1}b\ell) = X_j(\ell)^*F(a,b)X_j(\ell) \tag{3.14}$$

for $a, b, k \in K$, $\ell \in L^j$.

From (3.14) it follows that the function $\Phi : G \to \mathcal{B}(V_j)$ defined by $\Phi(\ell,a,b) = F(a,b)$ is contained in $\mathcal{C}_{\Delta,L^j,K_\omega,K_\omega}(G, \mathcal{B}(V_j))$, see (2.4). Thus, the function Φ yields $\psi_p \in \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$, which is nothing but the constant function with value $\mathrm{Id}_{\mathfrak{F}}$, and we are looking for $\varphi \in \mathcal{C}(K \times L/L_\omega)$ with $R_j\varphi = \Phi$. More generally, we take any $A \in \mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$, and form $\Phi_A : G \to \mathcal{B}(V_j)$,

$$\Phi_A(\ell,a,b) = B(a)A(\ell)B(b)^*. \tag{3.15}$$

The operator given by this kernel, i.e., $\xi \longmapsto (a \longmapsto \int_{K/K_\omega} \Phi(\ell, a, b)\xi(b)db)$ is just $A(\ell)$. By a straightforward computation it follows from (3.10), (3.11) and the definition of $\mathcal{C}_{L^jK}(L, \mathcal{B}(V_j))$, see (3.6), that $\Phi_A \in \mathcal{C}_{\Delta, L^j, K_\omega, K_\omega}(G, \mathcal{B}(V_j))$. Of course, in general Φ_A is not a G–finite vector; it is if the left L–translates of A are sitting in a finite–dimensional subspace, as it happens for instance in the case $A \equiv \mathrm{Id}_{\mathfrak{F}}$, which corresponds to the projector ψ_p. To find a $\varphi_A \in \mathcal{C}(K \times L/L_\omega)$ with $R_j\varphi_A = \Phi_A$ we use the recipe of (2.10) – (2.13). Again, for short we define $\Psi : H' \to \mathcal{B}(V_j)$ by

$$\Psi = (\tilde{\gamma} \circ \rho)(\Phi), \Psi(k, \ell) = \Phi_A(\ell, \ell^{-1}k\ell, 1) = B(\ell^{-1}k\ell)A(\ell)B(1)^*,$$

and get

$$(3.16) \qquad \begin{aligned} \varphi_A(k, [t]) &= \sum_{\ell \in L_\omega/L^j} n \,\mathrm{Tr}\, \Psi(k, t\,\ell) \\ &= \sum_{\ell \in L_\omega/L^j} n \,\mathrm{Tr}\, (B(\ell^{-1}t^{-1}k\,t\,\ell)A(t\,\ell)B(1)^*), \end{aligned}$$

where $n = \dim V_j$.

In particular, if $\mathfrak{p}' \in \mathcal{L}^1(K, \mathcal{C}(L/L_\omega))$ denotes the element corresponding to $\mathfrak{p} \in \mathcal{L}^1(K, Q)$, then $\mathfrak{p}'$ is a continuous function on $K \times L/L_\omega$ given by $\mathfrak{p}'(k, [t]) = \sum_{\ell \in L_\omega/L^j} n \,\mathrm{Tr}(B(\ell^{-1}t^{-1}k\,t\,\ell)B(1)^*)$. Our previous discussion yields the following result.

Proposition 3.17. *The subalgebra $\mathfrak{p}' * \mathcal{L}^1(K, \mathcal{C}(L/L_\omega)) * \mathfrak{p}'$ is equal to $\mathfrak{p}' * \mathcal{C}(K \times L/L_\omega) * \mathfrak{p}'$ and isomorphic to the C^*–algebra $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}))$.* □

Next, we are going to show that $\mathrm{Priv}_* \mathcal{L}^1(K, Q) = \mathrm{Priv}\, \mathcal{L}^1(K, Q)$, where $\mathrm{Priv}\, \mathcal{L}^1(K, Q)$ denotes the collection of primitive ideals in $\mathcal{L}^1(K, Q)$. Recall that for any (complex) algebra $\mathcal{A}$ a primitive ideal is, by definition, the annihilator of a simple left $\mathcal{A}$–module E. Simple means that $\mathcal{A}\,\xi = E$ for all non–zero $\xi \in E$; for a little more information on this notion see for instance the first pages of [12].

