Quantitative Erdös-Kac theorem for additive functions, a self-contained probabilistic approach

Joint work with X. Yang and L. Chen

Arturo Jaramillo Gil

Université du Luxembourg
National University of Singapore
Denote the set of primes by $\mathcal{P}$. 

Let $\omega : \mathbb{N} \to \mathbb{N}$ denote the prime factor counting function, $\omega(n) := |\{p \in \mathcal{P}; p \text{ divides } n\}|$. 

For instance, $\omega(54) = \omega(2 \times 3^2) = 2$. 

Let $J_n$ be a random variable with uniform distribution over $\{1, \ldots, n\}$. 

**Goal** 
- Study $\omega(J_n)$. 
- Describe as accurately as possible the asymptotic behavior of $\omega(J_n) - \mu_n \sigma_n$, for suitable chosen $\mu_n$ and $\sigma_n$. 
- What can be said when $\omega$ is replaced by a general function $\psi : \mathbb{N} \to \mathbb{N}$ only satisfying $\psi(ab) = \psi(a) + \psi(b)$ for $a, b \in \mathbb{N}$ coprime?
Denote the set of primes by $\mathcal{P}$. Let $\omega : \mathbb{N} \to \mathbb{N}$ denote the prime factor counting function,

$$\omega(n) := |\{p \in \mathcal{P}; p \text{ divides } n\}|.$$
Denote the set of primes by $\mathcal{P}$. Let $\omega : \mathbb{N} \rightarrow \mathbb{N}$ denote the prime factor counting function,

$$\omega(n) := |\{p \in \mathcal{P}; \ p \text{ divides } n\}|.$$

For instance, $\omega(54) = \omega(2 \times 3^2) = 2$. 

Goal

Study $\omega(J_n)$. Describe as accurately as possible the asymptotic behavior of $\omega(J_n) - \mu_n \sigma_n$, for suitable chosen $\mu_n$ and $\sigma_n$.

What can be said when $\omega$ is replaced by a general function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ only satisfying $\psi(ab) = \psi(a) + \psi(b)$ for $a, b \in \mathbb{N}$ coprime?
Denote the set of primes by $\mathcal{P}$. Let $\omega : \mathbb{N} \to \mathbb{N}$ denote the prime factor counting function,

$$\omega(n) := |\{p \in \mathcal{P}; \ p \text{ divides } n\}|.$$

For instance, $\omega(54) = \omega(2 \times 3^2) = 2$. Let $J_n$ be a random variable with uniform distribution over $\{1, \ldots, n\}$. 

• Study $\omega(J_n)$.

• Describe as accurately as possible the asymptotic behavior of $\omega(J_n) - \mu_n \sigma_n$, for suitable chosen $\mu_n$ and $\sigma_n$.

• What can be said when $\omega$ is replaced by a general function $\psi : \mathbb{N} \to \mathbb{N}$ only satisfying $\psi(ab) = \psi(a) + \psi(b)$ for $a, b \in \mathbb{N}$ coprime?
Denote the set of primes by $\mathcal{P}$. Let $\omega : \mathbb{N} \rightarrow \mathbb{N}$ denote the prime factor counting function,

$$\omega(n) := |\{p \in \mathcal{P}; \ p \text{ divides } n\}|.$$ 

For instance, $\omega(54) = \omega(2 \times 3^2) = 2$. Let $J_n$ be a random variable with uniform distribution over $\{1, \ldots, n\}$.

**Goal**

- Study $\omega(J_n)$. 
Denote the set of primes by $\mathcal{P}$. Let $\omega : \mathbb{N} \rightarrow \mathbb{N}$ denote the prime factor counting function,

$$\omega(n) := |\{p \in \mathcal{P}; p \text{ divides } n\}|.$$ 

For instance, $\omega(54) = \omega(2 \times 3^2) = 2$. Let $J_n$ be a random variable with uniform distribution over $\{1, \ldots, n\}$.

**Goal**

- Study $\omega(J_n)$. Describe as accurately as possible the asymptotic behavior of $\frac{\omega(J_n) - \mu_n}{\sigma_n}$, for suitable chosen $\mu_n$ and $\sigma_n$. 
Denote the set of primes by $\mathcal{P}$. Let $\omega : \mathbb{N} \to \mathbb{N}$ denote the prime factor counting function,

$$\omega(n) := |\{p \in \mathcal{P}; \ p \text{ divides } n\}|.$$

For instance, $\omega(54) = \omega(2 \times 3^2) = 2$. Let $J_n$ be a random variable with uniform distribution over $\{1, \ldots, n\}$.

**Goal**

- Study $\omega(J_n)$. Describe as accurately as possible the asymptotic behavior of $\frac{\omega(J_n) - \mu_n}{\sigma_n}$, for suitable chosen $\mu_n$ and $\sigma_n$.
- What can be said when $\omega$ is replaced by a general function $\psi : \mathbb{N} \to \mathbb{N}$ only satisfying $\psi(ab) = \psi(a) + \psi(b)$ for $a, b \in \mathbb{N}$ coprime?
1. Historical context

2. Main results

3. Ideas behind the proofs
   Simplifying the model
   Stein’s method
Historical context
Starting point: Paul Erdös and Mark Kac, proved that

\[ Z_n := \frac{\omega(J_n) - \log \log(n)}{\sqrt{\log \log(n)}} \]  

converge in distribution towards a standard Gaussian random variable \( \mathcal{N} \).
Classical Erdős-Kac theorem (1940)

Starting point: Paul Erdős and Mark Kac, proved that

\[ Z_n := \frac{\omega(J_n) - \log \log(n)}{\sqrt{\log \log(n)}} \] (1)

converge in distribution towards a standard Gaussian random variable \( \mathcal{N} \).

Some intuition: Denote \( \mathcal{P}_n := \mathcal{P} \cap [1, n] \).
Starting point: Paul Erdős and Mark Kac, proved that

\[ Z_n := \frac{\omega(J_n) - \log \log(n)}{\sqrt{\log \log(n)}} \] (1)

converge in distribution towards a standard Gaussian random variable \( \mathcal{N} \).

