L^p bounds for spectral multipliers on rank one NA-groups with roots not all positive

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Abstract

We consider a family of non-unimodular rank one NA-groups with roots not all positive, and we show that on these groups there exists at least a distinguished left invariant sub-Laplacian which admits a differentiable L^p functional calculus for every $p \geq 1$.

Keywords: sub-Laplacian, solvable Lie group, stratified group, exponential volume growth, multiplier theorem, differentiable functional calculus.

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0 Introduction

Let G be a real connected Lie group, let X_j, $j = 1, \dots d$, be some left invariant vector fields on G which generate the Lie algebra of G, and form the left invariant sub-Laplacian $\Delta = -\sum_{j=1}^{d} X_j^2$. On the space $L^2(G)$ relative to the right invariant Haar measure on G, the operator Δ is formally self-adjoint and non-negative. Then, from the spectral theorem, every Borel function m bounded on $\mathbb{R}^+$ determines a bounded operator on $L^2(G)$ via the formula $m(\Delta) = \int_{\mathbb{R}^+} m(\lambda)\mathrm{d}E_\lambda$, where $\Delta = \int_{\mathbb{R}^+} \lambda \mathrm{d}E_\lambda$ is the spectral resolution of Δ. A question which arises naturally is the following (see Hörmander [12] for $\mathbb{R}^n$): is it possible under certain conditions regarding the function m, to extend $m(\Delta)$ to a bounded operator on $L^p(G)$ for some $p \neq 2$?

We concentrate here on the case where G is a solvable Lie group with exponential volume growth (for Lie groups with polynomial volume growth, see Christ [2], and Alexopoulos [1]). Two classes of solvable Lie groups with exponential volume growth and invariant sub-Laplacians emerge in the miscellaneous works on that problem: Lie groups with sub-Laplacians which admit a differentiable L^p functional calculus on the one hand (see e.g. Hebisch [8], [9], [10], Cowling, Giulini, Hulanicki and Mauceri [4], Mustapha [14], Gnewuch [7]); Lie groups with sub-Laplacians of holomorphic L^p type on the other hand (see Christ and Müller [3], Ludwig and Müller [13], and Hebisch, Ludwig and Müller [11]). In this paper, we prove a multiplier theorem for groups and sub-Laplacians belonging to the first class.

We consider a family of Lie groups G such that each G is a semidirect product of a real nilpotent Lie group N (non necessarily Euclidean) with the real line $\mathbb{R}$, and the action is semisimple and has nonzero eigenvalues all positive but one. We show that on each G there is a left invariant sub-Laplacian with differentiable functional calculus on $L^p(G)$ for all $p \geq 1$. This result is new when N is non-Euclidean (for N Euclidean, see Hebisch [10]); in case of eigenvalues not all positive with N non-Euclidean, previous multiplier theorems concern exclusively invariant sub-Laplacians of holomorphic L^p type (see Christ and Müller [3], Ludwig and Müller [13], and Hebisch, Ludwig and Müller [11]).

1 Results

Let us begin by introducing some notations, and recalling some basic notions about stratified groups (those can be found in the book of Folland and Stein [6]).

Let H be a stratified group, that is a connected simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{h}$ is such that the following vector space decomposition holds

$$\mathfrak{h} = \oplus_{j=1}^{n} V_j$$

where the subspaces V_j satisfy

$$[V_1, V_j] = V_{j+1}, \qquad j = 1, \dots n-1.$$

We consider the family of algebra automorphisms $\{\sigma_t\}_{t\in\mathbb{R}}$ of $\mathfrak{h}$ defined by

$$\sigma_t(X_j) = e^{jt} X_j, \qquad X_j \in V_j.$$

This endows $\mathfrak{h}$ with a structure of dilations, which corresponds to a structure of dilations on H, given by the one-parameter group of automorphisms on the group H

$$\sigma_t = \exp_H \circ \sigma_t \circ \exp_H^{-1}, \qquad t \in \mathbb{R},$$

where $\exp_H$ denotes the exponential map from $\mathfrak{h}$ to H. With this structure of dilations, the nilpotent Lie group H is said to be a *homogeneous group* of *homogeneous dimension*

$$Q = \sum_{j=1}^{n} j \dim V_j.$$

Let α be real negative, and let $G = H \times \mathbb{R} \times \mathbb{R}$ be the Lie group whose product is

$$g_1 \cdot g_2 = (h_1, a_1, t_1) \cdot (h_2, a_2, t_2) = (h_1 \cdot \sigma_{t_1} h_2, a_1 + e^{\alpha t_1} a_2, t_1 + t_2)$$

where $g_i = (h_i, a_i, t_i) \in G$, $i = 1, 2$. Observe that the Lie group G is solvable with exponential volume growth, and that it is non-unimodular when α equals not $-Q$. We endow G with the right invariant Haar measure

$$\mathrm{d}g = \mathrm{d}h \mathrm{d}a \mathrm{d}t,$$

$\mathrm{d}h$ being a bi-invariant Haar measure on H, $\mathrm{d}a$ and $\mathrm{d}t$ being the Lebesgue measures on $\mathbb{R}$ corresponding respectively to the variables a and t.

In what follows, we identify $\mathfrak{h}$ with an ideal of the Lie algebra $\mathfrak{g}$ of G. Fix $\{e_1, \dots, e_d\}$ a basis of the vector space V_1. To each vector e_j, $j = 1, \dots d$, we associate a vector field $\widetilde{X}_j$ left invariant on H, by setting:

$$\widetilde{X}_j \phi(h) = \partial_s \phi(h \cdot \exp_H(s e_j))|_{s=0}, \qquad h \in H, \ \phi \in C^1(H),$$

and a vector field X_j left invariant on G, by setting:

$$X_j \phi(g) = \partial_s \phi(g \cdot \exp(s e_j))|_{s=0}, \qquad g \in G, \ \phi \in C^1(G),$$

where exp denotes the exponential map from $\mathfrak{g}$ to G. It is easy to see that $X_j = e^t \widetilde{X}_j$, $j = 1, \dots d$. We define two more left invariant vector fields on G

$$X_0 = \partial_t \quad \text{and} \quad X_{d+1} = e^{\alpha t} \partial_a.$$

Note that the system $\chi = \{X_0, \dots X_{d+1}\}$ satisfies Hörmander condition on G.

We consider now the operator $-\sum_{j=0}^{d+1} X_j^2$ defined on the set $C_0^\infty(G)$ of smooth functions compactly supported in G. Let Δ denote the Friedrichs extension of this operator on $L^2(G)$ (*i.e.* the smallest self-adjoint extension),

$$\Delta = -\sum_{j=0}^{d} X_j^2.$$

The so-defined operator Δ is a left invariant sub-Laplacian on G.

The aim of this paper is to prove the following multiplier theorem.

Theorem 1.1 *Let G and Δ be as above. Suppose that m is a real continuous function compactly supported in $]0, +\infty[$ which belongs to the Sobolev space $H^{Q+5+\epsilon}(\mathbb{R}^+)$ for some $\epsilon > 0$. Then the operator $m(\Delta)$ extends to an operator bounded on $L^p(G)$ for all $p \geq 1$.*

Remark 1.1 *The degree of regularity of our L^p multipliers is not expected to be sharp. The interest of our result is not quantitative but qualitative: we give the first example of sub-Laplacians which admit a differentiable L^p functional calculus, $p \neq 2$, on NA-groups with roots not all positive and for which $N \neq \mathbb{R}^n$.*

To prove Theorem 1.1, we estimate the heat kernel $\{p_z\}_{\Re z>0}$ associated with the sub-Laplacian Δ

$$e^{-z\Delta}\phi = \phi *_l p_z, \qquad \phi \in C_0^\infty(G),\ \Re z > 0,$$

where $*_l$ denotes the product of convolution in the space $L^2(G, \mathrm{d}^l g)$ relative to $\mathrm{d}^l g = e^{-(Q+\alpha)t}\mathrm{d}g$, the left invariant Haar measure on G. We show that p_{1+is} is uniformly bounded in $L^1(G)$ by a polynomial in $s \in \mathbb{R}$. It was proved by Hebisch [9], that $m(\Delta)$ is then bounded on $L^p(G)$ for all $p \geq 1$. Following Hebisch, Theorem 1.1 derives from the result below.

Theorem 1.2 *Let G and p_z be as above. There is a constant $C > 0$ such that*

$$\|p_{1+is}\|_{L^1(G)} \leq C(1+|s|)^{Q+\frac{9}{2}}, \qquad s \in \mathbb{R}.$$

Our purpose is thus to establish Theorem 1.2. In order to do so, we estimate the norm of p_{1+is} in $L^1(G)$ by means of its norm in the space $L^2(G)$ weighted by some weight ω. In section 2, we define ω, we give an estimate of p_{1+is} in $L^2(G, \omega \mathrm{d}g)$ (Theorem 2.1), and we show that it implies Theorem 1.2. The end of the paper is devoted to the proof of Theorem 2.1.

We use the variable constant convention, which means that in a sequence of equations, identical names will possibly be applied to different constants (whose dependence in the parameters of the equations is clear). The notations introduced in section 1 and in section 2 hold in the all paper, except in section 4.1 and the appendix, section 7; these two sections contain general results on Lie groups and have their own local notations.

2 Estimates on the heat kernel: L^1 through L^2

Let ρ be the Carnot-Carathéodory distance on G associated with the Hörmander system χ. We denote by $|g| = \rho(e, g)$ the distance from an element g in G to the origin e of G, and by $B_R = \{g \in G : |g| < R\}$, $R > 0$, the ball centered in e of radius R.