It is a general fact for symmetric Banach algebras $\mathcal{A}$ that $\mathrm{Priv}\, \mathcal{A}$ is contained in $\mathrm{Priv}_* \mathcal{A}$, because each simple module E can be "unitarized", see [11, Cor. 1, p. 307], i.e., there exists an irreducible involutive representation π in some Hilbert space and a non–zero $\mathcal{A}$–intertwining operator $E \to \mathfrak{H}$, necessarily with dense image, which implies that the annihilator of E is equal to $\ker \pi$.

On the other hand, if π is an irreducible involutive representation of an involutive Banach algebra $\mathcal{A}$ in $\mathfrak{H}$ such that $\pi(\mathcal{A})$ contains at least one non–zero operator of finite rank, then one may form the two–sided ideal $\mathcal{I}_\pi := \{a \in \mathcal{A} \mid \pi(a) \text{ is of finite rank }\}$; and, according to the arguments of [3, Théorème 2], the (dense) subspace $\mathfrak{H}' := \mathcal{I}_\pi\mathfrak{H}$ of $\mathfrak{H}$ is a simple $\mathcal{A}$–module, whose annihilator equals $\ker \pi$.

Since the above assumptions are met by $\mathcal{L}^1(K, Q)$ we obtain the following proposition. For this, it is not necessary to use the above constructed L–invariant projections, because in fact **all** G–finite vectors in $\mathcal{L}^1(K,Q)$ yield finite rank operators when represented irreducibly.

Proposition 3.18. $\operatorname{Priv}\mathcal{L}^1(K,Q) = \operatorname{Priv}_*\mathcal{L}^1(K,Q)$. □

3.19. *For each $i \in J$ we choose and fix a $\mathfrak{U}_i$–invariant, for the definition of $\mathfrak{U}_i$ see (3.2), finite–dimensional subspace $\mathfrak{F}_i$ of $\mathfrak{H}_i$, to which we find $\mathfrak{p}_i \in \mathcal{L}^1(K,Q)$ according to (3.3).*

Next, we prove the Wiener property of $\mathcal{L}^1(K,Q)$; for more information on this notion see [8, 10].

Theorem 3.20. *If $\mathcal{I}$ is a proper closed two–sided ideal in $\mathcal{L}^1(K,Q)$ then there exist $\ell \in L$ and $j \in J$ such that $\mathcal{I}$ is contained in $\ker_{\mathcal{L}^1(K,Q)} \ell\pi_j$.*

Proof. For short, put $\mathcal{A} := \mathcal{L}^1(K,Q)$. Suppose to the contrary that for any ℓ, j the image $(\ell\pi_j)(\mathcal{I})$ is non–zero. Then the closure of $(\ell\pi_j)(\mathcal{I})$ in $\mathcal{K}(\mathfrak{H}_j)$ is a non–trivial ideal, hence equal to $\mathcal{K}(\mathfrak{H}_j)$.

For any $i \in J$, fixed for the moment, consider the (closed) ideal $\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i = \mathcal{I} \cap \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$ in $\mathcal{A}_i := \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$. For any $\ell \in L$ the image $(\ell\pi_i)(\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i)$ is dense in $\mathcal{B}(\mathfrak{F}_i)$, hence equal to $\mathcal{B}(\mathfrak{F}_i)$. In particular, there exists $g_\ell \in \mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ with $(\ell\pi_i)(g_\ell) = \mathrm{Id}_{\mathfrak{F}_i}; f_\ell := g_\ell^* * g_\ell$ has the same properties.

