Complex Fourth Moment Theorems

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Introduction

Generic Fourth Moment Theorem:

\[ X_n \overset{d}{\to} X \sim \mu \iff P(E[X_n], E[X_n^2], E[X_n^3], E[X_n^4]) \to 0 \]

- Nualart-Peccati (2005): \( X_n = I_p(f_n), \mu \sim \mathcal{N}(0, \sigma^2), P = X_n^4 - 3 \)
- Peccati-Tudor (2005): Extension to multivariate case for \( \mu \sim \mathcal{N}_d(0, \Sigma) \)
- Nourdin-Peccati (2009): Quantitative FMT via Malliavin-Stein for \( \mu \sim \mathcal{N}(0, \sigma^2) \) and \( \mu \sim \text{Gamma}(\nu) \).
- Nourdin-Peccati-Revéillac (2010): Quantitative FMT via Malliavin-Stein for \( \mu \sim \mathcal{N}_d(0, \Sigma) \)
• Ledoux (2012): new, pathbreaking proofs by spectral methods
• Azmoodeh-C.-Poly (2014): FMT for chaotic eigenfunctions of generic Markov diffusion generators, $\mu \sim \mathcal{N}(0, \sigma^2)$, $\mu \sim \text{Gamma}(\nu)$ or $\mu \sim \text{Beta}(\alpha, \beta)$
• C.-Nourdin-Peccati-Poly (2015+): Multivariate extension for $\mu \sim \mathcal{N}_d(0, \Sigma)$
• In this talk: Extension to complex valued random variables and $\mu \sim \mathcal{CN}_d(0, \Sigma)$
Complex normal distribution

- \( Z \sim \mathcal{CN}_d(0, \Sigma) \) if its density \( f \) is given by
  \[
f(z) = \frac{1}{\pi^d |\det \Sigma|} \exp \left( -z^T \Sigma^{-1} z \right)
  \]

- \( E[ZZ^T] = \Sigma \) and \( E[ZZ^T] = 0 \)

- Completely characterized by its moments \( E \left[ \prod_j Z_j^{p_j} \bar{Z}_j^{q_j} \right] \)

- For \( Z \sim \mathcal{CN}_1(0, 1) \):
  \[
  E \left[ Z^p \bar{Z}^q \right] = \begin{cases} 
  p! & \text{if } p = q \\
  0 & \text{if } p \neq q.
  \end{cases}
  \]
Wirtinger calculus

\[ \partial_z = \frac{1}{2} (\partial_x - i\partial_y) \quad \text{and} \quad \partial_{\bar{z}} = \frac{1}{2} (\partial_x + i\partial_y) \]

Wirtinger derivatives

\[ \partial_z \text{ and } \partial_{\bar{z}} \text{ satisfy product and chain rules} \]

Heuristic: \( z \) and \( \bar{z} \) can be treated as algebraically independent variables when differentiating, for example:

\[ \partial_z z^p \bar{z}^q = pz^{p-1} \bar{z}^q \quad \text{and} \quad \partial_{\bar{z}} z^p \bar{z}^q = qz^p \bar{z}^{q-1} \]
Lemma

\( Z \sim \mathcal{CN}_1(0, 1) \) if, and only if,

\[
\mathbb{E}[\partial_z f(Z)] - \mathbb{E}[\overline{Z} f(Z)] = 0
\]

for suitable \( f : \mathbb{C} \rightarrow \mathbb{C} \).

Looks nice, but associated Stein equation can not be solved in general.
Lemma

$Z \sim \mathcal{CN}_1(0, 1)$ if, and only if,

$$2 \mathbb{E}[\partial_{zz} f(Z)] - \mathbb{E}[\bar{Z} \partial_{\bar{z}} f(Z)] - \mathbb{E}[Z \partial_z f(Z)] = 0$$

for suitable $f: \mathbb{C} \to \mathbb{C}$.

For $W \sim \mathcal{CN}_1(0, 1)$, associated Stein equation

$$2\partial_{z\bar{z}} f(z) - \bar{z} \partial_{\bar{z}} f(z) - z \partial_z f(z) = h(z) - \mathbb{E}[h(W)]$$

has nice solution for suitable $h$. 
Abstract setting

- Symmetric diffusion Markov generator $L$ acting on $L^2(E, \mathcal{F}, \mu)$
- Discrete spectrum

$$S = \{ \cdots < -\lambda_2 < -\lambda_1 < -\lambda_0 = 0 \}$$

- Spectral theorem:

$$L^2(E, \mathcal{F}, \mu) = \bigoplus_{k=0}^{\infty} \ker(L + \lambda_k \text{Id})$$

