Schwartz functions, tempered distributions, and kernel theorem on solvable Lie groups

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0 Introduction

The aim of the present paper is to define an analogue of Schwartz functions and tempered distributions on connected simply connected solvable Lie groups, and to generalize some classical results from Schwartz theory to that setting.

Let G be a connected simply connected solvable Lie group endowed with its right invariant Haar measure. We are interested in forming a (topological) $*$-algebra of smooth integrable functions on G, without any assumption of compactness on the support. Practically, we look for an intermediate function algebra between $C_0^\infty(G)$ and $L^1(G)$ modeled on the Euclidean Schwartz space, *i.e.* a space of smooth functions decreasing with all their derivatives rapidly at infinity on G.

If G is nilpotent there is a naturally defined Schwartz space $\mathcal{S}(G)$, having the characteristics that we want. In this case indeed, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism from the Lie algebra $\mathfrak{g}$ of G onto G. This allows to identify the Lie group G and its Lie algebra $\mathfrak{g}$, and the product law on $G = \mathfrak{g}$ is then given by the Campbell-Baker-Hausdorff product

$$X \cdot_{\text{CBH}} Y = X + Y + \frac{1}{2}[X, Y] + \dots, \qquad X, Y \in \mathfrak{g},$$

which is, in fact, a polynomial mapping since G is nilpotent. One defines the Schwartz space $\mathcal{S}(G)$ as the image under composition with the exponential map of the usual Schwartz space $\mathcal{S}(\mathfrak{g})$ of rapidly decreasing smooth functions on $\mathfrak{g}$ (seen as finite dimensional real vector space); see for example [How77] or [Cor81]. The space $\mathcal{S}(G)$ has the properties that we want by construction: it consists of smooth functions, it contains $C_0^\infty(G)$ as a dense subspace, and it is a dense $*$-subalgebra of $L^1(G)$.

If G is not nilpotent but only solvable, the natural definition for the Schwartz space on G is less clear. Even if we assume that the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism (in which case G is said to be an exponential solvable Lie group), there is no hope to define $\mathcal{S}(G)$ as the image of the Euclidean Schwartz space $\mathcal{S}(\mathfrak{g})$ under composition with the exponential map, because there are functions f in $\mathcal{S}(\mathfrak{g})$ such that $f \circ \exp^{-1}$ fails to be integrable on G. The problem on such a group G, is that the volume of the balls grows exponentially with respect to their radius, so the decay imposed on the function f is not rapid enough to imply the integrability of $f \circ \exp^{-1}$ on G.

In this paper, we define a property of rapid decay on a general connected simply connected solvable G, that takes into account the geometry of G (by asking exponential decay in certain directions). With respect to this notion, we form the space of smooth functions on G that decrease with their derivatives rapidly at infinity. We endow this space with a family of seminorms, and we obtain a dense Fréchet $*$-subalgebra of $L^1(G)$. If G is nilpotent, our algebra of rapidly decreasing smooth function agrees with the Schwartz space $\mathcal{S}(G)$, and if G is exponential, we recover the space $\mathcal{ES}(G)$ introduced by J. Ludwig in [Lud83] (see also [LMEMB03] - we use the notation there).

The next step is to investigate the properties of the dual space of our algebra of rapidly decreasing smooth functions. We find that every element of the dual space is a finite linear combination of slowly increasing functions and derivatives of such functions. In this respect, we retrieve the classical concept of tempered distributions.

The last point that we examine in the paper concerns Schwartz kernel theorem, a result that definitely contributed to the raise of interest for the theory of distributions developped by Laurent Schwartz. Essentially, Schwartz kernel theorem asserts that every continous linear map from the space of rapidly decrasing functions $\mathcal{S}_x(\mathbb{R}^n)$ into the space of tempered distributions $\mathcal{S}'_y(\mathbb{R}^m)$, is given by a unique distribution "in both variables x and y", and that the correspondance between linear forms and distributions is of topological nature. This is a fundamental result in functional analysis, because it enables to unify the class of integral operators and that of differential operators. We show that, for our spaces of rapidly decreasing functions and of tempered distributions, Schwartz kernel theorem remains valid on connected simply connected solvable Lie groups.

Guide to the reader In section 1, we introduce the notations that will hold along the paper and we state our first result: the existence on a general connected simply connected solvable Lie group G of a dense Fréchet $*$-subalgebra of $L^1(G)$ (with Fréchet topology stronger than L^1 topology). This result is proved in section 2, where we define a weigth function σ on G, and we form the space $\mathcal{S}_\sigma(G)$ of smooth functions on G that decrease, together with all their derivatives, rapidly at infinity with respect to σ. Equipped with a countable family of seminorms, $\mathcal{S}_\sigma(G)$ is a dense Fréchet $*$-subalgebra of $L^1(G)$. In section 4, we consider a list of examples, typical for solvable connected simply connected Lie groups. We show how the weight σ captures fundamental geometrical properties of the groups, and give explicit description of the algebra $\mathcal{S}_\sigma(G)$ in each case. We see in particular that our algebra $\mathcal{S}_\sigma(G)$ agrees with the usual Schwartz space $\mathcal{S}(G)$ for G nilpotent, and with J. Ludwig's space $\mathcal{ES}(G)$ for G exponential. In section 5, we study the dual space $\mathcal{S}'_\sigma(G)$ of $\mathcal{S}_\sigma(G)$ and we show that the linear forms on $\mathcal{S}_\sigma(G)$ can be described in terms of σ-slowly increasing functions and their derivatives. Finally in section 5, we prove a variant of L. Schwartz's kernel theorem on connected simply connected solvable Lie groups for the algebra $\mathcal{S}_\sigma$.

1 Toward Schwartz algebra – Preliminaries

1.1 Existence theorem

Let G be a connected simply connected solvable Lie group endowed with right invariant Haar measure dg, and $\mathfrak{g}$ be its Lie algebra. Fix $\{X_1, \dots, X_m\}$ a basis of the Lie algebra $\mathfrak{g}$. For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, we define the left invariant differential operator X^α on G by

$$X^\alpha \phi(g) = \frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \dots \frac{\partial^{\alpha_m}}{\partial t_m^{\alpha_m}} \phi\big(g \exp(t_1 X_1) \dots \exp(t_m X_m)\big)\Big|_{t_1 = \dots = t_m = 0}, \ \phi \in C^\infty(G).$$

Definition 1. *Let ς be a weight function on G,* i.e *a Borel function with values in $\mathbb{R}^+$. Define $\mathcal{S}_\varsigma(G)$ to be the set of C^∞ functions ϕ on G such that*

$$\|\phi\|_{k,\alpha}^\infty := \|\varsigma^k X^\alpha \phi\|_{L^\infty(G)} \ < \ \infty$$

for every $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$. Clearly, $\mathcal{S}_\varsigma(G)$ has a vector space structure. We call $\mathcal{S}_\varsigma(G)$ the space of smooth functions decreasing ς-rapidly at infinity on G, and we equip it with the topology of the seminorms $\|\cdot\|^{\infty}_{k,\alpha}$.