By continuity there exists a neighborhood W_ℓ of ℓ such that $\langle(\ell'\pi_i)(f_\ell)\xi,\xi\rangle \geq \frac{1}{2} < \xi, \xi >$ for all $\xi \in \mathfrak{F}_i$, $\ell' \in W_\ell$. Using the compactness of L we find a finite cover $(W_\mu)_{1\leq\mu\leq m}$ of L and positive elements f_μ in $\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ with $\langle(\ell\pi_i)(f_\mu)\xi,\xi\rangle \geq \frac{1}{2} < \xi, \xi >$ for all $\xi \in \mathfrak{F}_i$, $\ell \in W_\mu$. The element $f := \sum_{\mu=1}^m f_\mu \in \mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ has the property that

$$\langle(\ell\pi_i)(f)\xi,\xi\rangle \geq \frac{1}{2} < \xi, \xi > \text{ for all } \ell \in L, \xi \in \mathfrak{F}_i. \tag{3.21}$$

We claim that f is invertible in the algebra $\mathcal{A}_i$ with unit $\mathfrak{p}_i$. Suppose to the contrary that 0 is in the spectrum of f. Then 0 is in the left and in the right spectrum of f as $(\mathcal{A}_i f)^* = f^*\mathcal{A}_i = f\mathcal{A}_i$, f being hermitean. By [11, Ch. V, § 23, p. 311, V.] there exist an irreducible involutive representation θ of $\mathcal{A}_i$ in some Hilbert space $\mathfrak{K}$, and a non–zero $\eta \in \mathfrak{K}$ with $\theta(f)\eta = 0$. Observe that $\mathcal{A}_i$ is symmetric as a closed involutive subalgebra of $\mathcal{A}$. Above, (3.7) and (3.9), we computed the irreducible representations of $\mathcal{A}_i$. Hence, we may suppose that $\mathfrak{K} = \mathfrak{F}_i$ and $\theta = \ell\pi_i|_{\mathcal{A}_i,\mathfrak{F}_i}$ for some $\ell \in L$. The equation $\theta(f)\eta = 0$ contradicts (3.21).

Since $f \in \mathfrak{p}_i*\mathcal{I}*\mathfrak{p}_i$ is invertible in $\mathcal{A}_i$ it follows that $\mathfrak{p}_i \in \mathcal{I}$ (for all $i \in J$), hence $\mathcal{P} := \{\sum_{i\in J} \mathcal{A} * \mathfrak{p}_i * \mathcal{A}\}^- \subseteq \mathcal{I}$. The image of the two–sided ideal $\mathcal{P}$ in the C^*–hull $C^*(\mathcal{A}) = C^*(K, \mathcal{C}(L/L_\omega))$ is dense, because there are no (irreducible) involutive representations annihilating $\mathcal{P}$ (all C^*–algebras have the Wiener property). Again, by (1.10) and the Lemma in the introduction, observe that $\mathcal{P}$ is G–invariant, $\mathcal{P}^{(G)} = C^*(K, \mathcal{C}(L, L_\omega))^{(G)} = \mathcal{A}^{(G)}$. But $\mathcal{A}^{(G)}$ is dense in $\mathcal{A}$, whence $\mathcal{A} = \mathcal{P} = \mathcal{I}$. □

The proof that L–orbits are sets of synthesis follows the same trail.

Theorem 3.22. *Let $\mathcal{I}$ be a closed two–sided ideal in $\mathcal{A} = \mathcal{L}^1(K, Q)$. Suppose that the hull $h(\mathcal{I})$, i.e., $h(\mathcal{I}) := \{\mathcal{P} \in \mathrm{Priv}_*\mathcal{A} \mid \mathcal{P} \supseteq \mathcal{I}\}$, is equal to $\{\ker_{\mathcal{A}} \ell\pi_j \mid \ell \in L\}$ for a certain $j \in J$. Then $\mathcal{I} = \bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j$.*

Proof. Clearly, $\mathcal{I}$ is contained in $\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j$. Fix, for the moment, $i \in J$, $i \neq j$. By the definition of $h(\mathcal{I})$, for each $\ell \in L$ there exists $g \in \mathcal{I}$ with $(\ell\pi_i)(g) \neq 0$. Again, consider the ideal $\mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i = \mathcal{I} \cap \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$ in $\mathcal{A}_i = \mathfrak{p}_i * \mathcal{A} * \mathfrak{p}_i$.