Some intuition: Denote \( \mathcal{P}_n := \mathcal{P} \cap [1, n] \). The convergence in (1) is hinted by the decomposition

\[ \omega(J_n) = \sum_{p \in \mathcal{P}_n} \mathbb{1}_{\{p \text{ divides } J_n\}} \] (2)
One guesses that $\mathbb{1}_{\{p \text{ divides } J_n\}}$ are weakly dependent since for $d \in \mathbb{N}$,

$$
\mathbb{P}[d \text{ divides } J_n] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{\{d \text{ divides } k\}} = \frac{1}{n} \left\lfloor \frac{n}{d} \right\rfloor \approx \frac{1}{d}.
$$

(3)
Intuition about Erdös-Kac theorem

One guesses that $\mathbb{1}_{p \text{ divides } J_n}$ are weakly dependent since for $d \in \mathbb{N}$,

$$\mathbb{P}[d \text{ divides } J_n] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{d \text{ divides } k} = \frac{1}{n} \left\lfloor \frac{n}{d} \right\rfloor \approx \frac{1}{d}. \quad (3)$$

Thus, if $p_1, \ldots, p_r \in \mathcal{P}_n$ are different primes,

$$\mathbb{P}[\mathbb{1}_{p_1 \text{ divides } J_n} = 1, \ldots, \mathbb{1}_{p_r \text{ divides } J_n} = 1] \approx \frac{1}{p_1 \cdots p_r}$$

$$\approx \mathbb{P}[\mathbb{1}_{p_1 \text{ divides } J_n} = 1] \cdots \mathbb{P}[\mathbb{1}_{p_r \text{ divides } J_n} = 1]$$
Intuition about Erdős-Kac theorem

One guesses that $\mathbb{1}_{\{p \text{ divides } J_n\}}$ are weakly dependent since for $d \in \mathbb{N}$,

$$
\mathbb{P}[d \text{ divides } J_n] = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{d \text{ divides } k} = \frac{1}{n} \left\lfloor \frac{n}{d} \right\rfloor \approx \frac{1}{d}.
$$

(3)

Thus, if $p_1, \ldots, p_r \in \mathcal{P}_n$ are different primes,

$$
\mathbb{P}[\mathbb{1}_{\{p_1 \text{ divides } J_n\}} = 1, \ldots, \mathbb{1}_{\{p_r \text{ divides } J_n\}} = 1] \approx \frac{1}{p_1 \cdots p_r}
\approx \mathbb{P}[\mathbb{1}_{\{p_1 \text{ divides } J_n\}} = 1] \cdots \mathbb{P}[\mathbb{1}_{\{p_r \text{ divides } J_n\}} = 1]
$$

Warning: nowadays it is known that the r.v. $\mathbb{1}_{\{p \text{ divides } J_n\}}$, for $p \in \mathcal{P} \cap [1, n^{\frac{1}{\alpha_n}}]$ are approximately independent if $\alpha_n \to \infty$ is suitably chosen (example: $\alpha_n := 3 \log \log(n)^2$).
Intuition about Erdös-Kac theorem

One guesses that \( \mathbbm{1}_{\{p \text{ divides } J_n\}} \) are weakly dependent since for \( d \in \mathbb{N} \),

\[
\mathbb{P}[d \text{ divides } J_n] = \frac{1}{n} \sum_{k=1}^{n} \mathbbm{1}_{\{d \text{ divides } k\}} = \frac{1}{n} \left\lfloor \frac{n}{d} \right\rfloor \approx \frac{1}{d}.
\]

(3)

Thus, if \( p_1, \ldots, p_r \in \mathcal{P}_n \) are different primes,

\[
\mathbb{P}[\mathbbm{1}_{\{p_1 \text{ divides } J_n\}} = 1, \ldots, \mathbbm{1}_{\{p_r \text{ divides } J_n\}} = 1] \approx \frac{1}{p_1 \cdots p_r}
\]

\[
\approx \mathbb{P}[\mathbbm{1}_{\{p_1 \text{ divides } J_n\}} = 1] \cdots \mathbb{P}[\mathbbm{1}_{\{p_r \text{ divides } J_n\}} = 1]
\]

**Warning:** nowadays it is known that the r.v. \( \mathbbm{1}_{\{p \text{ divides } J_n\}} \), for \( p \in \mathcal{P} \cap [1, n^{\frac{1}{\alpha_n}}] \) are approximately independent if \( \alpha_n \to \infty \) is suitably chosen (example: \( \alpha_n := 3 \log \log(n)^2 \)). But \( \alpha_n \) cannot be equal to one!
Can the asymptotic Gaussianity of $Z_n$ be quantitatively assessed with respect to a suitable probability metric? such as distance $d_1$, defined as

$$d_K(X, Y) = \sup_{z \in \mathbb{R}} |\mathbb{P}[X \leq z] - \mathbb{P}[Y \leq z]|$$
Can the asymptotic Gaussianity of $Z_n$ be quantitatively assessed with respect to a suitable probability metric? such as distance $d_1$, defined as

$$d_K(X, Y) = \sup_{z \in \mathbb{R}} |\mathbb{P}[X \leq z] - \mathbb{P}[Y \leq z]|$$

or

$$d_1(X, Y) = \sup_{h \in \text{Lip}_1} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|,$$

where $\text{Lip}_1$ is the family of Lipschitz functions with Lipschitz constant at most one.
Can the asymptotic Gaussianity of $Z_n$ be quantitatively assessed with respect to a suitable probability metric? such as distance $d_1$, defined as

$$d_K(X, Y) = \sup_{z \in \mathbb{R}} |P[X \leq z] - P[Y \leq z]|$$

or

$$d_1(X, Y) = \sup_{h \in \text{Lip}_1} |E[h(X)] - E[h(Y)]|,$$

where $\text{Lip}_1$ is the family of Lipschitz functions with Lipschitz constant at most one. We define as well

$$d_{TV}(X, Y) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P[X \in A] - P[Y \in A]|.$$
LeVeque’s conjecture (1949)

LeVeque, showed that

\[ d_K(Z_n, \mathcal{N}) \leq C \frac{\log \log \log(n)}{\log \log(n)^{\frac{1}{4}}} , \]

for some constant \( C > 0 \) independent of \( n \).
LeVeque’s conjecture (1949)

LeVeque, showed that

\[ d_K(Z_n, \mathcal{N}) \leq C \frac{\log \log \log(n)}{\log \log(n)^{\frac{1}{4}}}, \]

for some constant \( C > 0 \) independent of \( n \). He also conjectured that

\[ d_K(Z_n, \mathcal{N}) \leq C \log \log(n)^{-\frac{1}{2}}. \]
LeVeque’s conjecture (1949)

