Travaux mathématiques, Volume 16 (2005), 229–236, © Université du Luxembourg

Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{g}$ is a real simple Lie algebra of real rank one

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Abstract

We classify Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$, where $\mathfrak{g}$ is a real simple Lie algebra of real rank 0 or 1. We also apply our results to classification of quasi-Poisson homogeneous spaces.

1 Introduction

Let G be a (quasi-) Poisson Lie group, $\mathfrak{g} = \operatorname{Lie} G$ the corresponding Lie (quasi-) bialgebra, $D(\mathfrak{g})$ the double corresponding to $\mathfrak{g}$. A subalgebra $\mathfrak{l} \subset D(\mathfrak{g})$ is called Lagrangian if $\mathfrak{l}$ is a maximal isotropic subspace with respect to the natural scalar product in $D(\mathfrak{g})$. Denote by Λ the set of all Lagrangian subalgebras in $D(\mathfrak{g})$. Let M be a G-homogeneous space. It follows from [2] and [6] that a (quasi-) Poisson G-homogeneous structure on M is equivalent to a G-equivariant map $M \to \Lambda$, $m \mapsto \mathfrak{l}_m$ such that $\mathfrak{l}_m \cap \mathfrak{g} = \mathfrak{g}_m$, where $\mathfrak{g}_m$ is the Lie algebra of the stabilizer subgroup of G at m. Thus in order to describe the set of (quasi-) Poisson G-homogeneous spaces up to local isomorphism it is enough to describe G-conjugacy classes of Lagrangian subalgebras in $D(\mathfrak{g})$.

Let $\mathfrak{g}$ be a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, and G a corresponding connected Lie group. Consider $D(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$ with the scalar product given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \frac{1}{2} \left(\langle x_1, y_1 \rangle_{\mathfrak{g}} - \langle x_2, y_2 \rangle_{\mathfrak{g}} \right), \tag{1.1}$$

where $x_1, x_2, y_1, y_2 \in \mathfrak{g}$. The structure of Λ in the case $\mathfrak{g}$ is complex simple was studied in [3, 5]. In this paper we describe orbits of diagonal G-action on Λ for the case $\mathfrak{g}$ is a real simple Lie algebra of real rank 0 (i.e. compact) or 1.

In Section 1 we discuss a structure of Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ for an arbitrary Lie algebra $\mathfrak{g}$ with an invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. In Section 2 the case of compact connected Lie group G is considered. First, we describe G-orbits of Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ (see Subsection 2.1). In Subsection 2.2 we give a description of the corresponding quasi-Poisson homogeneous G-spaces. Section 3 is devoted to simple Lie algebras $\mathfrak{g}$ of real rank one. We classify G-orbits on the set of all Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ (see Theorem 4.3 for the main result).

We are grateful to the referee for valuable comments that helped to improve the paper.

2 Generalities on Lagrangian subalgebras in $\mathfrak{g}\times\mathfrak{g}$

Let G be a connected Lie group, and $\mathfrak{g} = \operatorname{Lie} G$ is equipped with a non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Let us consider the Lie algebra $\mathfrak{g} \times \mathfrak{g}$ equipped with the invariant symmetric bilinear form (1.1).

Definition 2.1. A Lie subalgebra $\mathfrak{l} \subset \mathfrak{g} \times \mathfrak{g}$ is said to be *Lagrangian* if $\dim \mathfrak{l} = \dim \mathfrak{g}$, and $\mathfrak{l}$ is *isotropic*, i.e. $\langle x, y \rangle = 0$ for all $x, y \in \mathfrak{l}$.

Definition 2.2. A Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is called *coisotropic* if $\mathfrak{c}^{\perp} \subset \mathfrak{c}$.

If a subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is coisotropic, then $\mathfrak{c}^{\perp}$ is an ideal in $\mathfrak{c}$, and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ induces a non-degenerate invariant symmetric bilinear form on $\mathfrak{c}/\mathfrak{c}^{\perp}$.