- Eigenspaces closed under conjugation as $\overline{LF} = L\overline{F}$
Carré du champ operator

- Carré du champ operator $\Gamma$:

$$\Gamma(F, G) = \frac{1}{2} \left( L(FG) - F L \bar{G} - \bar{G} L F \right)$$

- Integration by parts:

$$\int L(FG) \, d\mu = \int L(1)F \bar{G} \, d\mu = 0$$

$$\int_E \Gamma(F, G) \, d\mu = - \int_E F L \bar{G} \, d\mu$$

- Diffusion property:

$$\Gamma(\varphi(F_1, \ldots, F_d), G) = \sum_{j=1}^{d} \left( \partial_{z_j} \varphi(F) \Gamma(F_j, G) + \partial_{\bar{z}_j} \varphi(F) \Gamma(\bar{F}_j, G) \right)$$
Pseudo inverse of the generator

- $L^{-1}$ pseudo-inverse of generator (compact)
- Bears its name as
  \[ LL^{-1}F = F - \int_E F \, d\mu \]
- In particular:

  \[
  \int_E \Gamma(F, - L^{-1} G) \, d\mu = \int_E F L L^{-1} \overline{G} \, d\mu = \int_E F \overline{G} \, d\mu - \int_E F \, d\mu \int_E \overline{G} \, d\mu
  \]
Quantitative bound for the Wasserstein distance

**Theorem**

Let $Z \sim \mathcal{CN}_1(0, 1)$ and denote by $F$ a centered smooth complex random variable. Then it holds that

$$d_W(F, Z) \leq \sqrt{2}$$
Theorem

Let $Z \sim \mathcal{CN}_d(0, \Sigma)$ and denote by $F$ a centered smooth complex random vector. Then it holds that

$$d_W(F, Z) \leq \sqrt{2} \left\| \Sigma^{-1} \right\|_{op} \left\| \Sigma \right\|_{op}^{1/2} \left( \frac{1}{2} \int_E \left\| \Gamma(\bar{F}, -L^{-1} F) \right\|_{HS}^2 \, d\mu ight)^{1/2} + \int_E \left\| \Gamma(F, -L^{-1} F) - \Sigma \right\|_{HS}^2 \, d\mu \right)^{1/2},$$

where $\Gamma(F, -L^{-1} F) = (\Gamma(F_j, -L^{-1} F_k))_{1 \leq j, k \leq d}$ and $\left\| A \right\|_{HS} = \text{tr}(A \bar{A}^T)$. 

Quantitative bound for the Wasserstein distance
Abstract Markov chaos

Definition

- $F \in \ker(L + \lambda_p \text{Id})$ and $G \in \ker(L + \lambda_q \text{Id})$ are jointly chaotic, if
  \[
  p+q \\
  FG \in \bigoplus_{j=0}^{p+q} \ker(L + \lambda_j \text{Id}) \quad \text{and} \quad \bar{FG} \in \bigoplus_{j=0}^{p+q} \ker(L + \lambda_j \text{Id}).
  \]

- $F \in \ker(L + \lambda_p \text{Id})$ is chaotic, if $F$ is jointly chaotic with itself.

- A vector of eigenfunctions is chaotic, if any two components are jointly chaotic.
Lemma

For chaotic eigenfunctions $F,G$ it holds that

$$\int_E |\Gamma(F, - L^{-1} G)|^2 \, d\mu \leq \int_E F \overline{G} \Gamma(F, - L^{-1} G) \, d\mu$$

Consequence of general principle from Azmoodeh-C.-Poly (2014).
Quantitative Fourth Moment Theorem

**Theorem**

For $Z \sim \mathcal{CN}_1(0, 1)$ and chaotic eigenfunction $F$, it holds that

$$d_W(F, Z) \leq \sqrt{\int_E \left( \frac{1}{2} |F|^4 - 2|F|^2 + 1 \right) \, d\mu}$$

Similar bound for $Z \sim \mathcal{CN}_d(0, \Sigma)$ and chaotic vector $F$ involving

$$\int_E F_j \bar{F}_k \, d\mu \quad \text{and} \quad \int_E |F_j F_k|^2 \, d\mu.$$
Corollary

For $Z \sim \mathcal{C}N_1(0, 1)$ and normalized sequence $F_n$ of chaotic eigenfunctions, the following two assertions are equivalent:

