Complex Fourth Moment Theorems

Simon Campese

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Università di Roma
Tor Vergata

Introduction

Generic Fourth Moment Theorem:

$$X_n \xrightarrow{d} X \sim \mu \qquad \Longleftrightarrow \qquad P\left(\mathrm{E}[X_n], \mathrm{E}\left[X_n^2\right], \mathrm{E}\left[X_n^3\right], \mathrm{E}\left[X_n^4\right]\right) \to 0$$

- Nualart-Peccati (2005): $X_n = I_p(f_n)$, $\mu \sim \mathcal{N}(0, \sigma^2)$, $P = X_n^4 - 3$
- Peccati-Tudor (2005): Extension to multivariate case for $\mu \sim \mathcal{N}_d(0, \Sigma)$
- Nualart-Ortiz-Latorre (2008): New proof using Malliavin calculus
- Nourdin-Peccati (2009): Quantitative FMT via Malliavin-Stein for $\mu \sim \mathcal{N}(0, \sigma^2)$ and $\mu \sim \mathit{Gamma}(\nu)$.
- Nourdin-Peccati-Revéillac (2010): Quantitative FMT via Malliavin-Stein for $\mu \sim \mathcal{N}_d(0, \Sigma)$

Introduction

- Ledoux (2012): new, pathbreaking proofs by spectral methods
- Azmoodeh-C.-Poly (2014): FMT for chaotic eigenfunctions of generic Markov diffusion generators, $\mu \sim \mathcal{N}(0, \sigma^2)$, $\mu \sim \text{Gamma}(\nu)$ or $\mu \sim \text{Beta}(\alpha, \beta)$
- C.-Nourdin-Peccati-Poly (2015+): Multivariate extension for $\mu \sim \mathcal{N}_d(0, \Sigma)$
- In this talk: Extension to complex valued random variables and $\mu \sim \mathcal{CN}_d(0, \Sigma)$

Complex normal distribution

- $Z \sim \mathcal{CN}_d(0, \Sigma)$ if its density f is given by

$$f(z) = \frac{1}{\pi^d |\det \Sigma|} \exp\left(-\bar{z}^T \Sigma^{-1} z\right)$$

- $\mathsf{E}[Z\overline{Z}^T] = \Sigma$ and $\mathsf{E}[ZZ^T] = 0$
- Completely characterized by its moments $\mathsf{E}\left[\prod_j Z_j^{p_j} \overline{Z}_j^{q_j}\right]$
- For $Z \sim \mathcal{CN}_1(0, 1)$:

$$\mathsf{E}\left[Z^p \, \overline{Z}^q\right] = \begin{cases} p! & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

Wirtinger calculus

- $\partial_z = \frac{1}{2}\left(\partial_x - \mathrm{i}\partial_y\right)$ and $\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + \mathrm{i}\partial_y\right)$ Wirtinger derivatives
- ∂_z and $\partial_{\bar{z}}$ satisfy product and chain rules
- Heuristic: z and $\bar{z}$ can be treated as algebraically independent variables when differentiating, for example:

$$\partial_z\, z^p \bar{z}^q = p z^{p-1} \bar{z}^q \qquad \text{and} \qquad \partial_{\bar{z}}\, z^p \bar{z}^q = q z^p \bar{z}^{q-1}$$

Stein's method for the complex normal distribution

Lemma

$Z \sim \mathcal{CN}_1(0,1)$ if, and only if,

$$\mathsf{E}[\partial_z f(Z)] - \mathsf{E}\left[\overline{Z} f(Z)\right] = 0$$

for suitable $f\colon \mathbb{C} \to \mathbb{C}$.

Looks nice, but associated Stein equation can not be solved in general.

Stein's method for the complex normal distribution

Lemma

$Z \sim \mathcal{CN}_1(0,1)$ if, and only if,

$$2\,\mathsf{E}[\partial_{z\bar z} f(Z)] - \mathsf{E}\left[\bar Z\,\partial_{\bar z} f(Z)\right] - \mathsf{E}[Z\,\partial_z f(Z)] = 0$$

for suitable $f\colon \mathbb{C} \to \mathbb{C}$.