As above one finds an element $f \in \mathfrak{p}_i * \mathcal{I} * \mathfrak{p}_i$ with $\langle (\ell\pi_i)(f)\xi, \xi\rangle \geq \frac{1}{2} < \xi, \xi >$ for all $\ell \in L$, $\xi \in \mathfrak{F}_i$, from which one concludes again that f is invertible in $\mathcal{A}_i$, whence $\mathfrak{p}_i \in \mathcal{I}$.

Therefore, the G–invariant ideal $\mathcal{P} := \left\{\sum_{\substack{i\in J\\ i\neq j}} \mathcal{A} * \mathfrak{p}_i * \mathcal{A}\right\}^-$ is contained in $\mathcal{I}$. Consider the closure $\mathcal{P}'$ of the image of $\mathcal{P}$ in $C^*(\mathcal{A}) = C^*(K, \mathcal{C}(L/L_\omega))$, which is also G–invariant, and invariant under the involution. Since all closed subsets of the primitive spectrum of any C^*–algebra are "sets of synthesis" we conclude that

$$\mathcal{P}' = \bigcap_{\ell\in L} \ker_{C^*(\mathcal{A})} \ell\pi_j.$$

We have the dense G–equivariant injections

$$\begin{array}{rcl} \mathcal{P} & \longrightarrow & \mathcal{P}', \text{ and} \\ \bigcap\limits_{\ell\in L} \ker_{\mathcal{A}} \ell\pi'_j & \longrightarrow & \mathcal{P}'. \end{array}$$

Again, by (1.10) and the Lemma in the introduction, the collections of their respective G–finite vectors coincide. In particular, $\mathcal{P}^{(G)} = \{\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j\}^{(G)}$. But $\{\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j\}^{(G)}$ is dense in $\bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j$, whence $\mathcal{P} = \bigcap_{\ell\in L} \ker_{\mathcal{A}} \ell\pi_j \subseteq \mathcal{I}$. $\square$

For illustration and later applications we consider the special case that L is a direct product $L = S \times K$, where S is a compact abelian Lie group, not necessarily connected. As we will see, in this case the collection of groups L^j, $j \in J$, is finite, and a little more can be said on the structure of the C^*–algebras $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$.

One has two projections $\mathrm{pr}_1 : L \to S$, $\mathrm{pr}_2 : L \to K$. With the subgroup L_ω of L there are associated four canonical subgroups, namely

$$(3.23)\qquad \begin{array}{rcl} S_\omega & = & L_\omega \cap S, K_\omega = K \cap L_\omega, S' = \mathrm{pr}_1(L_\omega) = S \cap L_\omega K, \\ K' & = & \mathrm{pr}_2(L_\omega) = K \cap SL_\omega. \end{array}$$

The subgroup K_ω is normal in K', hence $S_\omega \times K_\omega$ is normal in $S' \times K'$. The image of L_ω in $S'/S_\omega \times K'/K_\omega$ under the canonical map $\mu : S' \times K' \to S'/S_\omega \times K'/K_\omega$ is the graph of a certain isomorphism

(3.24)

$\kappa : S'/S_\omega \to K'/K_\omega$, and L_ω is its pre–image, $L_\omega = \mu^{-1}\{(s, \kappa(s)) \mid s \in S'/S_\omega\}$.

In particular, also K'/K_ω is a compact abelian Lie group. The connected component $(S'/S_\omega)_\circ$ of S'/S_ω will be written as S°/S_ω with a certain subgroup S° of S' of finite index. Likewise, we have $(K'/K_\omega)_\circ = K^\circ/K_\omega$, and clearly κ map S°/S_ω onto K°/K_ω.