Proposition 2.1 *If there exists a function ω non-negative on G such that the following inequalities hold*

$$\|(1+\omega)^{-\frac{1}{2}}\|_{L^2(B_R)} \leq C(1+R)^{\frac{3}{2}}, \qquad R > 0,$$
$$\|\omega^{\frac{1}{2}} p_{1+is}\|_{L^2(G)} \leq C(1+|s|)^{Q+\frac{3}{2}}, \qquad s \in \mathbb{R},$$

then Theorem 1.2 is true.

Proof This result is a rewriting of a result proved by Hebisch [9]. ■

To prove Theorem 1.2, we show that there exists a function ω on G which has the properties required by Proposition 2.1. Let $|\cdot|_H$ be a homogeneous norm on the homogeneous group H, that is a function continuous and non-negative on H, smooth away from the origin e_H of H, and which satisfies for all h in H: $|h|_H = |h^{-1}|_H$; $|\sigma_t h|_H = e^t |h|_H$ for every t in $\mathbb{R}$; $|h|_H = 0$ if and only if $h = e_H$. We put

$$\omega(g) = \omega(h, a, t) = |h|_H^Q |a|, \qquad g = (h, a, t) \in G.$$

Lemma 2.1 *There is a constant $C > 0$ such that*

$$\|(1+\omega)^{-\frac{1}{2}}\|_{L^2(B_R)} \leq C(1+R)^{\frac{3}{2}}, \qquad R > 0.$$

Proof We draw from the geometry of the Lie group G that there exists $C > 0$ such that

$$B_R \subset \{g = (h, a, t) : \ |h|_H \leq Ce^{CR},\ |a| \leq Ce^{CR},\ |t| \leq C(R+1)\}, \quad R > 0.$$

Lemma 2.1 follows from easy computations. ■

Theorem 2.1 *There is a constant $C > 0$ such that*

$$\|\omega^{\frac{1}{2}} p_{1+is}\|_{L^2(G)} \leq C(1+|s|)^{Q+\frac{3}{2}}, \qquad s \in \mathbb{R}.$$

Lemma 2.1 and Theorem 2.1 prove Theorem 1.2, by Proposition 2.1. So the point is now to demonstrate Theorem 2.1.

To prove Theorem 2.1, we show that the function $s \mapsto \|\omega^{\frac{1}{2}} p_{1+is}\|^2_{L^2(G)}$ satisfies a certain differential inequality which, once integrated, implies the expected estimate on $\|\omega^{\frac{1}{2}} p_{1+is}\|^2_{L^2(G)}$. The end of the paper is organized as follows. In section 3, we consider the functions

$$f_{k,\xi}(s) = \||h|_H^{\frac{k}{2}} |a|^{\frac{1}{2}} p_{\xi+is}\|^2_{L^2(G)}, \qquad s \in \mathbb{R},\ \xi \in D_{Q-k},\ k = 0, \dots Q,$$

where D_n, $n \in \mathbb{N}$, denotes the disc $\{z \in \mathbb{C} : |z-1| \leq 1 - \frac{1}{2^n}\}$. Observe that

$$f_{Q,\xi}(s) = f_Q(s) = \|\omega^{\frac{1}{2}} p_{1+is}\|^2_{L^2(G)}, \qquad s \in \mathbb{R},\ \xi \in D_0.$$

We fix $k \in [0, Q]$ and $\xi \in D_{Q-k}$, and we show that $\partial_s f_{k,\xi}$ is bounded by a certain quantity. We evaluate directly one part of this quantity (sections 4 and 5), and we estimate the other part by $f_{0,\xi_0}, \dots f_{k-1,\xi_{k-1}}$ with $\xi_j \in D_{Q-j}$ (section 6). In section 6, we inject these estimates in the estimate of $\partial_s f_{k,\xi}$ and we obtain a certain differential inequality that can be integrated using an argument of induction. This proves, modulo a pointwise estimate on the heat kernel, that $f_{k,\xi}$ is bounded by a polynomial in s; in the particular case where $k = Q$, it proves Theorem 2.1. In the appendix, section 7, we establish a pointwise estimate on the heat kernel with complex time on a general non-unimodular Lie group which completes the proof of Theorem 2.1.

Remark In the sequel, we shall denote by $< \cdot, \cdot >$ the scalar product in $L^2(G)$, and when no ambiguity is possible, by $\| \cdot \|_{L^2}$ the norm $\| \cdot \|_{L^2(G)}$.

3 First step towards a differential inequality

In this section, we prove the following proposition, which provides estimates of the derivatives $\partial_s f_{k,\xi}$ of the functions $f_{k,\xi}$ defined above.

Proposition 3.1 *Let k be an integer in $[0, Q]$. There exists a constant $C_k > 0$ such that, for any ξ in D_{Q-k} and s in $\mathbb{R}$,*

$$|\partial_s \| |h|_H^{\frac{k}{2}} |a|^{\frac{1}{2}} p_{\xi+is}\|^2_{L^2}| \leq \begin{cases} C_k \sum_{\epsilon=0}^{1} \sum_{l=1-\epsilon}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(l+\epsilon\alpha)t}{2}} |h|_H^{\frac{k-l}{2}} |a|^{\frac{1-\epsilon}{2}} p_{\zeta+is}\|^2_{L^2} & \text{when } k = 1, \dots Q, \\ C_0 \sup_{\zeta \in D_{Q+1}} \|e^{\frac{\alpha t}{2}} p_{\zeta+is}\|^2_{L^2} & \text{when } k = 0. \end{cases}$$

First we show some auxiliary results (Lemmata 3.1, 3.2 and 3.3). The proof of Proposition 3.1 is given at the end of the section.

Lemma 3.1 *Let X be a left invariant vector field in χ, λ be real, and n_1, n_2 be integers. Let ϕ be the function defined by $\phi(g) = |h|_H^{n_1} |a|^{n_2}$ where $g = (h, a, t) \in G$. Then there exists a constant $C > 0$ such that, for any $\Re z > \frac{1}{2^{Q+1}}$,*

$$\|e^{\frac{\lambda t}{2}} \phi^{\frac{1}{2}} X p_z\|^2_{L^2} \leq C \sup_{|\zeta - z| < \frac{1}{2^{Q+1}}} \|e^{\frac{\lambda t}{2}} \phi^{\frac{1}{2}} p_\zeta\|^2_{L^2} + \sum_{j=1}^{d+1} | < X_j p_z, e^{\lambda t} X_j(\phi) p_z > |.$$

Proof The proof is a slight modification of an argument of Hebisch [9]. We shall assume that $X \neq X_0$; the proof for $X = X_0$ is similar. From integration by parts

$$\begin{aligned}
&< \Delta p_z, e^{\lambda t}\phi p_z > \\
&= \sum_{j=0}^{d+1} \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} X_j p_z\|_{L^2}^2 + \lambda < X_0 p_z, e^{\lambda t}\phi p_z > + \sum_{j=1}^{d+1} < X_j p_z, e^{\lambda t} X_j(\phi) p_z > \\
&\geq \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} X_0 p_z\|_{L^2}^2 + \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} X p_z\|_{L^2}^2 - |\lambda| \cdot \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} X_0 p_z\|_{L^2} \cdot \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} p_z\|_{L^2} \\
&\quad - \sum_{j=1}^{d+1} | < X_j p_z, e^{\lambda t} X_j(\phi) p_z > | \\
&\geq \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} X p_z\|_{L^2}^2 - \frac{\lambda^2}{4}\|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} p_z\|_{L^2}^2 - \sum_{j=1}^{d+1} | < X_j p_z, e^{\lambda t} X_j(\phi) p_z > |.
\end{aligned}$$

Since p_z depends analytically on z, there exists, by the Cauchy formula, a constant C positive such that, for all $x > \frac{1}{2Q+1}$ and $y \in \mathbb{R}$,

$$\|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} \Delta p_{x+iy}\|_{L^2}^2 = \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} \partial_x p_{x+iy}\|_{L^2}^2 \leq C \sup_{|\zeta-(x+iy)|<\frac{1}{2Q+1}} \|e^{\frac{\lambda t}{2}}\phi^{\frac{1}{2}} p_\zeta\|_{L^2}^2.$$

The desired inequality follows. ■

For k, l integer, and X a left invariant vector field in χ, we set

$$I_{k,l,X}(\xi, s) = \int_G e^{(l+\alpha)t} |h|_H^{k-l} |p_{\xi+is}(h,a,t)|\, |X p_{\xi+is}(h,a,t)| \mathrm{d}h \mathrm{d}a \mathrm{d}t,$$

where ξ is positive and s is real.

Lemma 3.2 *Let k be an integer in $[0, Q]$, l in $[0, k]$, and let X be a left invariant vector field in χ. There exists a constant $C > 0$ such that for any ξ in D_{Q-k} and any real s,*

$$I_{k,l,X}(\xi, s) \leq C \sum_{m=l}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(m+\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} p_{\zeta+is}\|_{L^2}^2.$$

Proof Fix k in $[0, Q]$ and $X \in \chi$.