Remark 1. *The definition of the (topological) vector space $\mathcal{S}_\varsigma(G)$ is independent of the choice of the basis $\{X_1, \dots, X_m\}$, because all the left invariant vectors differential operators are finite linear combinations of X^α's by Poincaré-Birkhoff-Witt theorem.*

By construction, $\mathcal{S}_\varsigma(G)$ is a locally convex space, but an adequate choice of ς makes its structure much richer:

Theorem 1. *Let G, and dg be as above. There exists a weight function ς on G, such that $\mathcal{S}_\varsigma(G)$ is a dense Fréchet *-subalgebra of the convolution algebra $L^1(G, dg)$.*

To prove Theorem 1, we construct explicitly a weight function ς on the Lie group G with the required property. It is then convenient to consider some particular realization for G due to J. Ludwig, in which the weight function ς has a simple expression. For the sake of the reader, we recall below the main lines of Ludwig's construction, and we refer to [Lud, chap 4] for the details of the arguments.

1.2 Special realization of the Lie group G

Let G and $\mathfrak{g}$ be as in § 1.1, and let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. A linear form λ on the complexification $\mathfrak{g}_\mathbb{C}$ of the Lie algebra $\mathfrak{g}$, is called a root if there exists an ideal $\mathfrak{a}$ of $\mathfrak{g}$ and some non-zero $w \in \mathfrak{g}_\mathbb{C}/\mathfrak{a}$, such that $\mathrm{ad}(v)w - \lambda(v)w \in \mathfrak{a}$ for all $v \in \mathfrak{g}$. We denote by $\mathcal{R}$ the set of roots of $\mathfrak{g}$. Take $X \in \mathfrak{g}$ in general position with respect to $\mathcal{R}$ (which means that $\lambda(X) \neq \lambda'(X)$ for any two distinct roots λ, λ'). The set of $\lambda(X)$ ($\lambda \in \mathcal{R}$) is the set of all the (distinct) eigenvalues of the linear operator $\mathrm{ad}(X)$ on $\mathfrak{g}_\mathbb{C}$. We therefore have a Jordan decomposition of the space $\mathfrak{g}_\mathbb{C}$

$$\mathfrak{g}_\mathbb{C} = \bigoplus_{\lambda \in \mathcal{R}} \mathfrak{g}_{\mathbb{C},\lambda},$$

along the generalized eigenspaces of $\mathrm{ad}(X)$

$$\mathfrak{g}_{\mathbb{C},\lambda} = \big\{ Y \in \mathfrak{g}_\mathbb{C};\ (\mathrm{ad}(X) - \lambda(X)\mathrm{Id}_{\mathfrak{g}_\mathbb{C}})^{\dim \mathfrak{g}}(Y) = 0 \big\}.$$

By standard arguments, we have

$$[\mathfrak{g}_{\mathbb{C},\lambda}, \mathfrak{g}_{\mathbb{C},\lambda'}] \subset \mathfrak{g}_{\mathbb{C},\lambda+\lambda'} \qquad \lambda, \lambda' \in \mathcal{R},$$

which implies in particular that $\mathfrak{g}_0 := \mathfrak{g}_{\mathbb{C},0} \cap \mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$, and we have also

$$\bigoplus_{\lambda \in \mathcal{R}\setminus\{0\}} \mathfrak{g}_{\mathbb{C},\lambda} \subset [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}] \subset \mathfrak{n}_{\mathbb{C}},$$

where $\mathfrak{n}_{\mathbb{C}}$ denote the complexification of the nilradical $\mathfrak{n}$. It follows that

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{n}.$$

We choose a subspace $\mathfrak{t}$ in $\mathfrak{g}_0$ such that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}.$$

Let us now define a Lie group structure on $\boldsymbol{G} = \mathfrak{t} \times \mathfrak{n}$. Since $\mathfrak{t} \subset \mathfrak{g}_0$ nilpotent, the Campbell-Baker-Hausdorff product of two elements T and T' of $\mathfrak{t}$, is given by a finite expression

$$T \cdot_{\text{CBH}} T' = T + T' + \frac{1}{2}[T, T'] + \cdots = T + T' + P(T, T')$$

where $P : \mathfrak{t} \times \mathfrak{t} \to \mathfrak{g}_0 \cap \mathfrak{n}$ is a polynomial mapping. We endow $\boldsymbol{G} = \mathfrak{t} \times \mathfrak{n}$ with the product law

$$(t, u) \cdot (t', u') = (t + t', P(t, t') \cdot_{\text{CBH}} u \cdot_{\text{CBH}} e^{\text{ad} t} u'), \qquad (t, u), (t', u') \in \boldsymbol{G}. \quad (1)$$

It is easy to check that $(\boldsymbol{G}, \cdot)$ is a Lie group, which admits as Lie algebra $\mathfrak{t} \times \mathfrak{n}$ with the Lie bracket

$$[(T, U), (T', U')] = (0, [T, T'] + [U, U'] + [T, U'] + [T', U]). \qquad (2)$$

In particular, the Lie algebra of $\boldsymbol{G}$ is isomorphic to $\mathfrak{g}$. Since $\boldsymbol{G}$ is connected and simply connected by construction, we have $\boldsymbol{G} = G$ as Lie groups.

Convention on the notations In the rest of paper, $\mathfrak{n}$, $\mathfrak{t}$, and $P : \mathfrak{t} \times \mathfrak{t} \to \mathfrak{g}_0 \cap \mathfrak{n}$ will be as above, and we will use the realization $(\boldsymbol{G}, \cdot)$ for the Lie group G. So, from now on, $G = \mathfrak{t} \times \mathfrak{n}$ with the product law (1).

Remark 2. *The analytic subgroup $N = \{0\} \times \mathfrak{n}$ of G has a Lie algebra which isomorphic to $\mathfrak{n}$ (namely $\{0\} \times \mathfrak{n}$ with Lie bracket (2) for $(0, U)$ elements of $\mathfrak{t} \times \mathfrak{n}$). Since N is simply connected, N is by definition the nilradical of G. In what follows, we will use the notation N equally for the subgroup $\{0\} \times \mathfrak{n}$ of G and for the connected simply connected nilpotent Lie group $(\mathfrak{n}, \cdot_{\text{CBH}})$.*

1.3 More definitions and notations

Let G be as is § 1.1 and § 1.2; the notations there are preserved in the all remaining part of the paper.

Let $d \in \mathbb{N}^*, k \in \mathbb{N}$ be such that $\mathfrak{n} \cong \mathbb{R}^d$ and $\mathfrak{t} \cong \mathbb{R}^k$. We denote by dn and dt be the Lebesgue measures on $\mathfrak{n} \cong \mathbb{R}^d$ and $\mathfrak{t} \cong \mathbb{R}^k$ respectively. Then $dg = dt\, dn$ is a right invariant Haar measure on G, and $d^l g = e^{-\mathrm{ad} t} dt\, dn$ the corresponding left invariant one. We write $\delta(t, n) = e^{\mathrm{ad} t}$ the modular function on G.