Proposition 2.1. *The set of Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ is in a natural G-equivariant bijection with the set of all triples $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$, where $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are coisotropic subalgebras in $\mathfrak{g}$, and $\varphi : \mathfrak{c}_1/\mathfrak{c}_1^{\perp} \to \mathfrak{c}_2/\mathfrak{c}_2^{\perp}$ is an isomorphism preserving the form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.*

Proof. Let $\mathfrak{l} \subset \mathfrak{g} \times \mathfrak{g}$ be a Lagrangian subalgebra. Consider the projections $p_i : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $i = 1, 2$, given by $p_1(x, y) = x$, $p_2(x, y) = y$, $(x, y) \in \mathfrak{g}\times\mathfrak{g}$. Set $\mathfrak{c}_i = p_i(\mathfrak{l})$. Because of maximality of $\mathfrak{l}$, we see that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are coisotropic subalgebras. Let us consider the map $\varphi : \mathfrak{c}_1/\mathfrak{c}_1^{\perp} \to \mathfrak{c}_2/\mathfrak{c}_2^{\perp}$ given by $\varphi(x + \mathfrak{c}_1^{\perp}) = y + \mathfrak{c}_2^{\perp}$ for any $(x, y) \in \mathfrak{l}$. Then φ is a well-defined isomorphism of vector spaces. Moreover, φ is an isomorphism of Lie algebras, because $\mathfrak{l}$ is a subalgebra. Further, the subalgebra $\mathfrak{l}$ is Lagrangian, therefore φ preserves the bilinear form. Thus we get the triple $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ with the required properties.

Conversely, starting from a triple $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ one can define

$$\mathfrak{l} = \{(x, y) \,|\, x \in \mathfrak{c}_1,\ y \in \mathfrak{c}_2,\ \varphi(x + \mathfrak{c}_1^{\perp}) = y + \mathfrak{c}_2^{\perp}\}.$$

It can be easily checked that $\mathfrak{l}$ is an isotropic subalgebra. Further, $\dim \mathfrak{l} = \dim \mathfrak{c}_1/\mathfrak{c}_1^{\perp} + \dim \mathfrak{c}_1^{\perp} + \dim \mathfrak{c}_2^{\perp} = \dim \mathfrak{c}_1 + \dim \mathfrak{c}_2^{\perp} = \dim \mathfrak{g}$, thus $\mathfrak{l}$ is maximal.

Let $\mathfrak{l}$ be a Lagrangian subalgebra and $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ be the corresponding triple. Under the construction above the natural (diagonal) action of G on the set of Lagrangian subalgebras turns into $g \cdot (\mathfrak{c}_1, \mathfrak{c}_2, \varphi) = (\operatorname{Ad} g \cdot \mathfrak{c}_1, \operatorname{Ad} g \cdot \mathfrak{c}_2, \operatorname{Ad} g \circ \varphi \circ \operatorname{Ad} g^{-1})$. $\square$

3 Compact case

Let G be a connected compact semisimple Lie group, $\mathfrak{g} = \operatorname{Lie} G$, and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ a (positive or negative) definite invariant symmetric bilinear form on $\mathfrak{g}$. Denote by $\operatorname{Aut} \mathfrak{g}$ the group of all automorphisms of $\mathfrak{g}$ preserving the form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

3.1 Lagrangian subalgebras

The aim of this section is to describe G-conjugacy classes of Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$. We use Proposition 2.1. Since $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is definite, the only coisotropic subalgebra of $\mathfrak{g}$ is $\mathfrak{g}$ itself. Thus any Lagrangian subalgebra of $\mathfrak{g} \times \mathfrak{g}$ is of the form $\mathfrak{l}_\varphi = \{(x, \varphi(x)) \mid x \in \mathfrak{g}\}$, where $\varphi \in \operatorname{Aut} \mathfrak{g}$. Therefore to obtain a description of G-orbits in the set of Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$ it is enough to classify $\operatorname{Int} \mathfrak{g}$-conjugacy classes of $\operatorname{Aut} \mathfrak{g}$. For reader's convenience we present here the well-known answer to the latter question.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ and τ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ w.r.t. $\mathfrak{g}$. We identify automorphisms of $\mathfrak{g}$ with automorphisms of $\mathfrak{g}_{\mathbb{C}}$ commuting with τ.

Let f_i, e_i, h_i be a standard generating system of $\mathfrak{g}_{\mathbb{C}}$ such that $\tau(e_i) = -f_i$. Consider the Cartan subalgebra $\mathfrak{h}$ spanned by $h_1, \ldots, h_r$, where $r = \operatorname{rank} \mathfrak{g}_{\mathbb{C}}$. Let Δ be the set of simple roots of $\mathfrak{g}_{\mathbb{C}}$ corresponding to $h_1, \ldots, h_r$. Let $\Pi : \operatorname{Aut} \mathfrak{g}_{\mathbb{C}} \to \operatorname{Aut} \Delta$ be the canonical homomorphism. For any $\sigma \in \operatorname{Aut} \Delta$, set $\Theta_\sigma = \{\theta \in \Pi^{-1}(\sigma) \mid \theta\tau = \tau\theta\}$.