LeVeque showed that

\[ d_K(Z_n, \mathcal{N}) \leq C \frac{\log \log \log(n)}{\log \log(n)^{1/4}}, \]

for some constant \( C > 0 \) independent of \( n \). He also conjectured that

\[ d_K(Z_n, \mathcal{N}) \leq C \log \log(n)^{-1/2}. \]

This was shown to be true later by Rényi and Turán (1958).
LeVeque’s conjecture (1949)

LeVeque, showed that

\[ d_K(Z_n, \mathcal{N}) \leq C \frac{\log \log \log(n)}{\log \log(n)^{\frac{1}{4}}} , \]

for some constant \( C > 0 \) independent of \( n \). He also conjectured that

\[ d_K(Z_n, \mathcal{N}) \leq C \log \log(n)^{-\frac{1}{2}} . \]

This was shown to be true later by Rényi and Turán (1958). The approach consisted on approximating \( \mathbb{E}[e^{i\lambda \omega(J_n)}] \).
LeVeque’s conjecture (1949)

LeVeque showed that

\[ d_K(Z_n, N) \leq C \frac{\log \log \log(n)}{\log \log(n)^{\frac{1}{4}}}, \]

for some constant \( C > 0 \) independent of \( n \). He also conjectured that

\[ d_K(Z_n, N) \leq C \log \log(n)^{-\frac{1}{2}}. \]

This was shown to be true later by Rényi and Turán (1958). The approach consisted on approximating \( \mathbb{E}[e^{i\lambda \omega(J_n)}] \).

Main ingredients: Perron’s formula, Dirichlet series and some estimates on the Riemann zeta function \( \zeta \) around the vertical strip \( \{ z \in \mathbb{C} ; \Re(z) = 1 \} \).
Nowadays, a lot is known about $\mathbb{E}[e^{i\lambda \omega(J_n)}]$. 

This lead to

**Theorem**

There exists a constant $C > 0$, such that

$$
\text{d} \text{TV}(\omega(J_n), M_n) \leq C \log \log(n) - \frac{1}{2}.
$$

(4)
Mod-\(\phi\) convergence approach (2014)

Nowadays, a lot is known about \(E[e^{i\lambda\omega(J_n)}]\). This motivated studying the asymptotic properties of \(\omega(J_n)\) by analyzing a relation of the type

\[
\frac{E[e^{i\lambda\omega(J_n)}]}{E[e^{i\lambda M_n}]} \approx F(\lambda),
\]

where \(M_n\) is a random variable with Poisson distribution of parameter \(\log \log(n)\) and \(F(\lambda)\) is a (possibly non-trivial) function (work by Barbour, Kowalski and Nikeghbali in 2014).
Nowadays, a lot is known about $\mathbb{E}[e^{i\lambda \omega(J_n)}]$. This motivated studying the asymptotic properties of $\omega(J_n)$ by analyzing a relation of the type

$$\frac{\mathbb{E}[e^{i\lambda \omega(J_n)}]}{\mathbb{E}[e^{i\lambda M_n}]} \approx F(\lambda),$$

where $M_n$ is a random variable with Poisson distribution of parameter $\log \log(n)$ and $F(\lambda)$ is a (possibly non-trivial) function (work by Barbour, Kowalski and Nikeghbali in 2014). This lead to

**Theorem**

*There exists a constant $C > 0$, such that*

$$d_{TV}(\omega(J_n), M_n) \leq C \log \log(n)^{-\frac{1}{2}}. \quad (4)$$
Other approaches (Stein’s method)

Recall the heuristics that \( \mathbb{1}_{\{p \text{ divides } J_n\}} \), for \( p \in \mathcal{P} \cap [1, n^{1/\alpha_n}] \) are approximately independent.
Other approaches (Stein’s method)

Recall the heuristics that $\mathbbm{1}_{\{p \text{ divides } J_n\}}$, for $p \in P \cap [1, n^{\frac{1}{\alpha_n}}]$ are approximately independent.

**Stein’s method perspective**

Random variables with weak dependence are the main object in the theory of Stein’s method.
Other approaches (Stein’s method)

Recall the heuristics that $\mathbb{1}_{\{p \text{ divides } J_n\}}$, for $p \in \mathcal{P} \cap [1, n^{\frac{1}{\alpha_n}}]$ are approximately independent.

**Stein’s method perspective**

Random variables with weak dependence are the main object in the theory of Stein’s method.

Adam Harper (2009) used this so show

$$d_{TV}\left(\sum_{p \in \mathcal{P} \cap [1, n^{\frac{1}{\alpha_n}}]} \mathbb{1}_{\{p \text{ divides } J_n\}}, M_n\right) \leq \frac{1}{2 \log \log(n)} + \frac{5.2}{\log \log(n)^{\frac{3}{2}}} ,$$

where $\alpha_n := 3 \log \log(n)^2$. 
Other approaches (Stein’s method)

Recall the heuristics that $\mathbb{1}_{\{p \text{ divides } J_n\}}$, for $p \in \mathcal{P} \cap [1, n^{\frac{1}{\alpha_n}}]$ are approximately independent.

**Stein’s method perspective**
Random variables with weak dependence are the main object in the theory of Stein’s method.

Adam Harper (2009) used this so show

$$d_{TV} \left( \sum_{p \in \mathcal{P} \cap [1, n^{\frac{1}{\alpha_n}}]} \mathbb{1}_{\{p \text{ divides } J_n\}}, M_n \right) \leq \frac{1}{2 \log \log(n)} + \frac{5.2}{\log \log(n)^{\frac{3}{2}}},$$

where $\alpha_n := 3 \log \log(n)^2$. Consequence,

$$d_{\text*} \left( \omega(J_n), M_n \right) \leq \frac{C \log \log \log(n)}{\sqrt{\log \log(n)}}.$$
Another idea consists on comparing \((\mathbb{1}_{p \text{ divides } J_n} ; p \in \mathcal{P} \cap [1, n^{\frac{1}{\beta n}}])\) with independent random variables.

Consequence similar to Harper's result.
Another idea consists on comparing \((\mathbb{1}_{p \text{ divides } J_n}; \ p \in \mathcal{P} \cap [1, n^{\frac{1}{\beta_n}}])\) with independent random variables.