Denote by $|\cdot|_{\mathbb{R}^d}$, $|\cdot|_{\mathbb{R}^k}$, $|\cdot|_{\mathbb{R}^m}$ the Euclidean norms on $\mathfrak{n} \cong \mathbb{R}^d$, $\mathfrak{t} \cong \mathbb{R}^k$, $\mathfrak{g} = \mathfrak{t} \times \mathfrak{n} \cong \mathbb{R}^m$ respectively, and by $\|\cdot\|_{\mathrm{op}\, \mathfrak{n}}$, $\|\cdot\|_{\mathrm{op}\, \mathfrak{g}}$ the related operator norms on $\mathfrak{n}$ and $\mathfrak{g}$.

Let U be some symmetric compact neighbourhood of identity e in G. We define the *length* of an element g of G by

$$|g|_G = \inf\{j \in \mathbb{N};\ g \in U^j = U \cdots U\}$$

where by convention $U^0 = \{e\}$. Observe that the length function is clearly subadditive $|g \cdot h|_G \leq |g|_G + |h|_G\ \forall g, h \in G$, and since U is symmetric, it satisfies $|g^{-1}|_G = |g|_G\ \forall g \in G$.

Since G is connected, the nilradical N of G is a closed analytic subgroup of G and $V = U \cap N$ is a symmetric compact neighbourhood of identity in N. Besides the length $|n|_G$ of elements n of N seen as elements of G, we endow N with an intrisic length

$$|n|_N = \inf\{j \in \mathbb{N};\ n \in V^j = V \cdots V\}$$

These two lengths on N are related in the following way:

$$|n|_G \leq |n|_N \leq C \exp(C|n|_G), \qquad n \in N.$$

The left hand inequality is clear from the definitions of $|\cdot|_N$ and $|\cdot|_G$. The right hand inequality says that the nilradical of a connected simply connected (solvable) Lie group has exponential distorsion (see [Var94]).

For a function ϕ on G, we use the notation $\check{\phi}$ for $\check{\phi}(g) := \phi(g^{-1})$.

2 Rapidly decreasing functions on solvable Lie groups

2.1 Definition of the weight function and reformulation of Theorem 1

Let σ be the weight function on G defined by

$$\sigma(g) := \max\left(\|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}},\ \|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}\,\mathfrak{g}}\right) \cdot \left(1 + |g|_G + |n|_N\right), \quad g = (t, n) \in G,$$

where Ad denotes the adjoint representation of G on its Lie algebra $\mathfrak{g}$ (σ is well defined as weight function, since it is continuous on G, with real values greater than one). We reformulate Theorem 1 as follows.

Theorem 1′. *For G, dg, and σ as above, let $\mathcal{S}_\sigma(G)$ be the space of smooth functions decreasing σ-rapidly at infinity on G. Then $\mathcal{S}_\sigma(G)$ is a dense Fréchet $*$-subalgebra of $L^1(G, dg)$.*

2.2 Properties of the weight function

In this section, we establish some properties of the weight function σ, that we need to prove Theorem 1′.

Property 1. *The weight σ compensate the growth of the volume of G,* i.e. *there exists $p \in \mathbb{N}$ such that*

$$\int_G \frac{1}{\sigma^p(g)} dg < \infty .$$

Proof In the $\mathfrak{t}$ coordinate, the growth of the volume is compensated by the length of $g = (t, n)$ in G. It follows from the estimate

$$|t|_{\mathbb{R}^k} \leq C|g|_G, \qquad g = (t, n) \in G, \tag{3}$$

where $C > 0$ independent from g. Let us prove (3). Assume that $|g|_G = j$. By definition, there are j elements $(t_1, n_1), \ldots, (t_j, n_j)$ in the compact subset U of G, such that $g = (t_1, n_1) \cdots (t_j, n_j)$. But in the $\mathfrak{t}$ component, the product on $G = (\mathfrak{t} \times \mathfrak{n})$ is simply a sum, so that

$$|t|_{\mathbb{R}^k} = |t_1 + \ldots + t_j|_{\mathbb{R}^k} \leq j \cdot \sup\{|a|_{\mathbb{R}^k}; (a, b) \in U\} = C|g|_G, \qquad g = (t, n).$$

In the $\mathfrak{n}$ coordinate, the growth of the volume is compensated by $|n|_N$. Recall that as a nilpotent Lie group $N = (\mathfrak{n}, \cdot_{\text{CBH}})$. It is known since Dixmier [Dix60] that in this case, the length $|n|_N$ dominates $|n|_{\mathbb{R}^d}$ up to an integer power. In fact the reverse controle is also true, and there are $q \in \mathbb{N}^*$ and $C > 1$ such that

$$C^{-1}\, |n|_{\mathbb{R}^d}^{1/q} \,\leq\, |n|_N \,\leq\, C\left(|n|_{\mathbb{R}^d} + 1\right), \qquad n \in N = \mathfrak{n}, \tag{4}$$

(see *e.g.* [LMB95]).

Putting (3) and (4) together, we obtain that

$$\sigma(g) \geq 1 + |g|_G + |n|_N \geq C(1 + |t|_{\mathbb{R}^k} + |n|_{\mathbb{R}^d}^{1/q}) \geq C(1 + |t|_{\mathbb{R}^k} + |n|_{\mathbb{R}^d})^{1/q},$$

with $C > 0$ independent from $g = (t, n) \in G$. It follows that for $p \geq \frac{k+d+2}{q}$

$$\begin{aligned} \int_G \frac{1}{\sigma^p(g)} dg \;&\leq\; C \int_{\mathbb{R}^k} \int_{\mathbb{R}^d} \frac{1}{(1 + |t|_{\mathbb{R}^k} + |n|_{\mathbb{R}^d})^{r+d+2}} dndt \\ &\leq\; C \int_{\mathbb{R}^{k+d}} \frac{1}{(1 + |(t,n)|_{\mathbb{R}^{k+d}})^{k+d+2}} dtdn < \infty. \end{aligned}$$

□

Property 2. *The weight σ dominates the modular function on G up to the power m (the dimension of G):*

$$\delta(g) \leq \sigma^m(g), \qquad g \in G.$$

Proof The left and right Haar measures are obtained by translations of a nonvanishing differential form $\omega \in \bigwedge^m T_e^* G$. The action of G on $\bigwedge^m T_e^* G$ induces at a point $g \in G$ a corrective factor between the two measures, the modular function of the group, which is

$$\delta(g) = |\det \mathrm{Ad}(g)|.$$

Now this is an elementary property of the operator norm that it dominates the module of every eigenvalue. Then $|\det \mathrm{Ad}(g)| \leq \|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}}^m$, and hence

$$\delta(g) \leq \|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}}^m \leq \sigma^m(g), \qquad g \in G.$$

□

Property 3. *The weight σ and its inverse weight $\check{\sigma}$ dominate one another up to integer powers: there exist $C > 1$ and $r \in \mathbb{N}^*$ such that*

$$C^{-1}\sigma^{1/r}(g) \leq \check{\sigma}(g) \leq C\sigma^r(g), \qquad g \in G.$$

Proof The inequality on the left hand side follows from the one on the right hand side by change of variable $g \leftrightarrow g^{-1}$, so it suffices to show that σ dominates $\check{\sigma}$ up to integer power.