If $j \in J$ is given, the subgroup L^j is of finite index in L_ω, and L^j contains $S_\omega \times L_\omega$, hence

$$L^j = \mu^{-1}\{(s, \kappa(s)) \mid s \in S^j/S_\omega\} \tag{3.25}$$

for a certain group S^j, $S_\omega \subset S^j \subset S'$. As L^j is of finite index in L_ω, the group S^j has to contain S°. Thus, we see that $\{L^i \mid i \in J\}$ is a finite collection. Moreover, we define K^j by $\kappa(S^j/S_\omega) = K^j/K_\omega$. The group K^j can also be described as the stabilizer in K' of $\alpha_j \in K_\omega^\wedge$. Next we choose a measurable projective extension $\widetilde{\alpha}_j$ of α_j to K^j with a measurable cocycle $m_j : K^j/K_\omega \times K^j/K_\omega \to \mathbb{T}$,

$$\widetilde{\alpha}_j : K^j \longrightarrow \mathcal{B}(V_j), \widetilde{\alpha}_j(x)\widetilde{\alpha}_j(y) = m_j(x,y)\widetilde{\alpha}_j(x\,y). \tag{3.26}$$

The cocycles on compact abelian Lie groups are very well known; replacing m_j by a cohomologous one (and modifying $\widetilde{\alpha}_j$ accordingly), we may assume that m_j lives on the finite group K^j/K°. In particular, $\widetilde{\alpha}_j$ is then continuous.

The representation $\widetilde{\alpha}_j$ delivers an m_j–projective representation Z_j of K^j/K_ω in $\mathfrak{H}_j = \mathcal{L}^2_{K_\omega}(K, V_j)$ by

$$(Z_j(a)\xi)(k) = \widetilde{\alpha}_j(a)\xi(ka) \tag{3.27}$$

for $k \in K$ and $a \in K^j$ (or $a \in K^j/K_\omega$, clearly Z_j is constant on K_ω–cosets), and hence also a representation A_j of the isomorphic copy S^j/S_ω,

$$A_j(s) = \chi(s)Z_j(\kappa(s)) \tag{3.28}$$

for $s \in S^j/S_\omega$ with a certain unitary character χ on S^j/S_ω to be determined later. Also A_j is projective, the cocycle m_j' on S^j/S_ω being given by $m_j'(s,t) = m_j(\kappa(s), \kappa(t))$.

The intertwining operators $X_j(\ell)$, $Y_j(\ell)$, $\ell \in L^j$ in V_j resp. $\mathfrak{H}_j$, see (2.2), (2.18), can now be specified to be

$$X_j(\ell) = \widetilde{\alpha}_j(b)\chi(s) \;\text{ if }\; \ell = (s,b) \in L^j, (Y_j(\ell)\xi)(k) = X_j(\ell)\xi(\ell^{-1}k\,\ell). \tag{3.29}$$

The crucial properties of the representation A_j are the following.

3.30. *For any $s \in S^j/S_\omega$ and any $k \in K$ the operators $A_j(s)$ and $\pi_j'(k)$ commute. If $\ell = (s,b) \in L^j$ and $a \in K$, whence $\ell\,a = (s, b\,a)$, then*

$$Y_j(\ell)\pi_j'(a) = A_j(s)\pi_j'(b\,a).$$

The straightforward computations are omitted. This implies that the functions $f \in \mathcal{C}_{L^jK}(L, \mathcal{K}(\mathfrak{H}_j))$, see (2.19), can alternatively be described as those functions which are constant on S_ω–cosets, and which satisfy

$$(3.31) \qquad f(\ell(s,k)) = \pi'_j(k)^* A_j(s)^* f(\ell) A_j(s) \pi'_j(k) \text{ for } \ell \in L, k \in K, s \in S^j.$$