Let us start by estimating $I_{k,l,X}$ for $l = k$. For any $\xi \in D_{Q-k}$ and $s \in \mathbb{R}$, $\Re(\xi + is) \geq \frac{1}{2Q-k} > \frac{1}{2Q+1}$. Consequently, by Lemma 3.1,

$$I_{k,l,X}(\xi, s) = \int_G e^{(k+\alpha)t} |p_{\xi+is}(h,a,t)|\, |X p_{\xi+is}(h,a,t)| \mathrm{d}h \mathrm{d}a \mathrm{d}t$$

$$
\begin{aligned}
&\leq \|e^{\frac{(k+\alpha)t}{2}} p_{\xi+is}\|_{L^2} \cdot \|e^{\frac{(k+\alpha)t}{2}} X p_{\xi+is}\|_{L^2} \\
&\leq C \sup_{|\zeta-(\xi+is)|<\frac{1}{2Q+1}} \|e^{\frac{(k+\alpha)t}{2}} p_\zeta\|_{L^2}^2 \\
&= C \sup_{|\zeta-\xi|<\frac{1}{2Q+1}} \|e^{\frac{(k+\alpha)t}{2}} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}.
\end{aligned}
$$

For any ζ such that $|\zeta-\xi| < \frac{1}{2Q+1}$, one has $|\zeta - 1| \leq 1 - \frac{1}{2Q-k+1}$; thus

$$
I_{k,l,X}(\xi,s) \leq C \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(k+\alpha)t}{2}} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R},
$$

which is the expected estimate for $l = k$.

Let us now estimate $I_{k,l,X}$ for $l \in [0, k-1]$. Assume that there is $l \in [1, k]$ such that, for some $C > 0$,

$$
I_{k,l,X}(\xi,s) \leq C \sum_{m=l}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(m+\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R},
$$

and let us estimate $I_{k,l-1,X}$. Again by Lemma 3.1,

$$
\begin{aligned}
I_{k,l-1,X}(\xi,s) &= \int_G e^{(l-1+\alpha)t} |h|_H^{k-l+1} |p_{\xi+is}(h,a,t)|\, |X p_{\xi+is}(h,a,t)| \mathrm{d}h \mathrm{d}a \mathrm{d}t \\
&\leq \|e^{\frac{(l-1+\alpha)t}{2}} |h|_H^{\frac{k-l+1}{2}} p_{\xi+is}\|_{L^2} \cdot \|e^{\frac{(l-1+\alpha)t}{2}} |h|^{\frac{k-l+1}{2}} X p_{\xi+is}\|_{L^2} \\
&\leq \|e^{\frac{(l-1+\alpha)t}{2}} |h|_H^{\frac{k-l+1}{2}} p_{\xi+is}\|_{L^2}^2 + \|e^{\frac{(l-1+\alpha)t}{2}} |h|^{\frac{k-l+1}{2}} X p_{\xi+is}\|_{L^2}^2 \\
&\leq C \sup_{|\zeta-(\xi+is)|<\frac{1}{2Q+1}} \|e^{\frac{(l-1+\alpha)t}{2}} |h|_H^{\frac{k-l+1}{2}} p_\zeta\|_{L^2}^2 \\
&\quad + \sum_{j=1}^{d+1} |< X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is} >|, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}.
\end{aligned}
$$

We treat the terms of this inequality separately. First we observe that

$$
\begin{aligned}
\sup_{|\zeta-(\xi+is)|<\frac{1}{2Q+1}} \|e^{\frac{(l-1+\alpha)t}{2}} |h|_H^{\frac{k-l+1}{2}} p_\zeta\|_{L^2}^2 &= \sup_{|\zeta-\xi|<\frac{1}{2Q+1}} \|e^{\frac{(l-1+\alpha)t}{2}} |h|_H^{\frac{k-l+1}{2}} p_{\zeta+is}\|_{L^2}^2 \\
&\leq \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(l-1+\alpha)t}{2}} |h|_H^{\frac{k-l+1}{2}} p_{\zeta+is}\|_{L^2}^2.
\end{aligned}
$$

Next, using $X_{d+1}(|h|_H^{k-l+1}) = 0$, we find that

$$
|< X_{d+1} p_{\xi+is}, e^{(l-1+\alpha)t} X_{d+1}(|h|_H^{k-l+1}) p_{\xi+is} >| = 0.
$$

Now we estimate

$$
|< X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is} >|
$$

for $1 \le j \le d$, remembering that $X_j = e^t \widetilde{X}_j$. An argument of homogeneity on the homogeneous norm $|\cdot|_H$, proves that for all $n \ge 1$ there is $C > 0$ which satisfies $\widetilde{X}_j(|h|_H^n) \le C|h|_H^{n-1}$. Then

$$\begin{aligned} & |< X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is} >| \\ \le \quad & C < |X_j p_{\xi+is}|, e^{(l+\alpha)t}|h|_H^{k-l}|p_{\xi+is}| >, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}. \quad (1) \end{aligned}$$

We recognize $I_{k,l,X_j}(\xi, s)$, bounded by assumption on l. Thus

$$\begin{aligned} & |< X_j p_{\xi+is}, e^{(l-1+\alpha)t} X_j(|h|_H^{k-l+1}) p_{\xi+is} >| \\ \le \quad & C \sum_{m=l}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(m+\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}. \end{aligned}$$

Finally this implies the required estimate on $I_{k,l-1,X}$:

$$I_{k,l-1,X}(\xi, s) \le C \sum_{m=l-1}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(m+\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}.$$

Lemma 3.2 follows. ■

For k, l integer, and X a left invariant vector field in χ, we set

$$J_{k,l,X}(\xi, s) = \int_G e^{lt} |h|_H^{k-l} |a|\, |p_{\xi+is}(h, a, t)|\, |X p_{\xi+is}(h, a, t)| \mathrm{d}h \mathrm{d}a \mathrm{d}t$$

where ξ is positive and s is real.

Lemma 3.3 *Let k be an integer in $[0, Q]$, l in $[0, k]$, and let X be an invariant vector field in χ. There exists a constant $C > 0$ such that, for any ξ in D_{Q-k} and any real s,*

$$J_{k,l,X}(\xi, s) \le C \sum_{\epsilon=0}^{1} \sum_{m=l}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(m+\epsilon\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} |a|^{\frac{1-\epsilon}{2}} p_{\zeta+is}\|_{L^2}^2.$$

Proof The proof is similar to that of Lemma 3.2, modulo the fact that integrals $J_{k,l-1,X}$ are estimated via Lemma 3.1, not only by integrals $J_{k,l,Y}$ with $Y \in \chi$, but also by integrals $I_{k,l-1,Y}$. Lemma 3.2 is used to conclude. ■

Proof of Proposition 3.1 For all integer k in $[0, Q]$,

$$\begin{aligned} & \partial_s \||h|_H^{\frac{k}{2}} |a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2 \\ = \quad & 2\Re < -i\Delta p_{\xi+is}, |h|_H^k |a| p_{\xi+is} > \end{aligned}$$

$$
\begin{aligned}
&= -2\Im\sum_{j=0}^{d+1} < X_j^2 p_{\xi+is}, |h|_H^k |a| p_{\xi+is} > \\
&= 2\Im\sum_{j=0}^{d+1} \Big(\||h|_H^{\frac{k}{2}}|a|^{\frac{1}{2}} X_j p_{\xi+is}\|_{L^2}^2 + < X_j p_{\xi+is}, X_j(|h|_H^k|a|)p_{\xi+is} > \Big) \\
&= 2\sum_{j=1}^{d+1} \Im < X_j p_{\xi+is}, X_j(|h|_H^k|a|)p_{\xi+is} > .
\end{aligned}
$$

Then for $k=0$, $\xi \in D_Q$ and $s \in \mathbb{R}$,

$$
\begin{aligned}
|\partial_s \||a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2| &\leq 2\sum_{j=1}^{d+1} | < X_j p_{\xi+is}, X_j(|a|)p_{\xi+is} > | \\
&\leq 2 < |X_{d+1} p_{\xi+is}|, e^{\alpha t}|p_{\xi+is}| > \\
&= 2 I_{0,0,X_{d+1}}(\xi, s).
\end{aligned}
$$

Hence by Lemma 3.2, there exists $C_0 > 0$ such that

$$
|\partial_s \||a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2| \leq C_0 \sup_{\zeta \in D_{Q+1}} \|e^{\frac{\alpha t}{2}} p_{\zeta+is}\|_{L^2}^2, \qquad \xi \in D_Q,\ s \in \mathbb{R},
$$

which proves Proposition 3.1 for $k=0$.

Now assume that k is in $[1,Q]$. The argument used to deduce the estimate (1) in the proof of Lemma 3.2, implies that there is $C_k > 0$ such that, for every $\xi \in D_{Q-k}$ and $s \in \mathbb{R}$,

$$
\begin{aligned}
&|\partial_s \||h|_H^{\frac{k}{2}}|a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2| \\
&\leq 2\sum_{j=1}^{d+1} | < X_j p_{\xi+is}, X_j(|h|_H^k|a|)p_{\xi+is} > | \\
&\leq C_k \sum_{j=1}^{d} < |X_j p_{\xi+is}|, e^t |h|_H^{k-1}|a||p_{\xi+is}| > + 2 < |X_{d+1} p_{\xi+is}|, e^{\alpha t}|h|_H^k |p_{\xi+is}| > \\
&= C_k \sum_{j=1}^{d} J_{k,1,X_j}(\xi, s) + 2 I_{k,0,X_{d+1}}(\xi, s).
\end{aligned}
$$

By Lemmata 3.2 and 3.3, we have for all $k \in [1,Q]$

$$
\begin{aligned}
&|\partial_s \||h|_H^{\frac{k}{2}}|a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2 \\
&\leq C_k \Big(\sum_{\epsilon=0}^{1}\sum_{m=1}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(m+\epsilon\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} |a|^{\frac{1-\epsilon}{2}} p_{\zeta+is}\|_{L^2}^2 \\
&\quad + \sum_{m=0}^{k} \sup_{\zeta \in D_{Q-k+1}} \|e^{\frac{(m+\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} p_{\zeta+is}\|_{L^2}^2 \Big)
\end{aligned}
$$

$$\leq \quad C_k \sum_{\epsilon=0}^{1} \sum_{m=1-\epsilon}^{k} \sup_{\zeta \in D_{Q-k+1}} \| e^{\frac{(m+\epsilon\alpha)t}{2}} |h|_H^{\frac{k-m}{2}} |a|^{\frac{1-\epsilon}{2}} p_{\zeta+is} \|_{L^2}^2, \quad \xi \in D_{Q-k}, \; s \in \mathbb{R},$$

which ends the proof of Proposition 3.1. ■

4 Estimates on the heat kernel in $L^2(G)$ weighted by exponentials in t

In this section, we establish *explicit* weighted estimates on the heat kernel (Proposition 4.1, Corollary 4.1, Proposition 4.2, Corollary 4.2). Those estimates will allow us to initialize a process of successive integration of differentiable inequality (sections 5 and 6), that will lead to Theorem 2.1 (section 6).