Let $g = (t, n) \in G$. We have

$$\begin{aligned} \check{\sigma}(g) &= \sigma(g^{-1}) = \sigma(-t, -e^{\operatorname{ad} t} n) \\ &= \max\big(\|\operatorname{Ad}(g^{-1})\|_{\operatorname{op}\mathfrak{g}}, \|\operatorname{Ad}(g)\|_{\operatorname{op}\mathfrak{g}}\big) \cdot \big(1 + |g^{-1}|_G + |-e^{\operatorname{ad} t} n|_N\big). \end{aligned}$$

In this last expression, the factor on the left is unchanged by the transformation $g \mapsto g^{-1}$, and it is a property of the length function that $|g^{-1}|_G = |g|_G$. So we just have to estimate the term $|-e^{\operatorname{ad} t} n|_N$.

Since the product on $N = \mathfrak{n}$ is given by the Campbell-Baker-Hausdorff formula, the inverse of an element n in N is $-n$. Therefore $|-e^{\operatorname{ad} t} n|_N = |e^{\operatorname{ad} t} n|_N$. Now by (4)

$$|e^{\operatorname{ad} t} n|_N \leq C\big(|e^{\operatorname{ad} t} n|_{\mathbb{R}^d} + 1\big) \leq C\big(\|e^{\operatorname{ad} t}\|_{\operatorname{op}\mathfrak{n}} \cdot |n|_{\mathbb{R}^d} + 1\big), \quad (t, n) \in \mathfrak{t} \times \mathfrak{n}. \quad (5)$$

From the construction in § 1.2 follows that the operator $e^{\operatorname{ad} t} : \mathfrak{n} \to \mathfrak{n}$ is the restriction to $\mathfrak{n}$ of the adjoint operator $\operatorname{Ad}(t, 0)$, $(t, 0) \in \mathfrak{t} \times \mathfrak{n}$ (the restriction is well defined because $\mathfrak{n}$ is the nilradical of the Lie algebra $\mathfrak{g}$, hence an ideal of $\mathfrak{g}$, so it is stable by the action of $\operatorname{Ad}(h)$ $\forall h \in G$). Writing $(t, 0) = (0, -n) \cdot (t, n)$, we see that $\operatorname{Ad}(t, 0) = \operatorname{Ad}(0, -n) \circ \operatorname{Ad}(t, n)$ as operators on $\mathfrak{g}$. By restriction to the subspace $\mathfrak{n}$, we obtain that $e^{\operatorname{ad} t} = \operatorname{Ad}(0, -n)|_{\mathfrak{n}} \circ \operatorname{Ad}(t, n)|_{\mathfrak{n}}$.

Again by construction, the restriction to $\mathfrak{n}$ of the operator $\operatorname{Ad}(0, n)$ coincides with the adjoint operator $\operatorname{Ad}(n)$, where this last Ad denotes the adjoint representation of the Lie group N. But N is a nilpotent connected Lie group, so the length $|\cdot|_N$ bounds the Ad operator, that is there $C > 0$ and $q' \in \mathbb{N}$ such that

$$\|\operatorname{Ad}(n)\|_{\operatorname{op}\mathfrak{n}} \leq C|n|_N^{q'}, \qquad n \in N \qquad (6)$$

(see [Sch93, Theorem 1.4.3]). Since the operator norm is submultiplicative, it follows that

$$\begin{aligned} \|e^{\operatorname{ad} t}\|_{\operatorname{op}\mathfrak{n}} &\leq \big\|\operatorname{Ad}(0, -n)|_{\mathfrak{n}}\big\|_{\operatorname{op}\mathfrak{n}} \cdot \big\|\operatorname{Ad}(t, n)|_{\mathfrak{n}}\big\|_{\operatorname{op}\mathfrak{n}} \\ &\leq \|\operatorname{Ad}(-n)\|_{\operatorname{op}\mathfrak{n}} \cdot \|\operatorname{Ad}(t, n)\|_{\operatorname{op}\mathfrak{g}} \\ &\leq C|n|_N^{q'} \cdot \|\operatorname{Ad}(g)\|_{\operatorname{op}\mathfrak{g}}, \end{aligned} \qquad (7)$$

with $C > 0$ independent of $g = (t, n) \in G$. By injecting this estimate in (5), then using (4), we obtain that

$$\begin{aligned} |e^{\mathrm{ad}t} n|_N &\leq C\big(|n|_N^{q'} \cdot \|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}} \cdot |n|_{\mathbb{R}^d} + 1\big) \\ &\leq C\big(\|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}} \cdot |n|_N^{q+q'} + 1\big), \qquad g = (t,n) \in G. \end{aligned}$$

It implies that there are $C > 0$ and $q, q' \in \mathbb{N}$, such that

$$\begin{aligned} \check{\sigma}(g) &\leq C \max\left(\|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}}, \|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}\,\mathfrak{g}}\right)^2 \cdot \left(1 + |g|_G + |n|_N^{q+q'}\right) \\ &\leq C\sigma(g)^{\max(2, q+q')}, \end{aligned}$$

for every $g = (t, n) \in G$. That proves Property 3 with $r = \max(2, q+q')$. □

Property 4. *The weight σ is sub-polynomial,* i.e *there exist $C > 0$ and $s \in \mathbb{N}$ such that*

$$\sigma(g \cdot g') \leq C\sigma^s(g) \cdot \sigma^s(g'), \qquad g, g' \in G.$$

Proof Let $g = (t, n)$, $g' = (t', n') \in G$. By definition

$$\sigma(g \cdot g') = \max\left(\|\mathrm{Ad}(g \cdot g')\|_{\mathrm{op}\,\mathfrak{g}}, \|\mathrm{Ad}((g \cdot g')^{-1})\|_{\mathrm{op}\,\mathfrak{g}}\right) \cdot (1 + |g \cdot g'|_G + |n + e^{\mathrm{ad}t} n'|_N).$$

A first easy remark is that the function $g \mapsto \max\left(\|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}}, \|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}\,\mathfrak{g}}\right)$ is submulticative on G (because Ad is an homomorphism of G into $\mathrm{Aut}(\mathfrak{g})$, and the operator norm is submultiplicative). The first factor in the expression of $\sigma(g \cdot g')$ is then compatible with the sub-polynomiality of σ, and we just need care about the second factor. .

In the second factor now, the term $1 + |g \cdot g'|_G$ is harmless. Indeed by subadditivity of $|\cdot|_G$,

$$1 + |g \cdot g'|_G \leq (1 + |g|_G) \cdot (1 + |g'|_G).$$

So to prove the sub-polynomiality of σ, we just have to check that the term $|n + e^{\mathrm{ad}t} n'|_N$ is dominated by integer powers of $\sigma(g)$ and $\sigma(g')$.

We argue like in the proof of Property 3. By subadditivity of $|\cdot|_N$ and estimates (5), (7), (4), we have

$$\begin{aligned} |n + e^{\mathrm{ad}t} n'|_N &\leq |n|_N + |e^{\mathrm{ad}t} n'|_N \\ &\leq |n|_N + C(\|e^{\mathrm{ad}t}\|_{\mathrm{op}\,\mathfrak{n}} \cdot |n'|_{\mathbb{R}^d} + 1) \\ &\leq |n|_N + C(|n|_N^{q'} \cdot \|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{n}} \cdot |n'|_{\mathbb{R}^d} + 1) \\ &\leq C|n|_N^{\max(1,q')} \cdot \|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}} \cdot (1 + |n'|_N^q), \end{aligned}$$

with $C > 0$, $q, q' \in \mathbb{N}$ independent of $g = (n,t)$ and $g' = (n',t')$.