Finally, we choose, as above, a finite–dimensional $\mathfrak{U}_j$–invariant, see (3.1), subspace $\mathfrak{F}_j$ of $\mathfrak{H}_j$, which we now assume to be irreducible. $\mathfrak{U}_j$-invariant means, by (3.30), to be invariant under $A_j(S^j/S_\omega)$ and under $\pi'_j(K)$. As these two groups commute, $\mathfrak{F}_j$ decomposes into a tensor product,

3.32. $\mathfrak{F}_j = \mathfrak{P}_j \otimes \mathfrak{Q}_j$, *where* $\mathfrak{P}_j$ *is an irreducible* S^j/S_ω*–space, and* $\mathfrak{Q}_j$ *is an irreducible* K*–space.*

Since the cocycle m'_j lives on S^j/S°, the operators $A_j(s)$, $s \in S^\circ/S_\omega$, are scalar on $\mathfrak{P}_j$. Adapting the above χ to the chosen subspace $\mathfrak{F}_j$ we can arrange that $A_j(S^\circ/S_\omega)$ is trivial on $\mathfrak{P}_j$.

As we have seen earlier, (3.9), the C^*–hull of $\mathfrak{p}_j * \mathcal{L}^1(K,Q) * \mathfrak{p}_j$ is $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$, which is given analogous to (3.31), compare (2.19), (3.6). The functions in the latter algebra are determined by their restriction to S, i.e., $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$ can be identified with

$$(3.33) \qquad \begin{aligned} &\mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j \otimes \mathfrak{Q}_j)) \\ &\quad = \{f : S \to \mathcal{B}(\mathfrak{P}_j \otimes \mathfrak{Q}_j) \mid f \text{ is continuous, } f(st) = \\ &\quad\qquad A_j(t)^* \otimes \mathrm{Id}_{\mathfrak{Q}_j} \circ f(s) \circ A_j(t) \otimes \mathrm{Id}_{\mathfrak{Q}_j} \text{ for } s \in S, t \in S^j\} \\ &\quad \cong \mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j)) \otimes \mathcal{B}(\mathfrak{Q}_j), \end{aligned}$$

where $\mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j))$ has the obvious meaning. A tensor $\varphi \otimes B$ is mapped to the function $s \longmapsto \varphi(s) \otimes B$. Actually, the functions in $\mathcal{C}_{S^j}(S, \mathcal{B}(\mathfrak{P}_j))$ are constant on S°–cosets, i.e., this space may be written as $\mathcal{C}_{S^j/S^\circ}(S/S^\circ, \mathcal{B}(\mathfrak{P}_j))$. One should note that

3.34. *the* S*–isotypical components in* $\mathcal{C}_{S^j/S^\circ}(S/S^\circ, \mathcal{B}(\mathfrak{P}_j)) \otimes \mathcal{B}(\mathfrak{Q}_j), S$ *acts by left translations, are clearly finite–dimensional, whence they coincide with the* S*–isotypical components in* $\mathfrak{p}_j * \mathcal{L}^1(K,Q) * \mathfrak{p}_j$.

Also, the L–action on $\mathcal{C}_{L^jK}(L, \mathcal{B}(\mathfrak{F}_j))$ can easily be transferred into the new picture. An $\ell = (t,k) \in S \times K$ acts on the tensor $\varphi \otimes B \in \mathcal{C}_{S^j/S^\circ}(S/S^\circ, \mathcal{B}(\mathfrak{P}_j)) \otimes \mathcal{B}(\mathfrak{Q}_j)$ by $\ell(\varphi \otimes B) = \varphi' \otimes B'$, where $\varphi'(s) = \varphi(t^{-1}s)$, $B' = \widetilde{\pi}_j(k) \circ B \circ \widetilde{\pi}_j(k)^*$.

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Universität Bielefeld
Fakultät für Mathematik
Postfach 100131
33501 Bielefeld
Germany
email: poguntke@mathematik.uni-bielefeld.de