Proposition 4.1 *There is a constant C positive such that*

$$\| e^{\frac{\alpha t}{2}} p_{\xi+is} \|_{L^2}^2 \leq C(1+|s|)^2, \qquad \xi \in D_{Q+1}, \; s \in \mathbb{R}.$$

Proof This result is derived trivially from an estimate established by Hebisch in [9] (p 163). ■

An easy consequence of Proposition 4.1 is the following.

Corollary 4.1 *There is a constant C positive such that, for any integer k in $[\alpha, Q]$,*

$$\| e^{\frac{kt}{2}} p_{\xi+is} \|_{L^2}^2 \leq C(1+|s|)^2, \qquad \xi \in D_{Q+1}, \; s \in \mathbb{R}.$$

Proof Let k be an integer in $[\alpha, Q]$. For every $\xi \in D_{Q+1}$ and $s \in \mathbb{R}$,

$$\begin{aligned} \| e^{\frac{kt}{2}} p_{\xi+is} \|_{L^2}^2 &\leq \int_{-\infty}^{0} \left(\int_{H \times \mathbb{R}} e^{\alpha t} |p_{\xi+is}(h,a,t)|^2 \mathrm{d}h \mathrm{d}a \right) \mathrm{d}t \\ &\quad + \int_{0}^{+\infty} \left(\int_{H \times \mathbb{R}} e^{Qt} |p_{\xi+is}(h,a,t)|^2 \mathrm{d}h \mathrm{d}a \right) \mathrm{d}t \\ &\leq \| e^{\frac{\alpha t}{2}} p_{\xi+is} \|^2 + \int_G e^{Qt} |p_{\xi+is}(g)|^2 \mathrm{d}g. \end{aligned}$$

Now

$$\begin{aligned} \int_G e^{Qt} |p_{\xi+is}(g)|^2 \mathrm{d}g &= \int_G e^{-Qt} |p_{\xi+is}(g^{-1})|^2 \mathrm{d}(g^{-1}) \\ &= \int_G e^{-Qt} |\delta(g) p_{\xi+is}(g)|^2 \delta(g^{-1}) \mathrm{d}g \\ &= \int_G e^{-Qt} e^{(Q+\alpha)t} |p_{\xi+is}(g)|^2 \mathrm{d}g \\ &= \| e^{\frac{\alpha t}{2}} p_{\xi+is} \|_{L^2}^2. \end{aligned}$$

Thus by Proposition 4.1, there exists $C > 0$ such that

$$\|e^{\frac{kt}{2}} p_{\xi+is}\|_{L^2}^2 \leq 2\|e^{\frac{\alpha t}{2}} p_{\xi+is}\|^2 \leq C(1+|s|)^2, \qquad \xi \in D_{Q+1}, s \in \mathbb{R},$$

hence Corollary 4.1. ■

A refinement of the estimate of Hebisch [9] more sophisticated than Proposition 4.1, is given by the following proposition.

Proposition 4.2 *There is a constant C positive such that*

$$\|e^{\frac{\alpha t}{2}} p_{\frac{x}{2}+\eta s+i(y+s)}\|_{L^2}^2 \leq C\frac{s}{\eta^2}, \qquad x+iy \in D_Q,\ \eta \in]0,1[,\ s \geq 1.$$

The proof of Proposition 4.2 uses tools related with Schrödinger operators. In section 4.1 below, we introduce the results on Schrödinger operators we need; then we prove Proposition 4.2 in section 4.2. But before that, we complete the list of our explicit weighted estimates on the heat kernel by the following corollary of Proposition 4.2.

Corollary 4.2 *There is a constant C positive such that, for any integer k in $[1, Q]$,*

$$\|e^{\frac{kt}{2}} p_{\frac{x}{2}+\eta s+i(y+s)}\|_{L^2}^2 \leq C\frac{s}{\eta^2}, \qquad x+iy \in D_Q,\ \eta \in]0,1[,\ s \geq 1.$$

Proof The proof is analogous to that of Corollary 4.1. ■

4.1 On Schrödinger operators

Note that the results presented here are general, and are not specific to the groups introduced in section 1.

Let G_0 be a real connected simply connected Lie group, $\mathfrak{g}_0$ be its Lie algebra, and dg_0 be a right invariant Haar measure on G_0. We consider a family $\chi_0 = \{Y_1, \dots Y_n\}$ of left invariant vector fields on G_0, and we assume that it satisfies Hörmander condition. Let ϱ denote the Carnot-Carathéodory distance relative to χ_0, and τ the corresponding distance to the origin 0 of G_0

$$\tau(g_0) = \varrho(0, g_0), \qquad g_0 \in G_0.$$

Let $\{f_1, \dots f_n\}$ be a family of real functions in $C^1(G_0)$. We define the operators

$$U_j\phi = (Y_j + if_j)\phi, \qquad \phi \in C_0^\infty(G_0),\ j = 1, \dots n,$$

and their adjoint operators U_j^*, $j = 1, \dots n$, in $L^2(G_0)$. We consider the operator $\sum_{j=1}^n U_j^* U_j$ defined on the set $C_0^\infty(G_0)$. That operator is symmetric and non-negative on $L^2(G_0)$, and thus it admits a Friedrichs extension. Let H denote this extension,

$$H = \sum_{j=1}^{n} U_j^* U_j.$$

The operator H is a Schrödinger operator on $L^2(G_0)$. The semigroup e^{-zH} is well defined for $\Re z > 0$, and so is the kernel of the semigroup that we shall denote by $e^{-zH}\delta_0$.

Lemma 4.1 *Let K be a compact set in the half-plan $\{z \in \mathbb{C} : \Re z > 0\}$. There is a constant C positive independent of the family of functions $\{f_1, \ldots f_n\}$ such that for every real c, $x + iy$ in K, η in $]0, 1[$ and $s \geq 1$,*

$$\|e^{-(\frac{x}{2}+\eta s+i(y+s))H}\delta_0\|_{L^2(G_0, e^{2c\tau(g_0)}\mathrm{d}g_0)} \leq C \exp\left(C\frac{s}{\eta^2}c^2\right).$$

Proof It is analogous to the proof of Lemma 1.4 in Hebisch [9]. ■

4.2 Proof of Proposition 4.2

Let the notations be those of sections 1 and 2 again.

Let G_0 denote the Lie group $G_0 = H \times \mathbb{R}$, with H as in section 1, whose product is

$$g_0 \cdot \tilde{g}_0 = (h, t) \cdot (\tilde{h}, \tilde{t}) = (h \cdot \sigma_t \tilde{h}, t + \tilde{t}), \qquad g_0 = (h, t), \tilde{g}_0 = (\tilde{h}, \tilde{t}) \in G_0.$$

We equip G_0 with the right invariant Haar measure $\mathrm{d}g_0 = \mathrm{d}h\mathrm{d}t$, and we identify the left invariant vector fields $\{X_0, \ldots X_d\}$ on G to left invariant vector fields on G_0. It is then easy to show that $\chi_0 = \{X_0, \ldots X_d\}$ is a Hörmander system on G_0. Let ϱ denote the Carnot-Carathéodory distance on G_0 related to χ_0, and τ the corresponding distance from an element in G_0 to the origin 0 of G_0.

Now consider the operator $-\sum_{j=0}^{d} X_j^2 + e^{2\alpha t}a^2$, where a is a real parameter, defined on $C_0^\infty(G_0)$. That operator has the form $\sum_{j=1}^{n} U_j^* U_j$ described above in section 4.1. Thus it admits a Friedrichs extension on $L^2(G_0)$; let us denote by H_a this extension,

$$H_a = -\sum_{j=0}^{d} X_j^2 + e^{2\alpha t}a^2.$$

Let us fix $x + iy \in D_Q$, $\eta \in]0, 1[$ and $s \geq 1$. From the Plancherel formula

$$\|e^{\frac{\alpha t}{2}} p_{\frac{x}{2}+\eta s+i(y+s)}\|^2_{L^2(G)} \quad = \quad \int_{\mathbb{R}} \|e^{\frac{\alpha t}{2}} p_{\frac{x}{2}+\eta s+i(y+s)}\|^2_{L^2(G_0)} \mathrm{d}a$$

$$= \int_{\mathbb{R}} \|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\mathrm{d}a$$
$$= \int_{|a|<e^{-cs/\eta^2}} + \int_{e^{-cs/\eta^2}\leq|a|<1} + \int_{1\leq|a|},$$

whatever positive constant c is. Let us estimate each integral separately.