For $W \sim \mathcal{CN}_1(0,1)$, associated Stein equation

$$2\partial_{z\bar z} f(z) - \bar z\,\partial_{\bar z} f(z) - z\,\partial_z f(z) = h(z) - \mathsf{E}[h(W)]$$

has nice solution for suitable h.

Abstract setting

- Symmetric diffusion Markov generator L acting on $L^2(E, \mathcal{F}, \mu)$
- Discrete spectrum

$$S = \{\cdots < -\lambda_2 < -\lambda_1 < -\lambda_0 = 0\}$$

- Spectral theorem:

$$L^2(E, \mathcal{F}, \mu) = \bigoplus_{k=0}^{\infty} \mathrm{ker}(\mathsf{L} + \lambda_k \,\mathrm{Id})$$

- Eigenspaces closed under conjugation as $\overline{\mathsf{L} F} = \mathsf{L}\,\overline{F}$

Carré du champ operator

- Carré du champ operator Γ:

$$\Gamma(F,G) = \frac{1}{2}\big(\mathsf{L}(F\overline{G}) - F\,\mathsf{L}\,\overline{G} - \overline{G}\,\mathsf{L}\,F\big)$$

- Integration by parts: $\int \mathsf{L}(F\overline{G})\,\mathrm{d}\mu = \int \mathsf{L}(1)F\overline{G}\,\mathrm{d}\mu = 0$

$$\int_E \Gamma(F,G)\,\mathrm{d}\mu = -\int_E F\,\mathsf{L}\,\overline{G}\,\mathrm{d}\mu$$

- Diffusion property:

$$\Gamma(\varphi(F_1,\ldots,F_d),G) = \sum_{j=1}^{d}\Big(\partial_{z_j}\varphi(F)\,\Gamma(F_j,G) + \partial_{\overline{z}_j}\varphi(F)\,\Gamma(\overline{F}_j,G)\Big)$$

Pseudo inverse of the generator

- L^{-1} pseudo-inverse of generator (compact)
- Bears its name as

$$\mathsf{L}\,\mathsf{L}^{-1}\,F = F - \int_E F\,\mathrm{d}\mu$$

- In particular:

$$\begin{aligned}\int_E \Gamma(F, -\mathsf{L}^{-1}\,G)\,\mathrm{d}\mu &= \int_E F\,\mathsf{L}\,\mathsf{L}^{-1}\,\overline{G}\,\mathrm{d}\mu \\ &= \int_E F\overline{G}\,\mathrm{d}\mu - \int_E F\,\mathrm{d}\mu \int_E \overline{G}\,\mathrm{d}\mu\end{aligned}$$

Quantitative bound for the Wasserstein distance

Theorem

Let $Z \sim \mathcal{CN}_1(0,1)$ and denote by F a centered smooth complex random variable. Then it holds that

$$d_W(F,Z) \leq \sqrt{2} \left(\frac{1}{2} \int_E \left| \Gamma(\overline{F}, -\mathsf{L}^{-1} F \right|^2 \mathrm{d}\mu + \int_E \left(\Gamma(F, -\mathsf{L}^{-1} F) - 1 \right)^2 \mathrm{d}\mu \right)^{1/2}.$$

Quantitative bound for the Wasserstein distance

Theorem

Let $Z \sim C\mathcal{N}_d(0, \Sigma)$ and denote by F a centered smooth complex random vector. Then it holds that

$$d_W(F, Z) \leq \sqrt{2}\, \|\Sigma^{-1}\|_{op} \|\Sigma\|_{op}^{1/2} \left(\frac{1}{2} \int_E \left\| \Gamma(\overline{F}, -\mathsf{L}^{-1} F) \right\|_{HS}^2 \mathrm{d}\mu + \int_E \left\| \Gamma(F, -\mathsf{L}^{-1} F) - \Sigma \right\|_{HS}^2 \mathrm{d}\mu \right)^{1/2},$$

where $\Gamma(F, -\mathsf{L}^{-1} F) = \left(\Gamma(F_j, -\mathsf{L}^{-1} F_k) \right)_{1 \leq j,k \leq d}$ *and* $\|A\|_{HS} = \mathrm{tr}(A \overline{A}^T)$.