Let $\widetilde{\sigma} \in \operatorname{Aut} \mathfrak{g}_{\mathbb{C}}$ be defined by $\widetilde{\sigma}(e_i) = e_{\sigma(i)}$, $\widetilde{\sigma}(f_i) = f_{\sigma(i)}$, $\widetilde{\sigma}(h_i) = h_{\sigma(i)}$. Clearly, $\widetilde{\sigma} \in \Theta_\sigma$. Set $H = \{\exp(\operatorname{ad} x) \mid x \in \mathfrak{h}\}$. Let us identify H and $(\mathbb{C}^*)^\Delta$, i.e. $h \in H$ is identified with the set $\{h_\alpha\}_{\alpha \in \Delta}$ of eigenvalues of h on the root spaces $\mathfrak{g}_\alpha$. Choose a system Δ' of representatives of the σ-orbits in Δ. Set

$$H' = \{h \in H \mid h\tau = \tau h, \text{ and } h_\alpha = 1 \text{ for any } \alpha \notin \Delta'\}.$$

Theorem 3.1. *If* $\theta \in \Theta_\sigma$*, then there exists a unique* $h \in H'$ *such that* $g\theta g^{-1} = \widetilde{\sigma} h$ *for some* $g \in \operatorname{Int} \mathfrak{g}$*.* □

3.2 Quasi-Poisson homogeneous G-spaces

Our further investigations are based on the correspondence between (quasi-) Poisson homogeneous spaces and Lagrangian subalgebras (see [2] and [6] for details).

Let us consider the Manin quasi-triple $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}_\Delta, \mathfrak{g}_{-\Delta})$, where $\mathfrak{g} \times \mathfrak{g}$ is equipped with the invariant symmetric bilinear form (1.1), $\mathfrak{g}_\Delta = \{(x, x) \mid x \in \mathfrak{g}\}$, and $\mathfrak{g}_{-\Delta} = \{(x, -x) \mid x \in \mathfrak{g}\}$.[1] The corresponding Lie quasi-bialgebra structure on $\mathfrak{g}$ is given by $\delta = 0$, $\varphi = -[\Omega^{12}, \Omega^{23}]$, where $\Omega \in S^2\mathfrak{g}$ corresponds to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ (see details in [1, 6]). In this case the quasi-Poisson structure on G is given by $\pi_G = 0$ and φ aforesaid.

[1] Notice that $\mathfrak{g}_{-\Delta}$ is not a subalgebra in $\mathfrak{g} \times \mathfrak{g}$. It is easy to see that there is no subalgebra which is a Lagrangian complement of $\mathfrak{g}_\Delta$ in $\mathfrak{g} \times \mathfrak{g}$.

Theorem 3.2. *The set of G-conjugacy classes in* $\operatorname{Aut}\mathfrak{g}$*, where each class is equipped with the quasi-Poisson structure given by*

$$\pi(\theta) = ((r_\theta)_* \otimes (l_\theta)_* - (l_\theta)_* \otimes (r_\theta)_*)(\Omega), \tag{3.1}$$

$\theta \in \operatorname{Aut}\mathfrak{g}$*, is the complete system of representatives of quasi-Poisson G-homogeneous spaces up to local isomorphism.*

Proof. Let $\theta \in \operatorname{Aut}\mathfrak{g}$. Consider the Lagrangian subalgebra $\mathfrak{l}_\theta \subset \mathfrak{g} \times \mathfrak{g}$ defined by $\mathfrak{l}_\theta = \{(x, \theta(x)) \mid x \in \mathfrak{g}\}$. Then $\mathfrak{l}_\theta \cap \mathfrak{g}_\Delta \simeq \mathfrak{g}^\theta = \{x \in \mathfrak{g} \mid \theta(x) = x\}$. Consider $H_\theta = \{g \in G \mid \operatorname{Ad} g \cdot \theta \cdot \operatorname{Ad} g^{-1} = \theta\}$. It is clear that H_θ normalizes $\mathfrak{l}_\theta$, and $\operatorname{Lie} H_\theta = \mathfrak{g}^\theta$. By Theorem 3.2 in [6] we can conclude that the pair $(\mathfrak{l}_\theta, H_\theta)$ corresponds to the quasi-Poisson homogeneous space $O(\theta)$ which is the G-conjugacy class of θ in $\operatorname{Aut}\mathfrak{g}$. To finish the proof one has to show that the corresponding quasi-Poisson structure on $O(\theta)$ is given by (3.1). This can be calculated straightforwardly. □