Easy geometrical considerations on the Lie group G_0 allow us to show that there is a constant $C>0$ for which

$$|t| \leq C(\tau(g_0)+1), \qquad g_0=(h,t)\in G_0.$$

Hence there is $C>0$ such that, for every $a\in\mathbb{R}$,

$$\|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} \leq C\|e^{C\tau(g_0)} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}.$$

From which we draw by Lemma 4.1, that there is $C_1>0$ such that, for every $a\in\mathbb{R}$,

$$\|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} \leq C_1\exp\left(C_1\frac{s}{\eta^2}\right),\ x+iy\in D_Q,\ \eta\in]0,1[,\ s\geq 1.$$

Let us now choose the constant c equal to C_1 in the integrals to estimate. One then has for any $x+iy\in D_Q$, $\eta\in]0,1[$ and $s\geq 1$,

$$\int_{|a|<e^{-C_1 s/\eta^2}} \|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\mathrm{d}a \leq 2C_1. \tag{2}$$

For every $a\neq 0$ and every smooth function ϕ,

$$\|e^{\frac{\alpha t}{2}}\phi\|^2_{L^2(G_0)} \leq \frac{1}{|a|}\|e^{\frac{\alpha t}{2}}|a|^{\frac{1}{2}}\phi\|^2_{L^2(G_0)} \leq \frac{1}{|a|}(\|e^{\alpha t}a\phi\|^2_{L^2(G_0)}+\|\phi\|^2_{L^2(G_0)}).$$

The operator $-\sum_{j=0}^d X_j^2$ is non-negative on $L^2(G_0)$, thus for every $a\neq 0$ and for every function ϕ in the domain of H_a,

$$\begin{aligned}\|e^{\alpha t}a\phi\|^2_{L^2(G_0)} = < e^{2\alpha t}a^2\phi,\phi >_{L^2(G_0)} &\leq < H_a\phi,\phi >_{L^2(G_0)}\\ &\leq \|H_a\phi\|^2_{L^2(G_0)}+\|\phi\|^2_{L^2(G_0)},\end{aligned}$$

which implies

$$\|e^{\frac{\alpha t}{2}}\phi\|^2_{L^2(G_0)} \leq \frac{2}{|a|}(\|H_a\phi\|^2_{L^2(G_0)}+\|\phi\|^2_{L^2(G_0)}).$$

Let us take $\phi=e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0$ in the above inequality. We get

$$\begin{aligned}&\|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\\ &\leq \frac{2}{|a|}(\|H_a e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}+\|e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}).\end{aligned}$$

It is easy to check that the functions $s \mapsto \|H_a e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}$ and $s \mapsto \|e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}$ have non-positive derivatives on $\mathbb{R}^+$. Then they decrease on $\mathbb{R}^+$, and we have

$$\begin{aligned}
&\|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)} \\
&\quad \leq \frac{2}{|a|}(\|H_a e^{-(\frac{x}{2}+iy)H_a}\delta_0\|^2_{L^2(G_0)} + \|e^{-(\frac{x}{2}+iy)H_a}\delta_0\|^2_{L^2(G_0)}) \\
&\quad \leq \frac{2}{|a|}(\|H_a e^{-(\frac{x}{4}+iy)H_a}\|^2_{\mathcal{L}(L^2(G_0),L^2(G_0))} + \|e^{-(\frac{x}{4}+iy)H_a}\|^2_{\mathcal{L}(L^2(G_0),L^2(G_0))}) \\
&\qquad\qquad \times \|e^{-\frac{x}{4}H_a}\delta_0\|^2_{L^2(G_0)}.
\end{aligned}$$

We know from spectral theory that the operators $H_a e^{-(\frac{x}{4}+iy)H_a}$ and $e^{-(\frac{x}{4}+iy)H_a}$ are bounded on $L^2(G_0)$ uniformly in x, y for $x+iy \in D_Q$, and that the bound depends not on the parameter $a \in \mathbb{R}$. We deduce easily from the boundedness of $e^{-(\frac{x}{4}+iy)H_a}$ that $\|e^{-\frac{x}{4}H_a}\delta_0\|^2_{L^2(G_0)}$ is bounded uniformly in x for $x+iy \in D_Q$, and independently of a. As a consequence the second integral is such that, for any $x+iy \in D_Q$, $\eta \in]0,1[$, and $s \geq 1$,

$$\begin{aligned}
\int_{e^{-C_1 s/\eta^2} \leq |a| < 1} \|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\mathrm{d}a &\leq C\int_{e^{-C_1 s/\eta^2} \leq |a| < 1} \frac{1}{|a|}\mathrm{d}a \\
&\leq C\frac{s}{\eta^2}. \qquad (3)
\end{aligned}$$

And we have also

$$\begin{aligned}
\int_{1\leq|a|} \|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\mathrm{d}a &\leq C\int_{1\leq|a|} \frac{1}{|a|}\|e^{-\frac{x}{4}H_a}\delta_0\|^2_{L^2(G_0)}\mathrm{d}a \\
&\leq C\int_{\mathbb{R}} \frac{1}{|a|}\|e^{-\frac{x}{4}H_a}\delta_0\|^2_{L^2(G_0)}\mathrm{d}a \\
&= C\|p_{\frac{x}{4}}\|^2_{L^2(G)}.
\end{aligned}$$

It is a straightforward application of Theorem 7.1 (see Appendix, section 7) that $\|p_{\frac{x}{4}}\|_{L^2(G)}$ is bounded uniformly in x for $x+iy \in D_Q$. Then for any $x+iy \in D_Q$, $\eta \in]0,1[$, and $s \geq 1$,

$$\int_{1\leq|a|} \|e^{\frac{\alpha t}{2}} e^{-(\frac{x}{2}+\eta s+i(y+s))H_a}\delta_0\|^2_{L^2(G_0)}\mathrm{d}a \leq C. \qquad (4)$$

Using the estimates (2), (3), and (4), we find that there is $C > 0$ such that

$$\|e^{\frac{\alpha t}{2}} p_{\frac{x}{2}+\eta s+i(y+s)}\|^2_{L^2(G)} \leq C(1+\frac{s}{\eta^2}) \leq C\frac{s}{\eta^2}, \qquad x+iy \in D_Q,\ \eta \in]0,1[,\ s \geq 1,$$

which proves Proposition 4.2. ■

5 Estimates on the heat kernel in $L^2(G)$ weighted by polynomials in $|h|_H$

In this section, we derive new estimates on the heat kernel in $L^2(G)$ from those of section 4. The main result is given by the following proposition; we shall use it to estimate by polynomials in s some of the terms in the inequality in Proposition 3.1.

Proposition 5.1 *Let k be an integer in $[1, Q]$. There exists a constant C_k positive such that,*

$$\||h|_H^{\frac{k}{2}} p_{\xi+is}\|_{L^2}^2 \le C_k(1+|s|)^{3+k}, \qquad \xi \in D_Q,\ s \in \mathbb{R}.$$

We start by showing two technical lemmata before proving Proposition 5.1 at the end of the section.

Lemma 5.1 *Let k be an integer in $[2, Q]$. There exists a constant C_k positive such that, for any $x + iy$ in D_Q, η in $]0, 1[$, and $s \ge 1$,*

$$\partial_s \||h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s}\|_{L^2}^2 \le \begin{cases} C_k \dfrac{s^{\frac{2}{k}}}{\eta^{1+\frac{4}{k}}} \||h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s}\|_{L^2}^{\frac{2k-4}{k}} & \text{when } k \in [3, Q], \\ C_2 \dfrac{s}{\eta^3} & \text{when } k = 2. \end{cases}$$

Proof Let k integer be in $[2, Q]$. For every $x + iy \in D_Q$ and $\eta \in]0, 1[$,

$$\begin{aligned}
&\partial_s \||h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s}\|_{L^2}^2 \\
&= -2\Re \sum_{j=0}^{d+1} (i+\eta) < X_j^2 p_{\frac{x}{2}+iy+(i+\eta)s}, |h|_H^k p_{\frac{x}{2}+iy+(i+\eta)s} > \\
&= -2\eta \sum_{j=0}^{d+1} \||h|_H^{\frac{k}{2}} X_j p_{\frac{x}{2}+iy+(i+\eta)s}\|_{L^2}^2 \\
&\quad +2\Re \sum_{j=0}^{d+1} (i+\eta) < X_j p_{\frac{x}{2}+iy+(i+\eta)s}, X_j(|h|_H^k) p_{\frac{x}{2}+iy+(i+\eta)s} > \Big) \\
&\le -2\eta \sum_{j=0}^{d+1} \||h|_H^{\frac{k}{2}} X_j p_{\frac{x}{2}+iy+(i+\eta)s}\|_{L^2}^2 \\
&\quad +2|i+\eta| \sum_{j=1}^{d} \Big| < X_j p_{\frac{x}{2}+iy+(i+\eta)s}, X_j(|h|_H^k) p_{\frac{x}{2}+iy+(i+\eta)s} > \Big| \\
&\le -2\eta \sum_{j=0}^{d+1} \||h|_H^{\frac{k}{2}} X_j p_{\frac{x}{2}+iy+(i+\eta)s}\|_{L^2}^2
\end{aligned}$$

$$+2\sqrt{2}\sum_{j=1}^{d} < |X_j p_{\frac{x}{2}+iy+(i+\eta)s}|, |X_j(|h|_H^k)||p_{\frac{x}{2}+iy+(i+\eta)s}| >, \qquad s \geq 1.$$