Abstract Markov chaos

Definition

- $F \in \ker\big(\mathsf{L} + \lambda_p \operatorname{Id}\big)$ and $G \in \ker\big(\mathsf{L} + \lambda_q \operatorname{Id}\big)$ are jointly chaotic, if

$$FG \in \bigoplus_{j=0}^{p+q} \ker\big(\mathsf{L} + \lambda_j \operatorname{Id}\big) \qquad \text{and} \qquad F\overline{G} \in \bigoplus_{j=0}^{p+q} \ker\big(\mathsf{L} + \lambda_j \operatorname{Id}\big).$$

- $F \in \ker\big(\mathsf{L} + \lambda_p \operatorname{Id}\big)$ is chaotic, if F is jointly chaotic with itself.
- A vector of eigenfunctions is chaotic, if any two components are jointly chaotic.

Key lemma

Lemma

For chaotic eigenfunctions F,G it holds that

$$\int_E \left|\Gamma(F, -\mathsf{L}^{-1}\, G)\right|^2 \mathrm{d}\mu \leq \int_E F\overline{G}\, \Gamma(F, -\mathsf{L}^{-1}\, G)\, \mathrm{d}\mu$$

Consequence of general principle from Azmoodeh-C.-Poly (2014).

Quantitative Fourth Moment Theorem

Theorem

For $Z \sim \mathcal{CN}_1(0,1)$ and chaotic eigenfunction F, it holds that

$$d_W(F,Z) \le \sqrt{\int_E \left(\frac{1}{2}|F|^4 - 2|F|^2 + 1\right) \,\mathrm{d}\mu}$$

Similar bound for $Z \sim \mathcal{CN}_d(0,\Sigma)$ and chaotic vector F involving $\int_E F_j\overline{F}_k \,\mathrm{d}\mu$ and $\int_E |F_jF_k|^2 \,\mathrm{d}\mu$.

Quantitative Fourth Moment Theorem

Corollary

For $Z \sim \mathcal{CN}_1(0,1)$ and normalized sequence F_n of chaotic eigenfunctions, the following two assertions are equivalent:

(i) $F_n \xrightarrow{d} Z$

(ii) $\int_E |F_n|^4 \, \mathrm{d}\mu \to 2$

Proof of moment bound

By key lemma, diffusion property and integration by parts:

$$\begin{aligned}\int_E \Gamma(F, -\mathsf{L}^{-1} F)^2 \,\mathrm{d}\mu &\le \int_E F\bar{F}\,\Gamma(F, -\mathsf{L}^{-1} F)\,\mathrm{d}\mu \\ &= \frac{1}{2}\left(\int_E \Gamma(F^2\bar{F}, -\mathsf{L}^{-1} F)\,\mathrm{d}\mu - \int_E F^2\,\Gamma(\bar{F}, -\mathsf{L}^{-1} F)\,\mathrm{d}\mu\right) \\ &= \frac{1}{2}\int_E |F|^4\,\mathrm{d}\mu - \frac{1}{2}\int_E F^2\,\Gamma(\bar{F}, -\mathsf{L}^{-1} F)\,\mathrm{d}\mu\end{aligned}$$

Key lemma also implies that

$$\int_E \left|\Gamma(\bar{F}, -\mathsf{L}^{-1} F)\right|^2 \mathrm{d}\mu \le \int_E F^2\,\Gamma(\bar{F}, -\mathsf{L}^{-1} F)\,\mathrm{d}\mu.$$