4 Rank one case

Let $\mathfrak{g}$ be a simple Lie algebra of real rank one, G a connected Lie group such that $\operatorname{Lie} G = \mathfrak{g}$. Let $\langle\cdot,\cdot\rangle_\mathfrak{g}$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. Denote by $\operatorname{Aut}\mathfrak{g}$ the group of all automorphisms in $\mathfrak{g}$ preserving the form $\langle\cdot,\cdot\rangle_\mathfrak{g}$. Consider $\mathfrak{g} \times \mathfrak{g}$ equipped with the invariant symmetric bilinear form (1.1). We are going to describe the set of G-conjugacy classes of Lagrangian subalgebras in $\mathfrak{g} \times \mathfrak{g}$.

By Proposition 2.1, there is a bijection between the set of all Lagrangian subalgebras and the set of triples $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$, where $\mathfrak{c}_1, \mathfrak{c}_2$ are coisotropic subalgebras in $\mathfrak{g}$, and $\varphi : \mathfrak{c}_1/\mathfrak{c}_1^\perp \to \mathfrak{c}_2/\mathfrak{c}_2^\perp$ is an isomorphism preserving the form $\langle\cdot,\cdot\rangle_\mathfrak{g}$.

First, consider the triples of the form $(\mathfrak{g}, \mathfrak{g}, \varphi)$, where $\varphi \in \operatorname{Aut}\mathfrak{g}$. The corresponding G-conjugacy classes of such triples are parameterized by orbits of the $\operatorname{Int}\mathfrak{g}$-action on $\operatorname{Aut}\mathfrak{g}$ by conjugation. Let us denote a set of representatives of $\operatorname{Int}\mathfrak{g}$-orbits in $\operatorname{Aut}\mathfrak{g}$ by $\Phi(\mathfrak{g})$. The corresponding Lagrangian subalgebras are graphs of automorphisms in $\Phi(\mathfrak{g})$.

Further, we consider the case when the coisotropic subalgebras in the triples are proper subalgebras in $\mathfrak{g}$.

Proposition 4.1. *Any proper coisotropic subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is contained in a maximal parabolic subalgebra in $\mathfrak{g}$.*

Proof. Consider a maximal subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that $\mathfrak{c} \subset \mathfrak{q}$. According to Theorem 3.1 in [8], either the radical of $\mathfrak{q}$ is compact or $\mathfrak{q}$ is a maximal parabolic subalgebra. In the first case, $\mathfrak{q}$ is not coisotropic, neither is any of its subalgebras. Indeed, assume that $\mathfrak{q}^\perp \subset \mathfrak{q}$. Then $\mathfrak{q}^\perp$ is a solvable ideal of $\mathfrak{q}$ (because $\mathfrak{g}$ can be embedded into a suitable general linear algebra, and the restriction of the form $\langle X, Y\rangle = \operatorname{Tr} XY$ vanishes on $\mathfrak{q}^\perp$). Thus $\mathfrak{q}^\perp$ is contained in the radical of $\mathfrak{q}$. Note that any invariant bilinear form on $\mathfrak{g}$, being proportional to the Killing form, vanishes on $\mathfrak{q}^\perp$. Therefore $\mathfrak{q}^\perp$ is not compact, and we get a contradiction. □

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition, θ the Cartan involution, $\mathfrak{a}$ a maximal subalgebra of $\mathfrak{p}$. Recall that $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} = \dim \mathfrak{a}$. Denote by Θ the Cartan involution on G.