Then by the argument used to prove estimate (1) in Lemma 3.2, there is $C_k > 0$ such that, for every $x + iy \in D_Q$ and $\eta \in]0,1[$,

$$\begin{aligned}
&\partial_s \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \\
&\leq -2\eta \sum_{j=0}^{d+1} \| |h|_H^{\frac{k}{2}} X_j p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \\
&\quad + C_k \sum_{j=1}^{d} < |X_j p_{\frac{x}{2}+iy+(i+\eta)s}|, e^t |h|_H^{k-1} |p_{\frac{x}{2}+iy+(i+\eta)s}| > \\
&\leq -2\eta \sum_{j=1}^{d} \| |h|_H^{\frac{k}{2}} X_j p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \\
&\quad + C_k \sum_{j=1}^{d} \| |h|_H^{\frac{k}{2}} X_j p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2} \cdot \| e^t |h|_H^{\frac{k}{2}-1} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2} \\
&\leq \frac{C_k}{\eta} \| e^t |h|_H^{\frac{k}{2}-1} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2, \qquad s \geq 1.
\end{aligned}$$

For $k = 2$, this implies by Corollary 4.2 that

$$\begin{aligned}
\partial_s \| |h|_H p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 &\leq \frac{C_2}{\eta} \| e^t p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \\
&\leq C_2 \frac{s}{\eta^3}, \qquad x + iy \in D_Q,\ \eta \in]0,1[,\ s \geq 1,
\end{aligned}$$

which proves Lemma 5.1 for $k = 2$.

For $k \in [3, Q]$,

$$\begin{aligned}
&\partial_s \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \\
&\leq \frac{C_k}{\eta} \| e^t |h|_H^{\frac{k}{2}-1} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \\
&\leq \frac{C_k}{\eta} \| e^{2t} |p_{\frac{x}{2}+iy+(i+\eta)s}|^{\frac{4}{k}} \|_{L^{\frac{k}{2}}} \cdot \| |h|_H^{k-2} |p_{\frac{x}{2}+iy+(i+\eta)s}|^{\frac{2k-4}{k}} \|_{L^{\frac{k}{k-2}}} \\
&= \frac{C_k}{\eta} \| e^{\frac{k}{2}t} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^{\frac{4}{k}} \cdot \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^{\frac{2k-4}{k}}.
\end{aligned}$$

Thus by Corollary 4.2, for every $x + iy \in D_Q$, $\eta \in]0,1[$, $s \geq 1$,

$$\begin{aligned}
\partial_s \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 &\leq \frac{C_k}{\eta} \left(\frac{s}{\eta^2} \right)^{\frac{2}{k}} \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^{\frac{2k-4}{k}} \\
&\leq C_k \frac{s^{\frac{2}{k}}}{\eta^{1+\frac{4}{k}}} \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^{\frac{2k-4}{k}},
\end{aligned}$$

which proves Lemma 5.1 for $k \in [3, Q]$. ■

Lemma 5.2 *Let k be an integer in $[2, Q]$. There exists a constant C_k positive such that, for any $x+iy$ in D_Q, η in $]0,1[$, and $s \geq 1$,*

$$\| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \leq C_k \frac{s^{1+\frac{k}{2}}}{\eta^{2+\frac{k}{2}}}.$$

Proof We estimate $\| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2$ first for $k=2$, then for k in $[3, Q]$.

By Lemma 5.1,

$$\partial_s \| |h|_H p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \leq C_2 \frac{s}{\eta^3}, \qquad s \geq 1.$$

From which we draw that

$$\| |h|_H p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \leq \| |h|_H p_{\frac{x}{2}+\eta+i(y+1)} \|_{L^2}^2 + C_2 \frac{s^2}{\eta^3}, \qquad s \geq 1.$$

Now Theorem 7.1 implies that

$$\| |h|_H p_{\frac{x}{2}+\eta+i(y+1)} \|_{L^2}^2 \leq c, \qquad x+iy \in D_Q,\ \eta \in]0,1[.$$

Then

$$\| |h|_H p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \leq c + C_2 \frac{s^2}{\eta^3} \leq C_2 \frac{s^2}{\eta^3}, \qquad x+iy \in D_Q,\ \eta \in]0,1[,\ s \geq 1,$$

and Lemma 5.2 follows for $k=2$.

Now let us fix $k \in [3, Q]$. We write $\psi(s) = \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2$. Lemma 5.1 ensures that

$$\psi'(s) \leq C_k \frac{s^{\frac{2}{k}}}{\eta^{1+\frac{4}{k}}} \psi^{\frac{k-2}{k}}(s), \qquad s \geq 1.$$

Hence

$$\psi(s) \leq \left(\psi^{\frac{2}{k}}(1) + C_k \frac{s^{1+\frac{2}{k}}}{\eta^{1+\frac{4}{k}}} \right)^{\frac{k}{2}}, \qquad s \geq 1.$$

By Theorem 7.1,

$$\psi(1) = \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+\eta+i(y+1)} \|_{L^2}^2 \leq c_k, \qquad x+iy \in D_Q,\ \eta \in]0,1[.$$

In terms of the heat kernel, we have

$$\begin{aligned} \| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 &\leq \left(c_k + C_k \frac{s^{1+\frac{2}{k}}}{\eta^{1+\frac{4}{k}}} \right)^{\frac{k}{2}} \\ &\leq C_k \frac{s^{1+\frac{k}{2}}}{\eta^{2+\frac{k}{2}}}, \qquad x+iy \in D_Q,\ \eta \in]0,1[,\ s \geq 1, \end{aligned}$$

which proves Lemma 5.2 for $k = [3, Q]$. ■

Proof of Proposition 5.1 Let us estimate $\| |h|_H^{\frac{k}{2}} p_{\xi+is} \|_{L^2}^2$ for $k = 1$ first, and then for $k \in [2, Q]$.

Arguments similar to those used to prove Proposition 3.1 when $k = 0$, show that there is a constant $C_1 > 0$ for which

$$|\partial_s \| |h|_H^{\frac{1}{2}} p_{\xi+is} \|_{L^2}^2 | \leq C_1 \sup_{\zeta \in D_{Q+1}} \| e^{\frac{t}{2}} p_{\zeta+is} \|_{L^2}^2, \qquad \xi \in D_Q, \ s \in \mathbb{R}.$$

Hence by Corollary 4.1,

$$|\partial_s \| |h|_H^{\frac{1}{2}} p_{\xi+is} \|_{L^2}^2 | \leq C_1 (1 + |s|)^2, \qquad s \in \mathbb{R},$$

where C_1 is a constant positive independent on $\xi \in D_Q$. This implies that

$$\| |h|_H^{\frac{1}{2}} p_{\xi+is} \|_{L^2}^2 \leq \| |h|_H^{\frac{1}{2}} p_{\xi} \|_{L^2}^2 + C_1 (1 + |s|)^3, \qquad \xi \in D_Q, \ s \in \mathbb{R}.$$

By Theorem 7.1,

$$\| |h|_H^{\frac{1}{2}} p_{\xi+is} \|_{L^2}^2 \leq C_1 (1 + |s|)^3, \qquad \xi \in D_Q, \ s \in \mathbb{R},$$

which gives Proposition 5.1 for $k = 1$.

Now we assume that k is in $[2, Q]$. On the one hand, Theorem 7.1 implies that there exists $c_k > 0$ such that

$$\| |h|_H^{\frac{k}{2}} p_{\xi+is} \|_{L^2}^2 \leq c_k, \qquad \xi \in D_Q, \ |s| \leq 1.$$

On the other hand, we know from Lemma 5.2 that there is $C_k > 0$ positive such that,

$$\| |h|_H^{\frac{k}{2}} p_{\frac{x}{2}+iy+(i+\eta)s} \|_{L^2}^2 \leq C_k \frac{s^{1+\frac{k}{2}}}{\eta^{2+\frac{k}{2}}}, \qquad x + iy \in D_Q, \ \eta \in]0, 1[, \ s \geq 1.$$

For $\eta = \frac{x}{2s}$, this implies that

$$\| |h|_H^{\frac{k}{2}} p_{x+iy+is} \|_{L^2}^2 \leq C_k s^{1+\frac{k}{2}} \left(\frac{2s}{x} \right)^{2+\frac{k}{2}} \leq C_k s^{3+k}, \qquad x + iy \in D_Q, \ s \geq 1.$$

Now by a process analogous to the one used to established the above estimation, one proves that

$$\| |h|_H^{\frac{k}{2}} p_{x+iy+is} \|_{L^2}^2 \leq C_k |s|^{3+k}, \qquad x + iy \in D_Q, \ s \leq -1.$$

Combining the estimations for $|s| \leq 1$, $s \geq 1$ and $s \leq -1$, we obtain that

$$\| |h|_H^{\frac{k}{2}} p_{\xi+is} \|_{L^2}^2 \leq C_k (1 + |s|)^{3+k}, \qquad \xi \in D_Q, \ s \in \mathbb{R},$$

which proves Proposition 5.1 for $k \in [2, Q]$. ■

6 Proof of Theorem 2.1

In this section, we establish the following proposition which proves Theorem 2.1.

Proposition 6.1 *Let k be an integer in $[0,Q]$. There exits a constant C positive such that*

$$\||h|_H^{\frac{k}{2}} \cdot |a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2 \leq C(1+|s|)^{3+Q+k}, \qquad \xi \in D_{Q-k},\ s \in \mathbb{R}.$$

Denouement Theorem 2.1 follows from Proposition 6.1 with $k = Q$.