Proof of moment bound

Therefore,

$$\begin{aligned}
&\int_E \Big(\frac{1}{2}\big|\Gamma(\overline{F}, -\mathsf{L}^{-1} F)\big|^2 + \big(\Gamma(F, -\mathsf{L}^{-1} F) - 1\big)^2\Big)\,\mathrm{d}\mu \\
&\qquad = \int_E \left(\frac{1}{2}\big|\Gamma(\overline{F}, -\mathsf{L}^{-1} F)\big|^2 + \Gamma(F, -\mathsf{L}^{-1} F)^2 - 2|F|^2 + 1\right)\,\mathrm{d}\mu \\
&\qquad \le \int_E \left(\frac{1}{2}|F|^4 - 2|F|^2 + 1\right)\,\mathrm{d}\mu
\end{aligned}$$

Complex Peccati-Tudor Theorem

Theorem

Let $Z \sim \mathcal{CN}_d(0, \Sigma)$ and (F_n) be sequence of chaotic vectors satisfying $\mathrm{E}\left[F_n^2\right] \to 0$ and $\mathrm{E}\left[F_n \overline{F}_n\right] \to \Sigma$. Under some technical conditions on underlying generator, the following two assertions are equivalent:

(i) $F_n \xrightarrow{d} Z$ *jointly*

(ii) $F_n \xrightarrow{d} Z$ *componentwise*

Proof: Adaptation of real version in C.-Nourdin-Peccati-Poly (2015+)

Complex Ornstein-Uhlenbeck generator

- $S = -\mathbb{N}_0$, $\Gamma(F, G) = \langle DF, DG \rangle_H$
- Real and imaginary parts of any eigenfunction are themselves eigenfunctions of the real OU-generator.
- However, eigenspaces have much richer algebraic structure:

$$\ker(L + k\,\mathrm{Id}) = \bigoplus_{\substack{p,q \in \mathbb{N}_0 \\ p+q=k}} \mathcal{H}_{p,q}$$

with $\overline{\mathcal{H}_{p,q}} = \mathcal{H}_{q,p}$.

Complex Hermite Polynomials (Itô, 1952)

$$
\begin{aligned}
H_{p,q}(z) &= (-1)^{p+q} \mathrm{e}^{|z|^2} \, (\partial_z)^p \, (\partial_{\bar{z}})^q \, \mathrm{e}^{-|z|^2} \\
&= \sum_{j=0}^{p \wedge q} \binom{p}{j} \binom{q}{j} \, j! \, (-1)^j \, z^{p-j} \, \bar{z}^{q-j}
\end{aligned}
$$

First few:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & z & & \bar{z} & & \\
 & z^2 & & |z|^2 - 1 & & \bar{z}^2 & \\
z^3 & & z^2\bar{z} - 2z & & \bar{z}^2 z - 2\bar{z} & & \bar{z}^3
\end{array}
$$

Orthonormal basis for $\mathcal{H}_{p,q}$

- Let $\{Z(h) \colon h \in \mathfrak{H}\}$ be complex isonormal Gaussian process and (e_j) orthonormal basis of $\mathfrak{H}$.
- Orthonormal basis of $\mathcal{H}_{p,q}$ is given by

$$\left\{ \sqrt{m_p!\, m_q!} \prod_{j=1}^{\infty} H_{m_p(j),m_q(j)}(Z(e_j)) \colon (m_p, m_q) \in M_p \times M_q \right\}$$

- In particular: $Z(e_j)^p \in \mathcal{H}_{p,0}$
- Thus, $\mathcal{H}_{p,0}$ is sub-algebra of Dirichlet domain induced by Γ

Concluding remarks

- For OU generator and $d = 1$, a (non-quantitative) FMT and Peccati-Tudor Theorem have been proven by Chen-Liu (2014+) and Chen (2014+), respectively, by separating real and imaginary parts
- Our method can also yield FMT for other target laws (usual complexified Gamma and Beta distributions are not interesting as these are real valued)

Applications:

- Quantitative CLT for spin random fields (joint project with D. Marinucci and M. Rossi)
- New proof and generalization of de Reyna's complex Gaussian product inequality; advances for complex unlinking conjecture (forthcoming paper with G. Poly)