In the case of real rank one algebras all maximal parabolic subalgebras in $\mathfrak{g}$ are G-conjugate to $\mathfrak{q} = \mathfrak{q}_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda}$ (see [4] for details). We also set $\mathfrak{q}_- = \theta(\mathfrak{q}) = \mathfrak{g}_0 \oplus \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-2\lambda}$. Here $\mathfrak{g}_{\pm\lambda}$, $\mathfrak{g}_{\pm 2\lambda}$ are the root subspaces in $\mathfrak{g}$ (perhaps, $\mathfrak{g}_{\pm 2\lambda} = 0$). Set $\mathfrak{n} = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda}$ and $\mathfrak{n}_- = \theta(\mathfrak{n}) = \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-2\lambda}$. Obviously, $\mathfrak{q}^\perp = \mathfrak{n}$ and $\mathfrak{q}_-^\perp = \mathfrak{n}_-$. Therefore $\mathfrak{q}/\mathfrak{q}^\perp \simeq \mathfrak{q}_-/\mathfrak{q}_-^\perp \simeq \mathfrak{g}_0$. It is known that $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ (the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, see [9, §5.4.1]). Clearly, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is non-degenerate on $\mathfrak{g}_0$. Let $m = \dim \mathfrak{m}$.

Lemma 4.1. *Any coisotropic subalgebra in* $\mathfrak{g}$ *is* G*-conjugate to* $\mathfrak{c} = \mathfrak{c}_0 \oplus \mathfrak{q}^\perp$*, where* $\mathfrak{c}_0$ *is a coisotropic subalgebra in* $\mathfrak{g}_0$*.*

Proof. Let $\mathfrak{c}$ be a coisotropic subalgebra in $\mathfrak{g}$. By Proposition 4.1 and the fact that any proper parabolic subalgebra is G-conjugate to $\mathfrak{q}$, we may assume that $\mathfrak{c} \subset \mathfrak{q}$. Since $\mathfrak{q}^\perp \subset \mathfrak{c}^\perp \subset \mathfrak{c} \subset \mathfrak{q}$, we can consider $\mathfrak{c}_0 = \mathfrak{c}/\mathfrak{q}^\perp \subset \mathfrak{g}_0$. Obviously, $\mathfrak{c}_0$ is a coisotropic subalgebra in $\mathfrak{g}_0$. □

Denote by $\mathfrak{z}(\mathfrak{m})$ the center of $\mathfrak{m}$. It is not hard to show that $\dim \mathfrak{z}(\mathfrak{m}) \leq \operatorname{rank}_{\mathbb{R}} \mathfrak{g} = 1$. (In fact, the inequality $\dim \mathfrak{z}(\mathfrak{m}) \leq \operatorname{rank}_{\mathbb{R}} \mathfrak{g}$ holds for any real simple Lie algebra $\mathfrak{g}$.) Set $\mathfrak{m}' = [\mathfrak{m}, \mathfrak{m}]$. We have $\mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{z}(\mathfrak{m})$.

Lemma 4.2. *Any coisotropic subalgebra of* $\mathfrak{g}_0$ *has dimension* m *or* $m+1$*. In the case* $\mathfrak{z}(\mathfrak{m}) = 0$ *the only coisotropic subalgebra in* $\mathfrak{g}_0$ *is* $\mathfrak{g}_0$ *itself. In the case* $\dim \mathfrak{z}(\mathfrak{m}) = 1$ *there exist exactly two* m*-dimensional coisotropic subalgebras* $\mathfrak{c}_\pm$ *in* $\mathfrak{g}_0$*. Namely,* $\mathfrak{c}_\pm = \mathfrak{m}' \oplus \mathfrak{u}_\pm$*, where* $\mathfrak{u}_\pm$ *is spanned by* $x_0 \pm a$*,* x_0 *spans* $\mathfrak{z}(\mathfrak{m})$*, and* $a \in \mathfrak{a}$ *satisfies* $\langle a, a \rangle_{\mathfrak{g}} + \langle x_0, x_0 \rangle_{\mathfrak{g}} = 0$*.*

Proof. Note that the restriction of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ onto $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ is of signature $(1, m)$ (in particular, $\mathfrak{a}$ and $\mathfrak{m}$ are orthogonal w.r.t. $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$). Therefore any coisotropic subspace of $\mathfrak{g}_0$ has dimension m or $m+1$ (i.e. is equal to $\mathfrak{g}_0$ in the latter case). Now consider any one-dimensional isotropic subspace $\mathfrak{u} \neq 0$ in $\mathfrak{g}_0$. Let $a \in \mathfrak{a}$ and $x_0 \in \mathfrak{m}$ be such that $\mathfrak{u}$ is spanned by $a + x_0$; in particular, $\langle a + x_0, a + x_0 \rangle = 0$ and $x_0 \neq 0$, $a \neq 0$. Put $\mathfrak{c} = \mathfrak{u}^\perp$. It is clear that $\mathfrak{c}$ is a coisotropic subspace. Further, $\mathfrak{c}$ is a subalgebra if and only if $x_0 \in \mathfrak{z}(\mathfrak{m})$. This observation completes the proof. □