(i) $F_n \xrightarrow{d} Z$

(ii) $\int_E |F_n|^4 \, d\mu \rightarrow 2$
Proof of moment bound

By key lemma, diffusion property and integration by parts:

\[
\int_E \Gamma(F, - L^{-1} F)^2 \, d\mu \leq \int_E F\bar{F} \, \Gamma(F, - L^{-1} F) \, d\mu
\]

\[
= \frac{1}{2} \left( \int_E \Gamma(F^2\bar{F}, - L^{-1} F) \, d\mu - \int_E F^2 \, \Gamma(\bar{F}, - L^{-1} F) \, d\mu \right)
\]

\[
= \frac{1}{2} \int_E |F|^4 \, d\mu - \frac{1}{2} \int_E F^2 \, \Gamma(\bar{F}, - L^{-1} F) \, d\mu
\]

Key lemma also implies that

\[
\int_E |\Gamma(\bar{F}, - L^{-1} F)|^2 \, d\mu \leq \int_E F^2 \, \Gamma(\bar{F}, - L^{-1} F) \, d\mu.
\]
Proof of moment bound

Therefore,

\[
\int_E \left( \frac{1}{2} |\Gamma(\bar{F}, -L^{-1} F)|^2 + (\Gamma(F, -L^{-1} F) - 1)^2 \right) \, d\mu \\
= \int_E \left( \frac{1}{2} |\Gamma(\bar{F}, -L^{-1} F)|^2 + \Gamma(F, -L^{-1} F)^2 - 2|F|^2 + 1 \right) \, d\mu \\
\leq \int_E \left( \frac{1}{2} |F|^4 - 2|F|^2 + 1 \right) \, d\mu
\]
Complex Peccati-Tudor Theorem

Theorem

Let $Z \sim \mathcal{C}N_d(0, \Sigma)$ and $(F_n)$ be sequence of chaotic vectors satisfying $\mathbb{E}[F_n^2] \to 0$ and $\mathbb{E}[F_nF_n] \to \Sigma$. Under some technical conditions on underlying generator, the following two assertions are equivalent:

(i) $F_n \xrightarrow{d} Z$ jointly

(ii) $F_n \xrightarrow{d} Z$ componentwise

Complex Ornstein-Uhlenbeck generator

- $S = -\mathbb{N}_0$, $\Gamma(F, G) = \langle DF, DG \rangle_H$
- Real and imaginary parts of any eigenfunction are themselves eigenfunctions of the real OU-generator.
- However, eigenspaces have much richer algebraic structure:

\[
\ker(L + k \text{Id}) = \bigoplus_{p,q \in \mathbb{N}_0 \atop p+q=k} \mathcal{H}_{p,q}
\]

with $\overline{\mathcal{H}_{p,q}} = \mathcal{H}_{q,p}$. 
Complex Hermite Polynomials (Itô, 1952)

\[ H_{p,q}(z) = (-1)^{p+q} e^{\left|z\right|^2} (\partial_z)^p (\partial_{\bar{z}})^q e^{-\left|z\right|^2} = \sum_{j=0}^{\min(p,q)} \binom{p}{j} \binom{q}{j} j! (-1)^j \, z^{p-j} \, \bar{z}^{q-j} \]

First few:

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\[ z \quad \bar{z} \]

\[ z^2 \quad \left|z\right|^2 - 1 \quad \bar{z}^2 \]

\[ z^3 \quad z^2 \bar{z} - 2z \quad \bar{z}^2 z - 2\bar{z} \quad \bar{z}^3 \]
Orthonormal basis for $\mathcal{H}_{p,q}$

- Let $\{Z(h) : h \in \mathcal{H}\}$ be complex isonormal Gaussian process and $(e_j)$ orthonormal basis of $\mathcal{H}$.

- Orthonormal basis of $\mathcal{H}_{p,q}$ is given by

$$\left\{ \sqrt{m_p! m_q!} \prod_{j=1}^{\infty} H_{m_p(j), m_q(j)}(Z(e_j)) : (m_p, m_q) \in M_p \times M_q \right\}$$

- In particular: $Z(e_j)^p \in \mathcal{H}_{p,0}$

- Thus, $\mathcal{H}_{p,0}$ is sub-algebra of Dirichlet domain induced by $\Gamma$
Concluding remarks

- For OU generator and $d = 1$, a (non-quantitative) FMT and Peccati-Tudor Theorem have been proven by Chen-Liu (2014+) and Chen (2014+), respectively, by separating real and imaginary parts.
- Our method can also yield FMT for other target laws (usual complexified Gamma and Beta distributions are not interesting as these are real valued).

Applications:

- Quantitative CLT for spin random fields (joint project with D. Marinucci and M. Rossi).
- New proof and generalization of de Reyna’s complex Gaussian product inequality; advances for complex unlinking conjecture (forthcoming paper with G. Poly).