Kubilius (1964) showed that if \(\beta_n \to \infty\),

\[
d_{TV}\left(\left(\mathbb{1}_{p \text{ divides } J_n}; \ p \in \mathcal{P} \cap [1, n^{\frac{1}{\beta_n}}]\right), (B_p \in \mathcal{P} \cap [1, n^{\frac{1}{\beta_n}}])\right) \leq e^{-c\beta_n},
\]

where \(B_p\) are independent Bernoulli r.v. with \(\mathbb{P}[B_p] = 1/p\).
Another idea consists on comparing \((1_{p \text{ divides } J_n} ; p \in \mathcal{P} \cap [1, n^{1/\beta_n})\) with independent random variables.

Kubilius (1964) showed that if \(\beta_n \to \infty\),

\[
d_{TV}\left((1_{p \text{ divides } J_n} ; p \in \mathcal{P} \cap [1, n^{1/\beta_n}]), (B_p \in \mathcal{P} \cap [1, n^{1/\beta_n}])\right) \leq e^{-c\beta_n},
\]

where \(B_p\) are independent Bernoulli r.v. with \(\mathbb{P}[B_p] = 1/p\). Thus,

\[
\sum_{p \in \mathcal{P} \cap [1, n^{1/\beta_n}]} 1_{p \text{ divides } J_n} \approx \sum_{p \in \mathcal{P} \cap [1, n^{1/\beta_n}]} B_p
\]

Consequence similar to Harper’s result.
Arratia (2013) suggests comparing $J_n$ with a partial product of a biased permutation of factors $T_n$ and a random prime $P_n$. He proves that

$$d_{TV}(J_n, T_n Q_n) \leq C \frac{\log \log(n)}{\log(n)}.$$
The size biased permutation approach

Arratia (2013) suggests comparing \( J_n \) with a partial product of a biased permutation of factors \( T_n \) and a random prime \( P_n \). He proves that

\[
d_{TV}(J_n, T_n Q_n) \leq C \frac{\log \log(n)}{\log(n)}.
\]

This is used to show that if \( d_\Omega : \mathbb{N}^2 \to \mathbb{N} \) denotes the insertion deletion distance \( d_\Omega(\prod_{p \in \mathcal{P}} p^{\alpha_p}, \prod_{p \in \mathcal{P}} p^{\beta_p}) := \sum_{p \in \mathcal{P}} |\alpha_p - \beta_p| \), and \( d_{1,\Omega} \) the associated Wasserstein distance,
Arratia (2013) suggests comparing $J_n$ with a partial product of a biased permutation of factors $T_n$ and a random prime $P_n$. He proves that

$$d_{TV}(J_n, T_n Q_n) \leq C \frac{\log \log(n)}{\log(n)}.$$  

This is used to show that if $d_\Omega : \mathbb{N}^2 \rightarrow \mathbb{N}$ denotes the insertion deletion distance $d_\Omega(\prod_{p \in \mathcal{P}} p^{\alpha_p}, \prod_{p \in \mathcal{P}} p^{\beta_p}) := \sum_{p \in \mathcal{P}} |\alpha_p - \beta_p|$, and $d_{1, \Omega}$ the associated Wasserstein distance,

$$\lim_{n \rightarrow \infty} d_{1, \Omega}(J_n, \prod_{p \in \mathcal{P}_n} p^{\xi_p}) = 2,$$

where

$$\mathbb{P}[\xi_p = k] = p^{-k}(1 - 1/p),$$

for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. 


Main results
Let $\psi : \mathbb{N} \to \mathbb{N}$ be such that $\psi(ab) = \psi(a) + \psi(b)$ for $a, b$ co-prime.

**(H1)** We have that

$$\|\psi\|_P := \sup_{p \in \mathcal{P}} |\psi(p)| < \infty.$$ 

**(H2)** There exists a (possibly unbounded) function $\Psi : \mathcal{P} \to \mathbb{R}_+$ satisfying

$$\|\Psi\|_P := \left( \sum_{p \in \mathcal{P}} \frac{\Psi(p)^2}{p^2} \right)^{1/2} < \infty,$$

and such that for all $p \in \mathcal{P}_n$,

$$\|\psi(p^k + 2)\|_{L^2(\Omega)} \leq \Psi(p).$$
Main result for Kolmogorov distance

Let $\mu_n$ and $\sigma_n > 0$ be given by

$$\mu_n = \sum_{p \in \mathcal{P}_n} \mathbb{E}[\psi(p^\xi_\mathcal{P})] \quad \text{and} \quad \sigma_n^2 = \sum_{p \in \mathcal{P}_n} \text{Var}[\psi(p^\xi_\mathcal{P})].$$

(5)
Main result for Kolmogorov distance

Let $\mu_n$ and $\sigma_n > 0$ be given by

$$\mu_n = \sum_{p \in \mathcal{P}_n} \mathbb{E}[\psi(p^{\xi_p})] \quad \text{and} \quad \sigma_n^2 = \sum_{p \in \mathcal{P}_n} \text{Var}[\psi(p^{\xi_p})]. \tag{5}$$

**Theorem (Chen, Jaramillo, Yang)**

Suppose that $\psi$ satisfies (H1) and (H2). Then, if $X_p := \sigma_n^{-1} \psi(p^{\xi_p})$, and provided that $\sigma_n^2 \geq 3(\|\psi\|_P^2 + \|\Psi\|_P^2)$,

$$d_K \left( \frac{\psi(J_n) - \mu_n}{\sigma_n}, \mathcal{N} \right) \leq \frac{\kappa_1}{\sigma_n} + \kappa_2 \sum_{p \in \mathcal{P}_n} \mathbb{E}[|X_p|^3] + \frac{\kappa_3 \log \log(n)}{\log(n)},$$

where

$$\kappa_1 := 29.2\|\psi\|_P + 34.8\|\Psi\|_P \quad \kappa_2 := 97.2 \quad \kappa_3 := 61. \tag{6}$$
Main result for Wasserstein distance