Lemmas 4.1 and 4.2 imply the following

Proposition 4.2. *Any coisotropic subalgebra in* $\mathfrak{g}$ *is either parabolic or is* G*-conjugate to* $\mathfrak{c}_\pm \oplus \mathfrak{q}^\perp$*. The latter case is possible only for* $\mathfrak{z}(\mathfrak{m}) \neq 0$*.* □

Theorem 4.1. *Any triple* $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ *with* $\mathfrak{c}_1$ *and* $\mathfrak{c}_2$ *proper parabolic is* G*-conjugate to exactly one triple of the form* $(\mathfrak{q}, \mathfrak{q}_\pm, \varphi)$*, where* $\varphi|_{\mathfrak{a}} = \pm 1$*,* $\varphi|_{\mathfrak{z}(\mathfrak{m})} = \pm 1$ *(in the case* $\mathfrak{z}(\mathfrak{m}) \neq 0$*) and* $\varphi|_{\mathfrak{m}'} \in \Phi(\mathfrak{m}')$*.*

Proof. Any pair $(\mathfrak{c}_1, \mathfrak{c}_2)$ of proper parabolic subalgebras in $\mathfrak{g}$ is G-conjugate to a pair of the form $(\mathfrak{q}, \mathfrak{c})$. Further, we continue to conjugate $(\mathfrak{q}, \mathfrak{c})$ by elements of $N_G(\mathfrak{q})$. Clearly, the $N_G(\mathfrak{q})$-orbits on the set of all proper parabolic subalgebras are parameterized by $N_G(\mathfrak{q})\backslash G/N_G(\mathfrak{q})$.

Consider the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let $G = KAN$ be the Iwasawa decomposition of G, where $K \subset G$ is a maximal compact subgroup, $A = \exp(\mathfrak{a})$, and $N = \exp(\mathfrak{n})$ (see VII.2 in [7]). Set $M = Z_K(\mathfrak{a})$. Let $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group. In our case $|W| = 2$. We have the Bruhat decomposition $MAN\backslash G/MAN = W$.

Lemma 4.3. $N_G(\mathfrak{q}) = MAN$*.*

Proof. To show that $MAN \subset N_G(\mathfrak{q})$ it is enough to prove that M acts on $\mathfrak{q}$ by inner automorphisms. Indeed, by Theorem 7.66 in [7] the group M is connected unless $\dim \mathfrak{n} = 1$. The condition $\dim \mathfrak{n} = 1$ is equivalent to the condition $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. In the latter case $M = \{\pm E\}$ acts trivially on $\mathfrak{q}$.

Further, by the Bruhat decomposition we have $G = MAN \cup MAN\widetilde{\omega}MAN$, where $\widetilde{\omega} \in N_K(\mathfrak{a})$ represents the nontrivial element of W. It is evident that $MAN\widetilde{\omega}MAN$ does not normalize $\mathfrak{q}$. Hence we get $N_G(\mathfrak{q}) = MAN$. □

Therefore there are exactly two G-orbits in the set of pairs $(\mathfrak{c}_1, \mathfrak{c}_2)$, and their representatives are $(\mathfrak{q}, \mathfrak{q})$ and $(\mathfrak{q}, \mathfrak{q}_-)$. Each triple $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ is G-conjugate to a triple of the form $(\mathfrak{q}, \mathfrak{q}, \varphi)$ or $(\mathfrak{q}, \mathfrak{q}_-, \varphi)$.

Now let us describe $N_G(\mathfrak{q})$-conjugacy classes of orthogonal automorphisms $\varphi : \mathfrak{q}/\mathfrak{q}^\perp \to \mathfrak{q}/\mathfrak{q}^\perp$. We have $\mathfrak{q}/\mathfrak{q}^\perp \simeq \mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$. Since φ preserves $\langle \cdot, \cdot \rangle$, we have $\varphi(\mathfrak{a}) = \mathfrak{a}$, $\varphi(\mathfrak{m}) = \mathfrak{m}$. Since $\dim \mathfrak{a} = 1$, we have $\varphi|_{\mathfrak{a}} = \pm 1$. In the case $\dim \mathfrak{z}(\mathfrak{m}) = 1$ we have $\varphi|_{\mathfrak{z}(\mathfrak{m})} = \pm 1$. We consider φ as the pair $(\varphi|_{\mathfrak{a}}, \varphi|_{\mathfrak{m}}) = (\pm 1, \varphi|_{\mathfrak{m}})$ (or the triple $(\varphi|_{\mathfrak{a}}, \varphi|_{\mathfrak{z}(\mathfrak{m})}, \varphi|_{\mathfrak{m}'}) = (\pm 1, \pm 1, \varphi|_{\mathfrak{m}'})$ in the case $\dim \mathfrak{z}(\mathfrak{m}) = 1$).