Proof of Proposition 6.1 We shall demonstrate Proposition 6.1 by induction on k.

By arguments similar to those used to proved Proposition 5.1 when $k = 1$,

$$\||a|_H^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2 \leq C(1+|s|)^3, \qquad \xi \in D_Q,\ s \in \mathbb{R}.$$

Proposition 6.1 follows for $k = 0$.

Let us now fix k_0 integer in $[1,Q]$, and assume that Proposition 6.1 holds for every integer k in $[0, k_0 - 1]$. Proposition 3.1 ensures that there is $C > 0$ such that for any $\xi \in D_{Q-k_0}$,

$$\begin{aligned}
&|\partial_s \||h|_H^{\frac{k_0}{2}} |a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2| \\
&\leq C\Bigg(\sum_{l=0}^{k_0-1} \sup_{\zeta \in D_{Q-k_0+1}} \|e^{\frac{(l+\alpha)t}{2}} |h|_H^{\frac{k_0-l}{2}} p_{\zeta+is}\|_{L^2}^2 + \sup_{\zeta \in D_{Q-k_0+1}} \|e^{\frac{(k_0+\alpha)t}{2}} p_{\zeta+is}\|_{L^2}^2 \\
&\quad + \sum_{l=1}^{k_0} \sup_{\zeta \in D_{Q-k_0+1}} \|e^{\frac{lt}{2}} |h|_H^{\frac{k_0-l}{2}} |a|^{\frac{1}{2}} p_{\zeta+is}\|_{L^2}^2 \Bigg), \qquad s \in \mathbb{R}.
\end{aligned}$$

This gives an estimate of $|\partial_s \||h|_H^{\frac{k_0}{2}} |a|^{\frac{1}{2}} p_{\xi+is}\|_{L^2}^2|$ in terms of norms of the heat kernel in L^2 weighted. We shall evaluate the norms separately, depending on whether the weight is purely exponential in t, whether it is polynomial in $|h|_H$ and $|a|$, or whether it is polynomial in $|h|_H$ only.

There is only one term with weight purely exponential in t; it is the supremum of $\|e^{\frac{(k_0+\alpha)t}{2}} p_{\zeta+is}\|_{L^2}^2$ for $\zeta \in D_{Q-k_0+1}$. By Corollary 4.1, we have the following estimate

$$\sup_{\zeta \in D_{Q-k_0+1}} \|e^{\frac{(k_0+\alpha)t}{2}} p_{\zeta+is}\|_{L^2}^2 \leq C(1+|s|)^2, \qquad s \in \mathbb{R}.$$

Concerning terms with weights polynomial in $|h|_H$ and $|a|$, they are suprema of $\|e^{\frac{lt}{2}} |h|_H^{\frac{k_0-l}{2}} |a|^{\frac{1}{2}} p_{\zeta+is}\|_{L^2}^2$ for $\zeta \in D_{Q-k_0+1}$, with $l = 1, \ldots k_0$. Let us fix l in $[1,k_0]$. We have

$$\begin{aligned}
&\|e^{\frac{lt}{2}}|h|_H^{\frac{k_0-l}{2}}|a|^{\frac12}p_{\zeta+is}\|_{L^2}^2\\
&\le \int_{-\infty}^0\left(\int_{H\times\mathbb{R}}|h|_H^{k_0-l}|a||p_{\zeta+is}(h,a,t)|^2\mathrm{d}h\mathrm{d}a\right)\mathrm{d}t\\
&\quad+\int_0^{+\infty}\left(\int_{H\times\mathbb{R}}e^{(Q-k_0+l)t}|h|_H^{k_0-l}|a||p_{\zeta+is}(h,a,t)|^2\mathrm{d}h\mathrm{d}a\right)\mathrm{d}t\\
&\le \||h|_H^{\frac{k_0-l}{2}}|a|^{\frac12}p_{\zeta+is}\|_{L^2}^2+\int_G e^{(Q-k_0+l)t}|h|_H^{k_0-l}|a||p_{\zeta+is}(g)|^2\mathrm{d}g.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_G e^{(Q-k_0+l)t}|h|_H^{k_0-l}|a||p_{\zeta+is}(g)|^2\mathrm{d}g\\
&= \int_G e^{-(Q-k_0+l)t}|\sigma_{-t}(h^{-1})|_H^{k_0-l}|-e^{-\alpha t}a||p_{\zeta+is}(g^{-1})|^2\mathrm{d}(g^{-1})\\
&= \int_G e^{-(Q-k_0+l)t}e^{-(k_0-l)t}|h|_H^{k_0-l}e^{-\alpha t}|a||p_{\zeta+is}(g)|^2e^{(Q+\alpha)t}\mathrm{d}g\\
&= \||h|_H^{\frac{k_0-l}{2}}|a|^{\frac12}p_{\zeta+is}\|_{L^2}^2.
\end{aligned}$$

Thus, by assumption on k_0, there is $C>0$ such that,

$$\begin{aligned}
\sup_{\zeta\in D_{Q-k_0+1}}\|e^{\frac{lt}{2}}|h|_H^{\frac{k_0-l}{2}}|a|^{\frac12}p_{\zeta+is}\|_{L^2}^2 &\le 2\sup_{\zeta\in D_{Q-k_0+1}}\||h|_H^{\frac{k_0-l}{2}}|a|^{\frac12}p_{\zeta+is}\|_{L^2}^2\\
&\le C(1+|s|)^{3+Q+k_0-l},\qquad s\in\mathbb{R}.
\end{aligned}$$

Concerning terms with weights polynomial in $|h|_H$ only, they are suprema of $\|e^{\frac{(l+\alpha)t}{2}}|h|_H^{\frac{k_0-l}{2}}p_{\zeta+is}\|_{L^2}^2$ for $\zeta\in D_{Q-k_0+1}$, with $l=0,\dots k_0-1$. Fix l in $[0,k_0-1]$. We have

$$\begin{aligned}
\|e^{\frac{(l+\alpha)t}{2}}|h|_H^{\frac{k_0-l}{2}}p_{\zeta+is}\|_{L^2}^2 &\le \int_{|h[_H<1}\left(\int_{\mathbb{R}\times\mathbb{R}}e^{(l+\alpha)t}|p_{\zeta+is}(h,a,t)|^2\mathrm{d}a\mathrm{d}t\right)\mathrm{d}h\\
&\quad+\int_{|h[_H\ge1}\left(\int_{\mathbb{R}\times\mathbb{R}}e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(h,a,t)|^2\mathrm{d}a\mathrm{d}t\right)\mathrm{d}h\\
&\le \|e^{\frac{(l+\alpha)t}{2}}p_{\zeta+is}\|_{L^2}^2+\int_G e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2\mathrm{d}g.
\end{aligned}$$

On the one hand, by Corollary 4.1,

$$\sup_{\zeta\in D_{Q-k_0+1}}\|e^{\frac{(l+\alpha)t}{2}}p_{\zeta+is}\|_{L^2}^2\le C(1+|s|)^2,\qquad s\in\mathbb{R}.$$

On the other hand,

$$\begin{aligned}\int_G e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2\mathrm{d}g &= \int_G e^{-(l+\alpha)t}|\sigma_{-t}(h^{-1})|_H^{Q-l}|p_{\zeta+is}(g^{-1})|^2\mathrm{d}(g^{-1})\\ &= \int_G e^{-(l+\alpha)t}e^{-(Q-l)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2e^{(Q+\alpha)t}\mathrm{d}g\\ &= \||h|_H^{\frac{Q-l}{2}}p_{\zeta+is}\|_{L^2}^2.\end{aligned}$$

Thus by Proposition 5.1, there is $C>0$ such that,

$$\int_G e^{(l+\alpha)t}|h|_H^{Q-l}|p_{\zeta+is}(g)|^2\mathrm{d}g \le C(1+|s|)^{3+Q-l}, \qquad s\in\mathbb{R}.$$

Finally,

$$\begin{aligned}&|\partial_s\||h|_H^{\frac{k_0}{2}}|a|^{\frac{1}{2}}p_{\xi+is}\|_{L^2}^2|\\ &\le C\left(\sum_{l=0}^{k_0-1}(1+|s|)^{3+Q-l}+(1+|s|)^2+\sum_{l=1}^{k_0}(1+|s|)^{3+Q+k_0-l}\right)\\ &\le C(1+|s|)^{3+Q+k_0-1}, \quad \xi\in D_{Q-k_0},\ s\in\mathbb{R}.\end{aligned}$$

Then by Theorem 7.1, there exists $C>0$ such that,

$$\begin{aligned}\||h|_H^{\frac{k_0}{2}}|a|^{\frac{1}{2}}p_{\xi+is}\|_{L^2}^2| &\le \||h|_H^{\frac{k_0}{2}}|a|^{\frac{1}{2}}p_{\xi}\|_{L^2}^2|+C(1+|s|)^{3+Q+k_0}\\ &\le C(1+|s|)^{3+Q+k_0}, \qquad \xi\in D_{Q-k_0},\ s\in\mathbb{R},\end{aligned}$$

and Proposition 6.1 follows. ■

7 Appendix: Heat kernel with complex time on non-unimodular Lie groups

In this section, we prove a pointwise estimate on the heat kernel with complex time (Theorem 7.1) used in the previous sections. This estimate holds on general non-unimodular Lie groups, which seems to be new when the time is complex.

Let G be a real connected Lie group equipped with a right invariant Haar measure $\mathrm{d}g$. We denote by $\mathrm{d}^l g$ the corresponding left Haar measure and by δ the modular function of G

$$\mathrm{d}g=\delta(g)\mathrm{d}^l g.$$

From now on, we shall assume that the Lie group G is non-unimodular, that is $\delta\not\equiv 1$.