Theorem (Chen, Jaramillo, Yang)

\[ d_1 \left( \frac{\psi(J_n) - \mu_n}{\sigma_n}, \mathcal{N} \right) \leq \frac{\kappa_4}{\sigma_n} + \kappa_5 \sum_{p \in \mathcal{P}_n} \mathbb{E}[|X_p|^3] + \kappa_6 \frac{\log \log(n)^{3/2}}{\log(n)^{1/2}}, \quad (7) \]

where

\[ \kappa_4 := 16.6\|\psi\|_P + 11.3\|\Psi\|_P \quad \kappa_5 := 24 \quad \kappa_6 := 21\|\psi\|_P + 45. \]
Ideas behind the proofs
For a given $p \in \mathcal{P}$, define $\alpha_p : \mathbb{N} \rightarrow \mathbb{N}$, by

$$k = \prod_{p \in \mathcal{P}} p^{\alpha_p(k)}.$$  \hspace{1cm} (8)
Multiplicities of prime factors

For a given $p \in \mathcal{P}$, define $\alpha_p : \mathbb{N} \to \mathbb{N}$, by

$$k = \prod_{p \in \mathcal{P}} p^{\alpha_p(k)}.$$  \hfill (8)

For any $i \in \mathbb{N}$ and $k_1, \ldots, k_i \in \mathbb{N}_0$,

$$\bigcap_{j=1}^i \{ \alpha_{p_j}(J_n) \geq k_j \} = \bigcap_{j=1}^i \{ p_j^{k_j} \text{ divides } J_n \} = \left\{ \prod_{j=1}^i p_j^{k_j} \text{ divides } J_n \right\},$$

Question: can we use the $\xi_{p_j}$ to construct a r.v. equal in law to $J_n$?
Answer: not easily... but...
For a given $p \in \mathcal{P}$, define $\alpha_p : \mathbb{N} \to \mathbb{N}$, by

$$k = \prod_{p \in \mathcal{P}} p^{\alpha_p(k)}.$$  \hfill (8)

For any $i \in \mathbb{N}$ and $k_1, \ldots, k_i \in \mathbb{N}_0$,

$$\bigcap_{j=1}^{i} \{ \alpha_{p_j}(J_n) \geq k_j \} = \bigcap_{j=1}^{i} \{ p_j^{k_j} \text{ divides } J_n \} = \left\{ \prod_{j=1}^{i} p_j^{k_j} \text{ divides } J_n \right\},$$

so $\lim_{n \to \infty} \mathbb{P}[\alpha_{p_j}(J_n) \geq k_j \text{ for all } 1 \leq j \leq i] = p_1^{-k_1} \cdots p_i^{-k_i}$.
For a given \( p \in \mathcal{P} \), define \( \alpha_p : \mathbb{N} \to \mathbb{N} \), by

\[
k = \prod_{p \in \mathcal{P}} p^{\alpha_p(k)}.
\]

(8)

For any \( i \in \mathbb{N} \) and \( k_1, \ldots, k_i \in \mathbb{N}_0 \),

\[
\bigcap_{j=1}^{i} \{ \alpha_{p_j}(J_n) \geq k_j \} = \bigcap_{j=1}^{i} \{ p_j^{k_j} \text{ divides } J_n \} = \left\{ \prod_{j=1}^{i} p_j^{k_j} \text{ divides } J_n \right\},
\]

so \( \lim_{n \to \infty} \mathbb{P}[\alpha_{p_j}(J_n) \geq k_j \text{ for all } 1 \leq j \leq i] = p_1^{-k_1} \cdots p_i^{-k_i} \). This gives

\[
(\alpha_{p_1}(J_n), \ldots, \alpha_{p_i}(J_n)) \xrightarrow{\text{Law}} (\xi_{p_1}, \ldots, \xi_{p_i})
\]
For a given $p \in \mathcal{P}$, define $\alpha_p : \mathbb{N} \rightarrow \mathbb{N}$, by

$$k = \prod_{p \in \mathcal{P}} p^{\alpha_p(k)}.$$  

(8)

For any $i \in \mathbb{N}$ and $k_1, \ldots, k_i \in \mathbb{N}_0$,

$$\bigcap_{j=1}^{i} \{ \alpha_{p_j}(J_n) \geq k_j \} = \bigcap_{j=1}^{i} \{ p_j^{k_j} \text{ divides } J_n \} = \left\{ \prod_{j=1}^{i} p_j^{k_j} \text{ divides } J_n \right\},$$

so

$$\lim_{n \rightarrow \infty} \mathbb{P}[\alpha_{p_j}(J_n) \geq k_j \text{ for all } 1 \leq j \leq i] = p_1^{-k_1} \cdots p_i^{-k_i}.$$  

This gives

$$(\alpha_{p_1}(J_n), \ldots, \alpha_{p_i}(J_n)) \xrightarrow{\text{Law}} (\xi_{p_1}, \ldots, \xi_{p_i})$$

**Question:** can we use the $\xi_p$ to construct a r.v. equal in law to $J_n$?
For a given \( p \in \mathcal{P} \), define \( \alpha_p : \mathbb{N} \to \mathbb{N} \), by

\[
k = \prod_{p \in \mathcal{P}} p^{|\alpha_p(k)}|.
\] (8)

For any \( i \in \mathbb{N} \) and \( k_1, \ldots, k_i \in \mathbb{N}_0 \),

\[
\bigcap_{j=1}^{i} \{ \alpha_{p_j}(J_n) \geq k_j \} = \bigcap_{j=1}^{i} \{ p_j^{k_j} \text{ divides } J_n \} = \left\{ \prod_{j=1}^{i} p_j^{k_j} \text{ divides } J_n \right\},
\]

so \( \lim_{n \to \infty} \mathbb{P}[\alpha_{p_j}(J_n) \geq k_j \text{ for all } 1 \leq j \leq i] = p_1^{-k_1} \cdots p_i^{-k_i} \). This gives

\[
(\alpha_{p_1}(J_n), \ldots, \alpha_{p_i}(J_n)) \xrightarrow{\text{Law}} (\xi_{p_1}, \ldots, \xi_{p_i})
\]