Coming back to $\sigma(g \cdot g')$, we proved that

$$\begin{aligned} &\sigma(g \cdot g') \\ &\le \max\left(\|\mathrm{Ad}(g)\|_{\mathrm{op}\,\mathfrak{g}}, \|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}\,\mathfrak{g}}\right)^2 \cdot \max\left(\|\mathrm{Ad}(g')\|_{\mathrm{op}\,\mathfrak{g}}, \|\mathrm{Ad}(g'^{-1})\|_{\mathrm{op}\,\mathfrak{g}}\right) \\ &\quad \cdot (1 + |g|_G + |n|_N)^{\max(1,q')} \cdot (1 + |g'|_G + |n'|_N)^q \\ &\le C\sigma(g)^{\max(2,q')} \cdot \sigma(g')^q, \end{aligned}$$

with $C > 0$ and $q, q' \in \mathbb{N}$ independent of g and g'. □

2.3 Proof of Theorem 1′

Step one: $\mathcal{S}_\sigma(G)$ is a dense subset of $L^1(G, dg)$.

The inclusion follows immediately from the fact that the weight σ compensate the growth of the volume of the group G. Indeed if $\phi \in \mathcal{S}_\sigma(G)$, then by Property 1 there is $p \in \mathbb{N}$ and $C > 0$ such that

$$\int_G |\phi(g)| dg \le \int_G \frac{1}{\sigma^p(g)} dg \cdot \|\sigma^p \phi\|_{L^\infty(G)} = C\|\phi\|^\infty_{p,0} < \infty,$$

so $\phi \in L^1(G, dg)$.

The density is clear since the set $C_0^\infty(G)$ of differentiable compactly supported functions on G is already dense in $L^1(G, dg)$, and it is obvious that $C_0^\infty(G) \subset \mathcal{S}_\sigma(G)$.

Step two: $\mathcal{S}_\sigma(G)$ is a Fréchet space.

The family of seminorms $\|\cdot\|^\infty_{k,\alpha}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^m$, is countable and separating ($\|\phi\|^\infty_{0,0} = \|\phi\|_{L^\infty(G)} = 0 \Rightarrow \phi = 0$) on the vector space $\mathcal{S}_\sigma(G)$, so it defines a locally convex topology on $\mathcal{S}_\sigma(G)$. To prove that $\mathcal{S}_\sigma(G)$ is a Fréchet space, we just have to show that it is complete. The proof is modeled on the Euclidean case.

Let $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}_\sigma(G)$ be a Cauchy sequence for the seminorms $\|\cdot\|^\infty_{k,\alpha}$. For every $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$, the functions $\sigma^k X^\alpha \phi_n$ converge uniformly to a bounded function $\phi_{k,\alpha}$. If we show that

$$\phi_{k,\alpha} = \sigma^k X^\alpha \phi_{0,0}, \qquad k \in \mathbb{N},\ \alpha \in \mathbb{N}^m, \tag{8}$$

it will prove that $\phi_{0,0}$ belong to the space $\mathcal{S}_\sigma(G)$ and that ϕ_n converges to $\phi_{0,0}$ in $\mathcal{S}_\sigma(G)$, hence that $\mathcal{S}_\sigma(G)$ is complete.

Let us show (8). For $k = 0$ and α of length one, say $\alpha = \alpha_i$ with all coordinates equal to zero but the i^{th} equal to one, we have for all $t \in \mathbb{R}$

$$\phi_n(g \exp(tX_i)) = \phi_n(g) + \int_0^t X_i \phi_n(g \exp(sX_i)) ds\,.$$

Letting n go to infinity in the above equality gives

$$\phi_{0,0}(g \exp(tX_i)) = \phi_{0,0}(g) + \int_0^t \phi_{0,\alpha_i}(g \exp(sX_i)) ds\,.$$

We differentiate with respect to t at 0, it shows that $\phi_{0,0}$ is continously differentiable in the direction X_i with

$$X_i \phi_{0,0}(g) = \phi_{0,\alpha_i}(g), \qquad g \in G.$$

Repeating the argument shows that $\phi_{0,0} \in C^\infty(G)$ with $X^\alpha \phi_{0,0} = \phi_{0,\alpha}$ $\forall \alpha \in \mathbb{N}^m$. This implies that for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$, $\sigma^k X^\alpha \phi_n$ converges pointwise to $\sigma^k X^\alpha \phi_{0,0}$. But $\sigma^k X^\alpha \phi_n$ converges uniformly to $\phi_{k,\alpha}$ by hypothesis, so $\phi_{k,\alpha} = \sigma^k X^\alpha \phi_{0,0}$. Since $\phi_{k,\alpha} \in L^\infty(G)$, this proves that so $\phi_{0,0} \in \mathcal{S}_\sigma(G)$ and that ϕ_n converges to $\phi_{0,0}$ in $\mathcal{S}_\sigma(G)$. Thus $\mathcal{S}_\sigma(G)$ is complete, and therefore Fréchet.

Step three: Convolution is continuous from $\mathcal{S}_\sigma(G) \times \mathcal{S}_\sigma(G)$ to $\mathcal{S}_\sigma(G)$.

The convolution of two measurable functions ϕ, ψ on G, is defined by

$$\phi * \psi(g) = \int_G \phi(gh^{-1}) \psi(h)\, dh = \int_G \phi(h) \psi(h^{-1}g)\, d^l h, \qquad g \in G,$$

provided that the integrals converge. It is easy to check that since the differential operators X^α are left invariant, they act on the convolution in the following way:

$$X^\alpha(\phi * \psi) = \phi * X^\alpha \psi.$$

Let ϕ, $\psi \in \mathcal{S}_\sigma(G)$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$. By Property 4 and Property 1,

$$\begin{aligned}
\|\phi * \psi\|_{k,\alpha}^{\infty} &= \sup_{g\in G} \left| \sigma^k(g) \int_G \phi(gh^{-1}) X^\alpha \psi(h)\, dh \right| \\
&\leq \sup_{g\in G} \left| C \int_G \sigma^{ks}(gh^{-1}) \phi(gh^{-1}) \sigma^{ks}(h) X^\alpha \psi(h)\, dh \right| \\
&\leq C \|\phi\|_{ks,0}^{\infty} \int_G \frac{\sigma^{ks+p}(h) X^\alpha \psi(h)}{\sigma^p(h)}\, dh \\
&\leq C \|\phi\|_{ks,0}^{\infty} \|\psi\|_{ks+p,\alpha}^{\infty},
\end{aligned}$$

hence the continuity.

Step four: $L^1(G, dg)$-involution is continuous on $\mathcal{S}_\sigma(G)$

The involution on $L^1(G, dg)$ is defined by $\phi^*(g) = \overline{\phi(g^{-1})}\delta(g^{-1})$.