Let $\chi = \{X_1, \ldots, X_n\}$ be a system of left invariant vector fields on G that satisfies Hörmander condition. We form the sub-Laplacian left invariant on G

$$\Delta = \sum_{j=1}^{n} X_j^2.$$

We associate with Δ the semigroup of operators $T^z = e^{-z\Delta}$, $\Re z > 0$, and we denote by p_z the heat kernel

$$e^{-z\Delta}\phi = p_z *_l \phi, \qquad \phi \in C_0^\infty(G),\ \Re z > 0,$$

where $*_l$ denotes the convolution product in $L^2(G, \mathrm{d}^l g)$.

Theorem 7.1 *Let G, δ, p_z be as above, and λ be real positive. For any ϵ in $]0,1[$, there is a constant positive C such that*

$$|p_z(g)| \leq C(\Re z)^{-\frac{n}{2}} e^{\lambda \Re(z)} \delta^{-\frac{1}{2}}(g) \exp\left(-\Re\left(\frac{|g|^2}{(4+\epsilon)z}\right)\right), \qquad g \in G,\ \Re(z) > 0,$$

where n is the local dimension of the Lie group G.

Proof of Theorem 7.1 Theorem 7.1 is a straightforward consequence of Proposition 7.1 below. ■

To state Proposition 7.1, we need to introduce additional notations. Let us put

$$A = \delta^{\frac{1}{2}} \Delta \delta^{-\frac{1}{2}} + \lambda \mathrm{Id}.$$

The operator A is self-adjoint and non-negative on $L^2(G, \mathrm{d}^l g)$. We associate with A the semigroup of operators $S^z = e^{-zA}$, $\Re z > 0$. Elementary computations show that

$$S^z = e^{-\lambda z} \delta^{\frac{1}{2}} T^z \delta^{-\frac{1}{2}}, \qquad \Re z > 0.$$

It follows that, for every $\Re z > 0$,

$$q_z = e^{-\lambda z} \delta^{\frac{1}{2}} p_z \tag{5}$$

satisfies

$$S^z \phi = \phi *_l q_z, \qquad \phi \in C_0^\infty(G).$$

Proposition 7.1 *For any ϵ in $]0,1[$, there is a constant C positive such that*

$$|q_z(g)| \leq C(\Re z)^{-\frac{n}{2}} \exp\left(-\Re\left(\frac{|g|^2}{(4+\epsilon)z}\right)\right), \qquad g \in G,\ \Re(z) > 0.$$

Proof of Proposition 7.1 Following Davies [5] (Theorem 3.4.8), Lemmata 7.1 and 7.2 below prove Proposition 7.1.

Lemma 7.1 *For any ϵ in $]0,1[$, there is a constant C positive such that*

$$q_t(g) \leq Ce^{-\frac{\lambda t}{2}}(t)^{-\frac{n}{2}} \exp\left(-\frac{|g|^2}{(4+\epsilon)t}\right), \qquad g \in G,\ t > 0.$$

Proof By the pointwise estimate on the heat kernel of Varopoulos [15] (see Theorem IX.1.2)

$$p_t(g) \leq C\delta^{-\frac{1}{2}}(g)(\min(t,1))^{-\frac{n}{2}} \exp\left(-\frac{|g|^2}{(4+\epsilon)t}\right), \qquad g \in G,\ t > 0.$$

Lemma 7.1 follows, using (5) with $z = t \in]0, +\infty[$. ■

Lemma 7.2 *There exists a constant C positive such that*

$$|q_z(g)| \leq C(\Re z)^{-\frac{n}{2}}, \qquad g \in G,\ \Re(z) > 0.$$

Proof For any real x positive and any real y,

$$q_{x+iy} = S^{\frac{x}{3}} S^{\frac{x}{3}+iy} q_{\frac{x}{3}}.$$

So

$$\|q_{x+iy}\|_{L^\infty(G,\mathrm{d}^l g)} = \|S^{\frac{x}{3}}\|_{2\to\infty} \|S^{\frac{x}{3}+iy}\|_{2\to 2} \|q_{\frac{x}{3}}\|_{L^2(G,\mathrm{d}^l g)},$$

where

$$\|S^{\frac{x}{3}}\|_{2\to\infty} = \sup\{\|S^{\frac{x}{3}}\phi\|_{L^\infty(G,\mathrm{d}^l g)};\ \|\phi\|_{L^2(G,\mathrm{d}^l g)} \leq 1\},$$
$$\|S^{\frac{x}{3}+iy}\|_{2\to 2} = \sup\{\|S^{\frac{x}{3}+iy}\phi\|_{L^2(G,\mathrm{d}^l g)};\ \|\phi\|_{L^2(G,\mathrm{d}^l g)} \leq 1\}.$$

Let us estimate $\|S^{\frac{x}{3}}\|_{2\to\infty}$. For any $\phi \in C_0^\infty(G)$,

$$\|S^{\frac{x}{3}}\phi\|_{L^\infty(G,\mathrm{d}^l g)} \leq \|\phi *_l q_{\frac{x}{3}}\|_{L^\infty(G,\mathrm{d}^l g)} \leq \|\phi\|_{L^2(G,\mathrm{d}^l g)} \|q_{\frac{x}{3}}\|_{L^2(G,\mathrm{d}^l g)}.$$

Hence

$$\|S^{\frac{x}{3}}\|_{2\to\infty} \leq \|q_{\frac{x}{3}}\|_{L^2(G,\mathrm{d}^l g)},$$

which implies

$$\|q_{x+iy}\|_{L^\infty(G,\mathrm{d}^l g)} = \|S^{\frac{x}{3}+iy}\|_{2\to 2} \|q_{\frac{x}{3}}\|^2_{L^2(G,\mathrm{d}^l g)},$$

We observe that, for any $t > 0$,

$$q_t(g^{-1}) = e^{-\lambda t}\delta^{\frac{1}{2}}(g^{-1})p_t(g^{-1}) = e^{-\lambda t}\delta^{-\frac{1}{2}}(g)p_t(g)\delta(g) = q_t(g), \qquad g \in G.$$

Therefore

$$\|q_{\frac{x}{3}}\|^2_{L^2(G,\mathrm{d}^l g)} = \int_G q_{\frac{x}{3}}(g) q_{\frac{x}{3}}(g^{-1})\mathrm{d}^l g = q_{\frac{x}{3}} *_l q_{\frac{x}{3}}(e) = q_{\frac{2x}{3}}(e)$$

where e is the unity of the Lie group G. Then by Lemma 7.1, we have

$$\|q_{x+iy}\|_{L^\infty(G,\mathrm{d}^l g)} \leq Cx^{-\frac{n}{2}} \|S^{\frac{x}{3}+iy}\|_{2\to 2}, \qquad x > 0,\ y \in \mathbb{R},$$

which implies

$$\|q_{x+iy}\|_{L^\infty(G,\mathrm{d}^l g)} \leq Cx^{-\frac{n}{2}}, \qquad x > 0,\ y \in \mathbb{R}.$$

This completes the proof of Lemma 7.2. ■

References

[1] G. Alexopoulos. Spectral multipliers on Lie groups of polynomial growth. *Proc. Amer. Math. Soc.*, 120(3): 973–979, 1994.

[2] M. Christ. L^p bounds for spectral multipliers on nilpotent groups. *Trans. Amer. Math. Soc.*, 328(1): 73–81, 1991.

[3] M. Christ, D. Müller. On L^p spectral multipliers for a solvable Lie group. *Geom. Funct. Anal.*, 6(5): 860–876, 1996.

[4] M. Cowling, S. Giulini, A. Hulanicki, G. Mauceri. Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth. *Studia Math.*, 111(2): 103–121, 1994.

[5] E. B. Davies. *Heat kernels and spectral theory.* Cambridge University Press, Cambridge, 1989.

[6] G. B. Folland, E. M. Stein. *Hardy spaces on homogeneous groups.* Princeton University Press, Princeton, N.J., 1982.

[7] M. Gnewuch. Spectral multipliers for sub-Laplacians on amenable Lie groups with exponential volume growth. *Math. Z.*, 246(1-2): 69–83, 2004.

[8] W. Hebisch. The subalgebra of $L^1(AN)$ generated by the Laplacian. *Proc. Amer. Math. Soc.*, 117(2): 547–549, 1993.

[9] W. Hebisch. Boundedness of L^1 spectral multipliers for an exponential solvable Lie group. *Colloq. Math.*, 73(1): 155–164, 1997.

[10] W. Hebisch. Spectral multipliers on metabelian groups. *Rev. Mat. Iberoamericana*, 16(3): 597–604, 2000.

[11] W. Hebisch, J. Ludwig, D. Müller. Sub-Laplacians of holomorphic L^p-type on exponential solvable groups. Preprint, 2001.

[12] L. Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104: 93–140, 1960.

[13] J. Ludwig, D. Müller. Sub-Laplacians of holomorphic L^p-type on rank one AN-groups and related solvable groups. *J. Funct. Anal.*, 170(2): 366–427, 2000.

[14] S. Mustapha. Multiplicateurs de Mikhlin pour une classe particulière de groupes non-unimodulaires. *Ann. Inst. Fourier (Grenoble)*, 48(4): 957–966, 1998.

[15] N. Th. Varopoulos, L. Saloff-Coste, T. Coulhon. *Analysis and geometry on groups.* Cambridge University Press, Cambridge, 1992.