**Question:** can we use the \( \xi_p \) to construct a r.v. equal in law to \( J_n \)?

**Answer:** not easily
For a given $p \in \mathcal{P}$, define $\alpha_p: \mathbb{N} \to \mathbb{N}$, by

$$k = \prod_{p \in \mathcal{P}} p^{\alpha_p(k)}. \quad (8)$$

For any $i \in \mathbb{N}$ and $k_1, ..., k_i \in \mathbb{N}_0$,

$$\bigcap_{j=1}^{i} \{ \alpha_{p_j}(J_n) \geq k_j \} = \bigcap_{j=1}^{i} \{ p_j^{k_j} \text{ divides } J_n \} = \left\{ \prod_{j=1}^{i} p_j^{k_j} \text{ divides } J_n \right\},$$

so $\lim_{n \to \infty} \mathbb{P}[\alpha_{p_j}(J_n) \geq k_j \text{ for all } 1 \leq j \leq i] = p_1^{-k_1} \cdots p_i^{-k_i}$. This gives

$$(\alpha_{p_1}(J_n), \ldots, \alpha_{p_i}(J_n)) \xrightarrow{\text{Law}} (\xi_{p_1}, \ldots, \xi_{p_i})$$

**Question:** can we use the $\xi_p$ to construct a r.v. equal in law to $J_n$?

**Answer:** not easily... but...
Let $H_n$ be a r.v. with $\mathbb{P}[H_n = k] = \frac{1}{L_n k}$, where $L_n := \sum_{k=1}^{n} \frac{1}{k}$. 

A simplified model: Harmonic distribution $H_n$
Let $H_n$ be a r.v. with $\mathbb{P}[H_n = k] = \frac{1}{L_n k}$, where $L_n := \sum_{k=1}^{n} \frac{1}{k}$. Then,

**Proposition**

*Suppose that $n \geq 21$. Define the event

$$A_n := \{ \prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n \},$$

(9)

as well as the random vector $\tilde{\mathcal{C}}(n) := (\alpha_p(H_n); p \in \mathcal{P}_n)$. Then the random variables $Y_p := \psi(p^{\xi_p})$, indexed by $p \in \mathcal{P}_n$, satisfy

$$\mathcal{L}(\psi(H(n))) = \mathcal{L}(\sum_{p \in \mathcal{P}_n} Y_p | A_n).$$

(10)
Let \( \{Q(k)\}_{k \geq 1} \) be independent r.v. independent of \((J_n, H_n)\) with \(Q(k)\) uniformly distributed over

\[ P_k^* := \{1\} \cup P_k. \]
Let \( \{Q(k)\}_{k \geq 1} \) be independent r.v. independent of \((J_n, H_n)\) with \(Q(k)\) uniformly distributed over

\[
P_k^* := \{1\} \cup P_k.
\]

Let \( \pi(n) := |P \cap [1, n]|. \) Using the fact that for \( n \geq 229, \)

\[
\left| \pi(n) - \int_0^n \frac{1}{\log(t)} \, dt \right| \leq \frac{181n}{\log(n)^3}, \tag{11}
\]
Let \( \{Q(k)\}_{k \geq 1} \) be independent r.v. independent of \((J_n, H_n)\) with \(Q(k)\) uniformly distributed over

\[
P_k^* := \{1\} \cup P_k.
\]

Let \( \pi(n) := |P \cap [1, n]|. \) Using the fact that for \( n \geq 229, \)

\[
\left| \pi(n) - \int_0^n \frac{1}{\log(t)} dt \right| \leq \frac{181n}{\log(n)^3},
\]

(11)

**Lemma (Chen, Jaramillo and Yang)**

*The following bound (analogous to the one by Arratia) holds for \( n \geq 21 \)*

\[
d_{TV}(J_n, H_n Q(n/H_n)) \leq 61 \frac{\log \log n}{\log n}.
\]
Simplifying $\omega(J_n)$ to $\omega(H_n)$

By using the fact that $|\psi(H_nQ(n/H_n)) - \psi(H_n)| \leq ||\psi||_P$, we can easily show that $d_K(\psi(J_n) - \mu_n\sigma_n, N) \leq d_{TV}(J_n, H_nQ(n/H_n)) + d_K(\psi(H_nQ(n/H_n)) - \mu_n\sigma_n, \psi(H_n)) + d_K(\psi(H_n) - \mu_n\sigma_n, N)$.

New goal: bound $d_K(\psi(H_n) - \mu_n\sigma_n, N)$.
Simplifying $\omega(J_n)$ to $\omega(H_n)$

By using the fact that $|\psi(H_n Q(n/H_n)) - \psi(H_n)| \leq \|\psi\|_P$, we can easily show that

$$d_K\left(\frac{\psi(J_n) - \mu_n}{\sigma_n}, N\right) \leq d_{TV}(J_n, H_n Q(n/H_n))$$

$$+ d_K\left(\frac{\psi(H_n Q(n/H_n)) - \mu_n}{\sigma_n}, \frac{\psi(H_n) - \mu_n}{\sigma_n}\right)$$

$$+ d_K\left(\frac{\psi(H_n) - \mu_n}{\sigma_n}, N\right).$$
Simplifying $\omega(J_n)$ to $\omega(H_n)$

By using the fact that $|\psi(H_nQ(n/H_n)) - \psi(H_n)| \leq \|\psi\|_P$, we can easily show that

$$d_K \left( \frac{\psi(J_n) - \mu_n}{\sigma_n}, \mathcal{N} \right) \leq d_{TV} (J_n, H_n Q(n/H_n))$$

$$+ d_K \left( \frac{\psi(H_n Q(n/H_n)) - \mu_n}{\sigma_n}, \frac{\psi(H_n) - \mu_n}{\sigma_n} \right)$$

$$+ d_K \left( \frac{\psi(H_n) - \mu_n}{\sigma_n}, \mathcal{N} \right).$$

**New goal:** bound $d_K \left( \frac{\psi(H_n) - \mu_n}{\sigma_n}, \mathcal{N} \right)$
Simplifying $\omega(J_n)$ to $\omega(H_n)$

By using the fact that $|\psi(H_n Q(n/H_n)) - \psi(H_n)| \leq \|\psi\|_P$, we can easily show that

$$d_K \left( \frac{\psi(J_n) - \mu_n}{\sigma_n}, \mathcal{N} \right) \leq d_{TV} \left( J_n, H_n Q(n/H_n) \right)$$

$$+ d_K \left( \frac{\psi(H_n Q(n/H_n)) - \mu_n}{\sigma_n}, \frac{\psi(H_n) - \mu_n}{\sigma_n} \right)$$

$$+ d_K \left( \frac{\psi(H_n) - \mu_n}{\sigma_n}, \mathcal{N} \right).$$

**New goal:** bound $d_K \left( \frac{\psi(H_n) - \mu_n}{\sigma_n}, \mathcal{N} \right)$

**Methodology used**

Since $\psi(H_n)$ is conditionally equal to $\sum_{p \in \mathcal{P}_n} \psi(p^{\xi_p})$, we use Stein's method.
Lemma (Stein’s lemma)