Since $\operatorname{Ad} A = \exp(\operatorname{ad} \mathfrak{a})$, we see that A acts on $\mathfrak{m}$ identically. Therefore A acts on φ trivially. Now consider action of $n \in N$. Since $\operatorname{Ad} N = \exp(\operatorname{ad} \mathfrak{n})$, we conclude that $\operatorname{Ad} n$ carries $\mathfrak{a} \oplus \mathfrak{m}$ to $\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n}$ leaving the $\mathfrak{a} \oplus \mathfrak{m}$ component unchanged. Thus N acts on φ trivially as well. Finally, arguing as in the proof of Lemma 4.3, we see that M acts by inner automorphisms on $\mathfrak{g}_0$.

Now consider the case of a triple $(\mathfrak{q}, \mathfrak{q}_-, \varphi)$.

Lemma 4.4. *The normalizer of the pair* $(\mathfrak{q}, \mathfrak{q}_-)$ *in* G *is* MA*.*

Proof. Clearly, the normalizer of this pair is $N_G(\mathfrak{q}) \cap N_G(\mathfrak{q}_-)$. By Lemma 4.3, we have $N_G(\mathfrak{q}) = MAN$. Similarly, $N_G(\mathfrak{q}_-) = MAN_-$, where $N_- = \Theta N$. Further, $MAN \cap N_- = \{1\}$ (see Lemma 7.64 in [7]). Thus $N_G(\mathfrak{q}) \cap N_G(\mathfrak{q}_-) = MAN \cap MAN_- = MA$. □

By the above computation, we see that only the M-part of the normalizer of $(\mathfrak{q}, \mathfrak{q}_-)$ can act non-trivially on φ, and this action is by inner automorphisms of $\mathfrak{m}$. This completes the proof of the theorem. $\square$

Theorem 4.2. *Let $\mathfrak{z}(\mathfrak{m}) \neq 0$. Each triple $(\mathfrak{c}_1, \mathfrak{c}_2, \varphi)$ with $\mathfrak{c}_1$ and $\mathfrak{c}_2$ non-parabolic is G-conjugate to exactly one triple of the form $(\mathfrak{c}_\alpha \oplus \mathfrak{q}^\perp, \mathfrak{c}_\beta \oplus \mathfrak{q}_\gamma^\perp, \varphi)$, where $\varphi \in \Phi(\mathfrak{m}')$, and $\alpha, \beta, \gamma \in \{+, -\}$.*

Proof. By Proposition 4.1 we know that $\mathfrak{c}_1 \subset \mathfrak{q}_1$, $\mathfrak{c}_2 \subset \mathfrak{q}_2$, where $\mathfrak{q}_1, \mathfrak{q}_2$ are proper parabolic subalgebras in $\mathfrak{g}$. Using the same argument as in the proof of Theorem 4.1, we may assume, up to G-conjugation, that $(\mathfrak{q}_1, \mathfrak{q}_2)$ is equal to $(\mathfrak{q}, \mathfrak{q})$ or $(\mathfrak{q}, \mathfrak{q}_-)$. By Lemma 4.2, we may assume that $(\mathfrak{c}_1, \mathfrak{c}_2) = (\mathfrak{c}_\alpha \oplus \mathfrak{q}^\perp, \mathfrak{c}_\beta \oplus \mathfrak{q}_\gamma^\perp)$, where $\alpha, \beta, \gamma \in \{+, -\}$.