For $\psi \in C^\infty(G)$ and $\alpha = \alpha_j$ (with the notation of *Step Two*),

$$\begin{aligned}
X^\alpha \check{\psi}(g) &= \frac{d}{dt} \check{\psi}\big(g \exp(tX_j)\big)\Big|_{t=0} = \frac{d}{dt} \psi\big(\exp(-tX_j) g^{-1}\big)\Big|_{t=0} \\
&= \frac{d}{dt} \psi\big(g^{-1} \exp(-t \mathrm{Ad}(g) X_j)\big)\Big|_{t=0} = -(\mathrm{Ad}(g) X_j \psi)(g^{-1}) \\
&= -\sum_{1\leq i\leq m} (\mathrm{Ad}(g)_{ij} X_i \psi)(g^{-1}), \qquad g \in G,
\end{aligned}$$

where $\mathrm{Ad}(g)_{ij}$ denotes the ij-matrix coefficient of $\mathrm{Ad}(g)$ in the basis $\{X_1, \ldots, X_m\}$.

For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ with length greater than one the formula is similar, but we have in addition to re-order the derivations, a process that makes polynomial p_β in the matrix coefficients appear:

$$\begin{aligned}
X^\alpha \check{\psi}(g) &= (-1)^{|\alpha|} \big((\mathrm{Ad}(g) X_1)^{\alpha_1} \cdots (\mathrm{Ad}(g) X_m)^{\alpha_m} \psi \big)(g^{-1}) \\
&= \sum_{|\beta|\leq|\alpha|} \big(p_\beta(\mathrm{Ad}(g)_{ij}) X^\beta \psi \big)(g^{-1}), \qquad g \in G.
\end{aligned}$$

We want to apply this formula to $\psi = \overline{\phi}\delta$ with $\phi \in \mathcal{S}_\sigma(G)$. Since the modular function is multiplicative, it is an eigenfunction of the left invariant differential operators X^α (see [VSCC92]). Let $\lambda_\alpha \in \mathbb{R}$ be such that $X^\alpha \delta =$

$\lambda_\alpha \delta$ ($\alpha \in \mathbb{N}^m$). By using Leibniz rule and re-ordering the derivations, we obtain

$$X^\beta(\bar{\phi}\delta) = \sum_{|\beta_1+\beta_2| \le |\beta|} C_{\beta_1,\beta_2} X^{\beta_1}(\bar{\phi}) X^{\beta_2}\delta = \sum_{|\beta_1+\beta_2| \le |\beta|} C_{\beta_1,\beta_2}\, \lambda_{\beta_2}\, \delta\, \overline{X^{\beta_1}\phi}.$$

This shows that

$$X^\alpha(\bar{\phi}\delta)\check{}(g) = \delta(g^{-1}) \sum_{|\beta| \le |\alpha|} \left(\tilde{p}_\beta(\mathrm{Ad}(g)_{ij})\overline{X^\beta \phi}\right)(g^{-1}), \qquad g \in G,$$

with $\tilde{p}_\beta$ polynomials in the matrix coefficients. It follows that

$$\begin{aligned} \|\phi^*\|_{k,\alpha}^\infty &= \sup_{g\in G} \left|\sigma^k(g) X^\alpha(\bar{\phi}\delta)\check{}(g)\right| \\ &\le \sum_{|\beta|\le|\alpha|} \sup_{g\in G} \left|\sigma^k(g)\delta(g^{-1})\tilde{p}_\beta(\mathrm{Ad}(g)_{ij})\overline{X^\beta\phi(g^{-1})}\right| \\ &= \sum_{|\beta|\le|\alpha|} \sup_{g\in G} \left|\check{\sigma}^k(g)\delta(g)\tilde{p}_\beta(\mathrm{Ad}(g^{-1})_{ij})X^\beta\phi(g)\right| . \end{aligned}$$

Now, on the one hand by Properties 2 and 3,

$$\check{\sigma}^k(g)\delta(g) \le C\sigma^{rk+m}(g), \qquad g \in G.$$

And on the other hand, the operator norm dominates the modulus of the matrix coefficients, so that there exists $C_\alpha > 0$ and $M_\alpha \in \mathbb{N}$ for which

$$\tilde{p}_\beta(\mathrm{Ad}(g^{-1})_{ij}) \le C_\alpha \|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}\,\mathfrak{g}}^{M_\alpha}, \qquad g \in G,\ |\beta| \le |\alpha|.$$

It follows that

$$\|\phi^*\|_{k,\alpha}^\infty \le C_\alpha \sum_{|\beta|\le|\alpha|} \sup_{g\in G} \left|\sigma^{rk+m+M_\alpha}(g)X^\beta\phi(g)\right|, \tag{9}$$

which proves the continuity.

□

3 On the arbitrary in our construction

In this section, we show that what appears to be arbitrary choices in our definition of $\mathcal{S}_\sigma(G)$, do not show on the resulting function algebra.

3.1 Playing around with the translation invariances

3.1.1 The L^1 algebra – does the invariance of the Haar measure play a role in Theorem $1'$?

Since there no reason why the Lie group G should be unimodular, it is natural to ask whether the result in Theorem $1'$ is typical for the algebra $L^1(G, dg)$ of functions integrable with respect to the right invariant Haar measure on G, or if a similar result holds for $L^1(G, d^l g)$. The answer is that $\mathcal{S}_\sigma(G)$ does not distinguish between left and right Haar measure:

Proposition 1. *$\mathcal{S}_\sigma(G)$ is a dense Fréchet $*$-subalgebra of $L^1(G, d^l g)$.*

Proof Since σ dominates the modular function of G (Property 2), the argument in *Step one* of the proof of Theorem $1'$, adapts almost without changes to show that $\mathcal{S}_\sigma(G)$ is a dense subset of $L^1(G, d^l g)$. Then to prove Property 1, one just needs to check that $L^1(G, d^l g)$-involution $\star$ maps $\mathcal{S}_\sigma(G)$ continuously into itself.

The two involutions $\star$ on $L^1(G, d^l g)$ and $*$ on $L^1(G, dg)$ differ by a factor which is the square of the modular function

$$\phi^\star(g) := \overline{\phi(g^{-1})}\delta(g) = \phi^*(g)\delta^2(g).$$

Using Property 2 again, the continuity of $L^1(G, dg)$-involution $*$ on $\mathcal{S}_\sigma(G)$ implies that of $L^1(G, d^l g)$-involution $\star$. □

3.1.2 The definition of seminorms – what if one consider right invariant derivations?

Although we defined the space $\mathcal{S}_\sigma(G)$ by the mean of seminorms involving left invariant derivations on G, the above paragraph suggests that $\mathcal{S}_\sigma(G)$ might be independent from a specific choice of translation invariance. We show below that replacing left invariant derivations by right invariant ones in the seminorms, defines the same topology on $\mathcal{S}_\sigma(G)$.

Let $\widetilde{X}_1, \ldots, \widetilde{X}_m$ be the right invariant vector fields on G that agree with the $X_1, \ldots, X_m$ at the origin e of G. Denote by $\widetilde{X}^\alpha$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, the right invariant differential operators $\widetilde{X}^\alpha = \widetilde{X}_1^{\alpha_1} \ldots \widetilde{X}_m^{\alpha_m}$ on G.