For every smooth $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[f'(N)] = \mathbb{E}[Nf(N)]$$

Stein’s heuristics:

If $X$ is an $\mathbb{R}$-valued random variable such that $\mathbb{E}[f'(X)] \approx \mathbb{E}[Xf(X)]$, for a large class of functions $f$, then $Z$ is close to $N$ in some meaningful sense.
Lemma (Stein’s lemma)
For every smooth $f : \mathbb{R} \rightarrow \mathbb{R}$, 

$$\mathbb{E}[f'(N)] = \mathbb{E}[Nf(N)]$$

Stein’s heuristics: if $X$ is an $\mathbb{R}$-valued random variable such that 

$$\mathbb{E}[f'(X)] \approx \mathbb{E}[Xf(X)],$$

for a large class of functions $f$, then $Z$ is close to $N$ in some meaningful sense.
**Lemma**

Let $h_r : \mathbb{R} \to \mathbb{R}$ be given by $h_r(x) := \mathbb{1}_{(-\infty, r]}(x)$, for some $r \in \mathbb{R}$. Then, the equation

$$f'(x) - xf(x) = h_r(x) - \mathbb{E}[h_r(N)]$$

has a unique solution $f = f_r$, satisfying

$$\sup_{w \in \mathbb{R}} |f'_r(w)| \leq 2 \quad \text{and} \quad f_r(w) \leq \sqrt{\pi}/2 \quad (12)$$
Lemma
Let $h_r : \mathbb{R} \to \mathbb{R}$ be given by $h_r(x) := 1_{(-\infty,r]}(x)$, for some $r \in \mathbb{R}$. Then, the equation

$$f'(x) - xf(x) = h_r(x) - \mathbb{E}[h_r(N)]$$

has a unique solution $f = f_r$, satisfying

$$\sup_{w \in \mathbb{R}} |f'_r(w)| \leq 2 \quad \text{and} \quad f_r(w) \leq \sqrt{\pi}/2 \quad \text{(12)}$$

Thus, if $X$ is some r.v.

$$d_K(X, N) \leq \sup_f |\mathbb{E}[f'(X) - Xf(X)]|$$

where $f$ ranges over the functions satisfying (12)
Stein’s method for $\psi(H_n)$

As before, $h_r = 1_{(-\infty, r]}$, $f_r$ is Stein’s solution and $Y_p := \psi(p^\xi_p)$. 
Stein’s method for $\psi(H_n)$

As before, $h_r = 1_{(-\infty,r]}$, $f_r$ is Stein’s solution and $Y_p := \psi(p^{\xi_p})$. By the identity

$$\mathcal{L}(\psi(H(n))) = \mathcal{L}(\sum_{p \in \mathcal{P}_n} Y_p | \prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n),$$

New goal: estimate $\mathbb{E}[(f_r'(W) - Wf_r(W))I] = I_n := 1_{\{p \in \mathcal{P}_n \mid p^{\xi_p} \leq n\}}$.  

20
Stein’s method for $\psi(H_n)$

As before, $h_r = \mathbb{1}_{(-\infty, r]}$, $f_r$ is Stein’s solution and $Y_p := \psi(p^{\xi_p})$. By the identity

$$\mathcal{L}(\psi(H(n))) = \mathcal{L}\left(\sum_{p \in \mathcal{P}_n} Y_p | \prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n\right),$$

we have,

$$\mathbb{E}\left[h_r \left(\frac{\psi(H_n) - \mu_n}{\sigma_n}\right) - \mathbb{E}[h_r(N)]\right] = \frac{\mathbb{E}[(f'_r(W) - Wf_r(W))I]}{\mathbb{P}[\prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n]},$$

where

$$W = W_n := \sigma_n^{-1}\left(\sum_{p \in \mathcal{P}_n} \psi(p^{\xi_p}) - \mu_n\right)$$

and

$$I = I_n := \mathbb{1}_{\prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n}.$$
Stein’s method for $\psi(H_n)$

As before, $h_r = \mathbb{1}_{(-\infty, r]}$, $f_r$ is Stein’s solution and $Y_p := \psi(p^{\xi_p})$. By the identity

$$\mathcal{L}(\psi(H(n))) = \mathcal{L}(\sum_{p \in \mathcal{P}_n} Y_p \mid \prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n),$$

we have,

$$\mathbb{E}\left[h_r\left(\frac{\psi(H_n) - \mu_n}{\sigma_n}\right) - \mathbb{E}[h_r(N)]\right] = \frac{\mathbb{E}[(f'_r(W) - Wf_r(W))I]}{\mathbb{P}[\prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n]},$$

where

$$W = W_n := \sigma_n^{-1}\left(\sum_{p \in \mathcal{P}_n} \psi(p^{\xi_p}) - \mu_n\right)$$

$$I = I_n := \mathbb{1}\{\prod_{p \in \mathcal{P}_n} p^{\xi_p} \leq n\}.$$ 

New goal: estimate

$$\mathbb{E}[(f'_r(W) - Wf_r(W))I].$$
Let \( \{\xi_p\}_{p \in \mathcal{P}} \) be an independent copy of \( \{\xi_p\}_{p \in \mathcal{P}} \), and \( \Theta \) a random variable uniformly distributed over \( \mathcal{P}_n \) and independent of \( \{(\xi_p', \xi_p)\}_{p \in \mathcal{P}} \).
Let \( \{\xi'_p\}_{p \in \mathcal{P}} \) be an independent copy of \( \{\xi_p\}_{p \in \mathcal{P}} \), and \( \Theta \) a random variable uniformly distributed over \( \mathcal{P}_n \) and independent of \( \{(\xi'_p, \xi_p)\}_{p \in \mathcal{P}} \). For each \( n \in \mathbb{N} \), set

\[
W' = \sigma_n^{-1}(\psi(\Theta^{\xi\Theta}) \sum_{p \in \mathcal{P}_n \setminus \{\Theta\}} \psi(p^{\xi_p}) - \mu_n)
\]

\[
l' = 1_{\{\theta^{\xi'_\theta} \prod_{p \in \mathcal{P}_n \setminus \{\theta\}} p^{\xi_p} \leq n\}}.
\]

Then \((W, l), (W', l')) \overset{Law}{=} ((W', l'), (W, l))\).
Let \( \{\xi'_p\}_{p \in \mathcal{P}} \) be an independent copy of \( \{\xi_p\}_{p \in \mathcal{P}} \), and \( \Theta \) a random variable uniformly distributed over \( \mathcal{P}_n \) and independent of \( \{(\xi'_p, \xi_p)\}_{p \in \mathcal{P}} \).