Lemma 4.5. *$N_G(\mathfrak{c}_\pm \oplus \mathfrak{q}^\perp) = MAN$.*

Proof. The same argument as in Lemma 4.3 shows that $MAN \subset N_G(\mathfrak{c}_\pm \oplus \mathfrak{q}^\perp)$. Conversely, let $\mathrm{Ad}_g(\mathfrak{c}_\pm \oplus \mathfrak{q}^\perp) = \mathfrak{c}_\pm \oplus \mathfrak{q}^\perp$ for some $g \in G$. It is easy to see that $\mathfrak{c}_\pm$ is reductive and $\mathfrak{q}^\perp$ is nilpotent. Thus $\mathrm{Ad}_g \mathfrak{q}^\perp = \mathfrak{q}^\perp$. Since Ad_g preserves $\langle \cdot, \cdot \rangle$, we have $\mathrm{Ad}_g \mathfrak{q} = \mathfrak{q}$ and $g \in N_G(\mathfrak{q}) = MAN$. $\square$

The proof of the theorem can now be finished along the lines of the proof of Theorem 4.1. $\square$

Now let us summarize our investigations:

Theorem 4.3. *Any Lagrangian subalgebra in $\mathfrak{g} \times \mathfrak{g}$ is G-conjugate to exactly one subalgebra of the following list:*

1. $\mathfrak{l} = \{(x, \theta(x)) \,|\, x \in \mathfrak{g}\}$, *where $\theta \in \Phi(\mathfrak{g})$;*
2. $\mathfrak{l} = \{(x, \varphi(x)) \,|\, x \in \mathfrak{g}_0\} \oplus (\mathfrak{q}^\perp, 0) \oplus (0, \mathfrak{q}_\pm^\perp)$, *where $\varphi|_\mathfrak{a} = \pm 1$, $\varphi|_{\mathfrak{z}(\mathfrak{m})} = \pm 1$ (in the case $\mathfrak{z}(\mathfrak{m}) \neq 0$), and $\varphi|_{\mathfrak{m}'} \in \Phi(\mathfrak{m}')$;*
3. (*case $\mathfrak{z}(\mathfrak{m}) \neq 0$ only*). $\mathfrak{l} = \{(x, \varphi(x)) \,|\, x \in \mathfrak{m}'\} \oplus (\mathfrak{u}_\alpha \oplus \mathfrak{q}^\perp, 0) \oplus (0, \mathfrak{u}_\beta \oplus \mathfrak{q}_\gamma^\perp)$, *where $\alpha, \beta, \gamma \in \{+, -\}$, $\varphi \in \Phi(\mathfrak{m}')$, and $\mathfrak{u}_\pm$ are defined in Lemma 4.2.* $\square$

Remark. It rarely happens that $\mathfrak{g}_\Delta$ has a Lagrangian complement in $\mathfrak{g} \times \mathfrak{g}$. Indeed, if $\mathfrak{l}$ is the graph of $\theta \in \mathrm{Aut}\,\mathfrak{g}$, then $\mathfrak{l} \cap \mathfrak{g}_\Delta = \mathfrak{g}_\Delta^\theta \neq 0$. For $\mathfrak{l}$ described in parts 2 and 3 of Theorem 4.3 we see that $\mathfrak{l} \cap \mathfrak{g}_\Delta \supset (\mathfrak{m}')_\Delta^\varphi$, which is non-zero for $\mathfrak{m}' \neq 0$ (cf. [4, §3.3.4]). It is not hard to check that the only real rank one algebras with $\mathfrak{m}' = 0$ are $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ (here $\mathfrak{m} = 0$), $\mathfrak{g} = \mathfrak{su}(1, 2)$, and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})_\mathbb{R}$ (in the latter cases $\mathfrak{m} = \mathbb{R}$). Using Theorem 4.3 it is clear that for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ there is a unique up to G-conjugation Lagrangian complement $\mathfrak{a}_{-\Delta} \oplus (\mathfrak{q}^\perp, 0) \oplus (0, \mathfrak{q}_-^\perp)$, while for $\mathfrak{g} = \mathfrak{su}(1, 2)$ or $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})_\mathbb{R}$ there are three G-conjugacy classes of Lagrangian complements with representatives $\mathfrak{a}_{-\Delta} \oplus \mathfrak{m}_{-\Delta} \oplus (\mathfrak{q}^\perp, 0) \oplus (0, \mathfrak{q}_-^\perp)$, $(\mathfrak{u}_+ \oplus \mathfrak{q}^\perp, 0) \oplus (0, \mathfrak{u}_- \oplus \mathfrak{q}_-^\perp)$, $(\mathfrak{u}_- \oplus \mathfrak{q}^\perp, 0) \oplus (0, \mathfrak{u}_+ \oplus \mathfrak{q}_-^\perp)$.

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