Proposition 2. *Let*

$$\|\phi\|'^{\infty}_{k,\alpha} := \|\sigma^k \widetilde{X}^\alpha \phi\|_{L^\infty(G)}, \qquad \phi \in C^\infty(G)$$

with $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^m$. *The collection of* $\|\cdot\|'^{\infty}_{k,\alpha}$ *forms a family of continous seminorms on* $\mathcal{S}_\sigma(G)$*, which induces the same topology on* $\mathcal{S}_\sigma(G)$ *than the initial one.*

Proof We start by proving that the $\|\cdot\|'^{\infty}_{k,\alpha}$ are continuous seminorms on $\mathcal{S}_\sigma(G)$. Each $\|\cdot\|'^{\infty}_{k,\alpha}$ being clearly positive homogeneous and subadditive, we only need to show the continuity.

The left invariant differentiable operators X^α are related to the $\widetilde{X}^\alpha$'s in the following way:

$$\widetilde{X}^\alpha \phi(g) = (-1)^{|\alpha|} X^\alpha \check{\phi}(g^{-1}) = (-1)^{|\alpha|} X^\alpha (\overline{\phi^*}\delta)(g^{-1}), \qquad \phi \in C^\infty(G).$$

This means that to go from the seminorms relative to right invariant derivations, to the seminorms relative to left invariant derivations (and vice and versa), one needs to handle three operations: involution, multiplication by the modular function, and taking the inverse. By Theorem 1$'$ (and its proof), these operations are all continuous on $\mathcal{S}_\sigma(G)$. It implies that each $\|\cdot\|'^{\infty}_{k,\alpha}$. is continuous on $\mathcal{S}_\sigma(G)$.

To prove that the seminorms $\|\cdot\|'^{\infty}_{k,\alpha}$ actually defined the same topology than the $\|\cdot\|^{\infty}_{k,\alpha}$, we have to show the reverse control than that above, *i.e.* that the $\|\cdot\|^{\infty}_{k,\alpha}$ are continuous with respect to the $\|\cdot\|'^{\infty}_{k,\alpha}$. We leave the reader verify that involution, multiplication by the modular function, and taking the inverse are continuous with respect to the $\|\cdot\|'^{\infty}_{k,\alpha}$, which gives the result.

□

3.2 L^q seminorms

An other choice that we made to define $\mathcal{S}_\sigma(G)$, is that of using L^∞ seminorms. In the case $G = \mathbb{R}^n$, it is well known the classical Schwartz space $\mathcal{S}(\mathbb{R}^n)$ can be equipped with families seminorms that are in any space L^q for $1 \leq q \leq \infty$, and that all those families of seminorms define the same topology on $\mathcal{S}(\mathbb{R}^n)$. We show that the same is true for $\mathcal{S}_\sigma(G)$:

Theorem 2. *Let* $1 \leq q < \infty$. *We set*

$$\|\phi\|^q_{k,\alpha} := \|\sigma^k X^\alpha \phi\|_{L^q(G,dg)}, \qquad \phi \in C^\infty(G)$$

with $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^m$. *The collection of* $\|\cdot\|^q_{k,\alpha}$ *forms a family of continous seminorms on* $\mathcal{S}_\sigma(G)$*, which induces the same topology on* $\mathcal{S}_\sigma(G)$ *than the initial one.*

Proof We proceed like in the proof of Proposition 2. Again, the $\|\cdot\|^q_{k,\alpha}$ are clearly positive homogeneous and subadditive, so we just need to check the continuity. This is follows immediately from Property 1, since

$$\begin{aligned}\|\phi\|^q_{k,\alpha} = \|\sigma^k X^\alpha \phi\|_{L^q(G,dg)} &= \left(\int_G \frac{1}{\sigma^{qp}(g)} \sigma^{q(k+p)}(g)\, |X^\alpha \phi(g)|^q \, dg\right)^{1/q} \\ &\leq \|\sigma^{k+p} X^\alpha \phi\|_{L^\infty(G)} \left(\int_G \frac{1}{\sigma^{qp}(g)} dg\right)^{1/q} \leq C\|\phi\|^\infty_{k+p,\alpha}, \qquad \phi \in \mathcal{S}_\sigma(G).\end{aligned}$$

Hence the $\|\cdot\|^q_{k,\alpha}$ are continous seminorms on $\mathcal{S}_\sigma(G)$,

We now show that seminorms $\|\cdot\|^\infty_{l,\beta}$ are continuous with respect to the $\|\cdot\|^q_{k,\alpha}$, which will prove that the L^∞ and L^q seminorms define the same topology on $\mathcal{S}_\sigma(G)$. By Hölder inequality and Property 1, it is enough to prove the continuity with respect to the seminorms $\|\cdot\|^1_{k,\alpha}$.

We need a special case of Sobolev embedding (because the result is of local nature, the Euclidean argument adapts without problem and there is little interest in repeating it here. We refer to [Hör60, Lemma 1.1] for the proof in $\mathbb{R}^m$).

Lemma 1. *There exist a relatively compact open neighbourhood Ω of identity e in G and a constant $C > 0$, such that for every $\phi \in C^\infty(G)$*

$$|\phi(e)| \;\leq\; C \sum_{|\beta| \leq m} \int_\Omega \left|X^\beta \phi(g)\right| dg.$$

Let $\phi \in \mathcal{S}_\sigma(G)$. Fix $h \in G$, and denote by ϕ_h the translated function $\phi_h(g) := \phi(hg)$. We apply Lemma 1 to ϕ_h. By left invariance of the X_j's and right invariance of the Haar measure dg, we have that

$$\begin{aligned}|\phi(h)| = |\phi_h(e)| &\leq C \sum_{|\beta| \leq m} \int_\Omega \left|X^\beta \phi_h(g)\right| dg \\ &\leq C \sum_{|\beta| \leq m} \int_\Omega \left|X^\beta \phi(hg)\right| dg \\ &\leq C \sum_{|\beta| \leq m} \int_{h\Omega} \delta(h^{-1}) \left|X^\beta \phi(g)\right| dg,\end{aligned}$$

with $C > 0$ independent from $h \in G$, $\phi \in \mathcal{S}_\sigma(G)$. By Property 2 and Property 3, $\delta(h^{-1}) \leq C\sigma^{rm}(h)$. It follows that

$$|\phi(h)| \leq C\sigma^{rm}(h) \sum_{|\beta| \leq m} \int_{h\Omega} \left|X^\beta \phi(g)\right| dg.$$

Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$. Applying the above inequality to $\sigma^k X^\alpha \phi$, and reordering the derivations (using Poincaré-Birkhoff-Witt theorem) yields

$$\begin{aligned} |\sigma^k(h) X^\alpha \phi(h)| &\leq C\sigma^{k+rm}(h) \sum_{|\beta| \leq m} \int_{h\Omega} \left|X^\beta X^\alpha \phi(g)\right| dg \\ &\leq C\sigma^{k+rm}(h) \sum_{|\gamma| \leq m+|\alpha|} \int_{h\Omega} |X^\gamma \phi(g)|\, dg. \end{aligned}$$