For each \( n \in \mathbb{N} \), set

\[
W' = \sigma_n^{-1}(\psi(\Theta^{\xi\Theta}) \sum_{p \in \mathcal{P}_n \setminus \{\Theta\}} \psi(p^{\xi_p}) - \mu_n)
\]

\[
l' = \mathbb{1}_{\{\theta^{\xi'_\theta} \prod_{p \in \mathcal{P}_n \setminus \{\theta\}} p^{\xi_p} \leq n\}}.
\]

Then \( ((W, l), (W', l')) \overset{\text{Law}}{=} ((W', l'), (W, l)) \). By exchangeability,

\[
-2\mathbb{E}[(W' - W)f_r(W)l] = \mathbb{E}[(W' - W)(f_r(W')l' - f_r(W)l)].
\]
Handling \(-2\mathbb{E}[(W' - W)f_r(W)I]\)

We observe that \(LHS := -2\mathbb{E}[(W' - W)f_r(W)I]\) satisfies

\[
LHS = -\frac{2}{\pi(n)} \mathbb{E} \left[ \left( \sum_{\theta \in \mathcal{P}_n} Y'_\theta - \mu_n \right) - \left( \sum_{\theta \in \mathcal{P}_n} Y_\theta - \mu_n \right) \sigma_n \right] f_r(W)I
\]

\[
= \frac{2}{\pi(n)} \mathbb{E}[Wf_r(W)I] - \frac{2}{\pi(n)} \mathbb{E}[W] \mathbb{E}[f_r(W)I],
\]
We observe that \( \text{LHS} := -2\mathbb{E}[(W' - W)f_r(W)I] \) satisfies

\[
\text{LHS} = -\frac{2}{\pi(n)} \mathbb{E} \left[ \frac{\left( \sum_{\theta \in \mathcal{P}_n} Y'_\theta - \mu_n \right) - \left( \sum_{\theta \in \mathcal{P}_n} Y_\theta - \mu_n \right)}{\sigma_n} f_r(W)I \right] 
\]

\[
= \frac{2}{\pi(n)} \mathbb{E}[Wf_r(W)I] - \frac{2}{\pi(n)} \mathbb{E}[W] \mathbb{E}[f_r(W)I],
\]

so

\[
\text{LHS} = \frac{2}{\pi(n)} \mathbb{E}[Wf_r(W)I],
\]
Define \( X_p := \sigma_n^{-1}Y_p \) and

\[
RHS := \mathbb{E}[(W' - W)(f_r(W')I' - f_r(W)I)],
\]
Define $X_p := \sigma_n^{-1} Y_p$ and

$$RHS := \mathbb{E}[(W' - W)(f_r(W')I' - f_r(W)I)]$$

To estimate $RHS$ we formalize the approximation

$$RHS \approx \frac{1}{\pi(n)} \sum_{p \in \mathcal{P}_n} \mathbb{E}[(X'_p - X_p)^2 f'_r(W)I]$$

$$\approx \frac{1}{\pi(n)} \sum_{p \in \mathcal{P}_n} \mathbb{E}[(X'_p - X_p)^2] \mathbb{E}[f'_r(W)I]$$

$$= \frac{2 \text{Var}(W)}{\pi(n)} \mathbb{E}[f'_r(W)I],$$
Define $X_p := \sigma_n^{-1} Y_p$ and

$$RHS := \mathbb{E}[(W' - W)(f_r(W')I' - f_r(W)I)],$$

To estimate $RHS$ we formalize the approximation

$$RHS \approx \frac{1}{\pi(n)} \sum_{p \in P_n} \mathbb{E}[(X_p' - X_p)^2 f'_r(W)I]$$

$$\approx \frac{1}{\pi(n)} \sum_{p \in P_n} \mathbb{E}[(X_p' - X_p)^2] \mathbb{E}[f'_r(W)I]$$

$$= \frac{2 \text{Var}(W)}{\pi(n)} \mathbb{E}[f'_r(W)I],$$

to obtain

$$RHS \approx \frac{2}{\pi(n)} \mathbb{E}[f'_r(W)I],$$
We conclude that

\[ 0 = |RHS - LHS| \approx \left| \frac{2}{\pi(n)} \left( \mathbb{E}[Wf_r(W)I] - \mathbb{E}[f'_r(W)I] \right) \right|. \]

Thus, the result follows by a careful analysis of the approximations.
Poisson case

Theorem (Chen, Jaramillo and Yang)

Let $M_n$ be a Poisson distribution with parameter $\log \log(n)$ and define $\Omega : \mathbb{N} \to \mathbb{N}$ by $\Omega(m) := \sup_{p \in \mathcal{P}_n} \alpha_p(m)$. Then we have

$$d_{TV}(\omega(J_n), M_n) \leq 7.2 \sqrt{\log \log(n)} + 6.4 \log \log(n) \log(n).$$

$$d_{TV}(\Omega(J_n), M_n) \leq 14 \sqrt{\log \log(n)}. $$
Theorem (Chen, Jaramillo and Yang)

Let $M_n$ be a Poisson distribution with parameter $\log \log(n)$ and define $\Omega : \mathbb{N} \to \mathbb{N}$ by $\Omega(m) := \sup_{p \in \mathcal{P}_n} \alpha_p(m)$. Then we have

$$d_{TV}(\omega(J_n), M_n) \leq \frac{7.2}{\sqrt{\log \log(n)}} + 67.4 \frac{\log \log(n)}{\log(n)}$$

$$d_{TV}(\Omega(J_n), M_n) \leq \frac{14}{\sqrt{\log \log(n)}}.$$