Let $W_n = \{g \in G; n < \sigma(g) \leq n+1\}$. One has $G = \bigcup_{n\in\mathbb{N}} W_n$, with $W_0 = \{e\}$ and $Wn \bigcap W_m = \emptyset$ for $n \neq m$. For $n \geq 1$,

$$\begin{aligned} \sup_{h\in W_n} \left|\sigma^k(h) X^\alpha \phi(h)\right| &\leq C \sup_{h \in W_n} \left| \sigma^{k+rm}(h) \sum_{|\gamma| \leq m+|\alpha|} \int_{h\Omega} |X^\gamma \phi(g)|\, dg \right| \\ &\leq C(n+1)^{k+rm} \sum_{|\gamma| \leq m+|\alpha|} \int_{W_n\Omega} |X^\gamma \phi(g)|\, dg. \qquad (10) \end{aligned}$$

By Property 4, for $g = h\omega \in W_n\Omega$,

$$\sigma(g) \cdot \sigma(\omega^{-1}) = \sigma(h\omega) \cdot \sigma(\omega^{-1}) \geq C\sigma^{1/s}(h) \geq Cn^{1/s}.$$

Since Ω is relatively compact, $\sigma(\omega^{-1})$ is bounded on Ω, and it follows that

$$\sigma(g) \geq Cn^{1/s}.$$

It follows from (10) that

$$\begin{aligned} \sup_{h\in W_n} \left|\sigma^k(h) X^\alpha \phi(h)\right| &\leq \widetilde{C} n^{k+rm} \sum_{|\gamma| \leq m+|\alpha|} \int_{W_n\Omega} |X^\gamma \phi(g)|\, dg \\ &\leq \widetilde{C} \sum_{|\gamma| \leq m+|\alpha|} \int_{W_n\Omega} \sigma(g)^{s(k+rm)} |X^\gamma \phi(g)|\, dg \\ &\leq \widetilde{C} \sum_{|\gamma| \leq m+|\alpha|} \int_{G} \sigma(g)^{s(k+rm)} |X^\gamma \phi(g)|\, dg. \end{aligned}$$

Hence

$$\|\phi\|_{k,\alpha}^{\infty} = \sup_{g\in G} \left|\sigma^k(h) X^\alpha \phi(h)\right| \;\leq\; \widetilde{C} \sum_{|\gamma|\leq m+|\alpha|} \int_G \sigma(g)^{s(k+rm)} \left|X^\gamma \phi(g)\right| dg$$
$$= C \sum_{|\gamma|\leq m+|\alpha|} \|\phi\|_{s(k+rm),\gamma}^{1}, \qquad \phi \in \mathcal{S}_\sigma(G).$$

Therefore the seminorms $\|\cdot\|_{k,\alpha}^{\infty}$ are continuous with respect to the $\|\cdot\|_{k,\alpha}^{1}$ ones, which concludes the proof of Theorem 2

□

4 Specific cases

In this section, we consider three classes of solvable connected simply connected Lie groups with different geometrical properties. For each class of groups G, we evaluate the weight function σ and we see how the function algebra $\mathcal{S}_\sigma(G)$ reflects their specificities.

4.1 G nilpotent

Let G be a connected simply connected nilpotent Lie group. In this case G is equal to its nilradical $N \cong \mathbb{R}^d$ so the points in G have only nilradical coordinates, and $dg = dn$ the Lebesgue measure on $\mathbb{R}^d$.

By definition,

$$\sigma(g) = \max\left(\|\mathrm{Ad}(n)\|_{\mathrm{op}\,\mathfrak{n}}, \|\mathrm{Ad}(-n)\|_{\mathrm{op}\,\mathfrak{n}}\right) \cdot \left(1 + 2\,|n|_N\right)., \qquad g = (n) \in G.$$

Then by (4) and (6), there are $C > 1$ and $q, q' \in \mathbb{N}^*$

$$C^{-1}|g|_{\mathbb{R}^d}^{1/q} \leq \sigma(g) \leq C(|g|_{\mathbb{R}^d} + 1)^{q'+1}, \qquad g \in G,$$

so the weight function σ is comparable to powers of the Euclidean norm on $\mathbb{R}^d$.

Since the product on $G = (\mathfrak{n}, \cdot_{\mathrm{CBH}})$ is given by a polynomial mapping in the coordinates on $\mathbb{R}^d$, the left invariant vector fields X_j $(1 \leq j \leq d)$ have expressions of the form:

$$X_j\big|_n = \sum_{i=1}^{d} P_i^j(n) \left.\frac{\partial}{\partial n_i}\right|_n$$

where the $P_i^j : \mathbb{R}^d \to \mathbb{R}$ are polynomial maps. Furthermore, the matrix $\left(P_i^j\right)_{1\le i,j\le d}$ has determinent one (the computation is obvious if $\{X_1,\ldots,X_d\}$ is a Jordan-Hölder basis of $\mathfrak{g}$, because in this case the matrix $\left(P_i^j\right)_{1\le i,j\le d}$ is lower triangular with one's on the diagonal; of course, the value of the determinent is preserved by change of basis). It implies that the coefficients of the inverse matrix $\left(Q_i^j\right)_{1\le i,j\le d}$ of $\left(P_i^j\right)_{1\le i,j\le d}$ are also polynomial maps $Q_i^j : \mathbb{R}^d \to \mathbb{R}$ such that

$$\left.\frac{\partial}{\partial n_j}\right|_n = \sum_{i=1}^d Q_i^j(n)\, X_i|_n$$

It follows that the Fréchet algebra $\mathcal{S}_\sigma(G)$ of smooth functions decreasing σ-rapidly at infinity on G agrees with Euclidean Schwartz space on $\mathbb{R}^d$, so that we recover the usual notion of Schwartz space on G ([How77], [Cor81]).

5 Tempered distributions

Definition 2. *We call the dual space $\mathcal{S}'_\sigma(G)$ of $\mathcal{S}_\sigma(G)$, the space of σ-tempered distributions on G. For $T \in \mathcal{S}'_\sigma(G)$ and $\phi \in \mathcal{S}_\sigma(G)$, we denote the evaluation of T on ϕ by $< T, \phi >$.*

The terminology *tempered* is still justified in our setting. Like in the Euclidean case, it refers to a certain growth restriction at infinity, as we will now see.

Definition 3. *We say that a measurable function f on G is σ-slowly increasing at infinity, if there is $k \in \mathbb{N}$ such that*

$$\|\sigma^{-k} f\|_{L^\infty(G)} < \infty.$$

It is completely standard verification that polynomially bounded functions f determine tempered distributions T_f, via correspondance of the form:

$$T_f : \phi \mapsto \int_G \phi(g) f(g) dg.$$

It turns out that such functions, together with their distributional derivatives, exhaust $\mathcal{S}'_\sigma(G)$.

Theorem 3. *Let $T \in \mathcal{S}'_\sigma(G)$. There exists $M \in \mathbb{N}$, and a family $\{f_\alpha\}$ of continuous functions on G σ-slowly increasing at infinity, such that*

$$< T, \phi >= \sum_{\substack{|\alpha| \leq M \\ \alpha \in \mathbb{N}^m}} \int_G X^\alpha f_\alpha(g)\phi(g)dg, \qquad \phi \in \mathcal{S}_\sigma(G).$$

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