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Mathematics Research Unit
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Exercises to the lecture course

Riemann Surfaces

Exercise 1: A bijective holomorphic map between two domains in $\mathbb{C}$ is biholomorphic.

Exercise 2: To be analytically equivalent is an equivalence relation on the set of all complex atlases on a Riemann surface.

Exercise 3: 1) Check that the complex charts on $\hat{\mathbb{C}}$ introduced in the lecture are holomorphically compatible and constitute a complex atlas on $\hat{\mathbb{C}}$.

2) Prove that $\hat{\mathbb{C}}$ is homeomorphic to the complex projective line $\mathbb{P}_1 = \mathbb{P}_1(\mathbb{C})$.

Exercise 4: Let $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.

1) Fill in the gaps in the definition of the complex structure on $\mathbb{C}/\Gamma$.

2) Let S^1 denote the real 1-sphere. Show that $\mathbb{C}/\Gamma$ is homeomorphic to $S^1 \times S^1$.
Hint: *Let p_1, p_2 be the $\mathbb{R}$-basis of $\mathrm{Hom}(\mathbb{C}, \mathbb{R})$ dual to ω_1, ω_2. Consider the map $\mathbb{C}/\Gamma \to S^1 \times S^1$, $[z] \mapsto (\exp(2\pi i p_1(z)), \exp(2\pi i p_2(z)))$. Here $[z]$ denotes the equivalence class of a complex number z in $\mathbb{C}/\Gamma$.*

Exercise 5: Let X be a Riemann surface and let Y be an open subset in X. Check that the set $\mathcal{O}_X(Y)$ of holomorphic functions on Y is a $\mathbb{C}$-algebra.

Exercise 6: (Examples of morphisms of Riemann surfaces)
1) The quotient map $\mathbb{C} \to \mathbb{C}/\Gamma$, where Γ is a lattice in $\mathbb{C}$, is a holomorphic map.

2) Let Γ and Γ' be two lattices in $\mathbb{C}$. Let $\alpha \in \mathbb{C}^*$ and assume that $\alpha \cdot \Gamma \subset \Gamma'$. Show that the map

$$\mathbb{C}/\Gamma \to \mathbb{C}/\Gamma', \quad [z] \mapsto [\alpha z],$$

is a well-defined holomorphic map. Show that it is an isomorphism if and only if $\alpha \cdot \Gamma = \Gamma'$.

Exercise 7: 1) Let X be a Riemann surface and let Y be an open subset in X. Check that the set $\mathcal{M}_X(Y)$ of meromorphic functions on Y is a $\mathbb{C}$-algebra and $\mathcal{O}_X(Y)$ is naturally included in $\mathcal{M}_X(Y)$.

2) Let $f \in \mathcal{M}(X)$ and let P be the set of poles of f. Define $\hat{f} : X \to \hat{\mathbb{C}}$ by

$$\hat{f}(z) := \begin{cases} f(z) & \text{if } z \notin P, \\ \infty & \text{if } z \in P. \end{cases}$$

Show that $\hat{f}$ is a continuous map. Note that it is enough to check the continuity at poles $p \in P$.

Exercise 8: 1) Let $X \xrightarrow{f} Y$ be a non-constant holomorphic map of Riemann surfaces and let $a \in X$. Show that the multiplicity of f at a is uniquely determined, i. e., does not depend on the choice of local charts.

2) Let $f(z) \in \mathbb{C}[z]$ be a polynomial of degree k. This gives a holomorphic map $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\hat{f}(\infty) = \infty$. Show that $\hat{f}$ has multiplicity k at ∞.

Exercise 9: Show that the set of meromorphic functions on $\hat{\mathbb{C}}$ coincide with the set of rational functions

$$\left\{ \frac{f(z)}{g(z)} \mid f, g \in \mathbb{C}[z] \text{ (polynomials in } z), g \not\equiv 0 \right\}.$$

Hint: One could follow the following steps. Let F, $F \not\equiv 0$, be a meromorphic function on $\hat{\mathbb{C}}$.

- Note that F has only finitely many zeros and poles.
- There are two possibilities: ∞ is either a pole of F or not.
- If ∞ is not a pole of F, consider the poles $a_1, \dots, a_n$ of F. Consider the principal parts h_ν of F at a_ν, $\nu = 1, \dots, n$, and observe that $F - \sum_{\nu=1}^{n} h_\nu$ is a holomorphic function on $\hat{\mathbb{C}}$. So it must be constant and hence F is a rational function.
- If ∞ is a pole of F, consider the function $\frac{1}{F}$ and show as above that it is rational.

Exercise 10: Deduce the following fundamental facts from the statements proven in the lecture.

1) *Liouville's Theorem:* Any bounded holomorphic function $\mathbb{C} \to \mathbb{C}$ is constant.

2) *The Fundamental Theorem of Algebra:* Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geqslant 1$. Then there exists $a \in \mathbb{C}$ such that $f(a) = 0$.

Exercise 11: Let Γ be a lattice in $\mathbb{C}$. Then a meromorphic function $f \in \mathcal{M}_{\mathbb{C}}(\mathbb{C})$ is called doubly periodic (or elliptic) with respect to Γ if $f(z) = f(z + \gamma)$ for all $z \in \mathbb{C}$ and for all $\gamma \in \Gamma$.

1) Show that there is a one-to-one correspondence between elliptic functions on $\mathbb{C}$ with respect to Γ and meromorphic functions on $\mathbb{C}/\Gamma$.

2) Show that there are only constant holomorphic doubly periodic functions.

3) Show that every non-constant elliptic function attains every value $b \in \hat{\mathbb{C}}$.

Exercise 12: Show that Riemann surfaces are path-connected.
Hint: *For a point x_0 of a Riemann surface X consider the set S of all points that can be connected with x_0 by a path. Show that S is non-empty, closed and open.*

Exercise 13: 1) Let a and b be two points in a topological space X. Check that the homotopy is an equivalence relation on the set of all curves from a to b.

2) Fill in the gaps and check the technical details in the definition of the fundamental group from the lecture.

Exercise 14: 1) Let a and b be two points in a topological space X. Assume there exists a path $\gamma : I \to X$ with $\gamma(0) = a$, $\gamma(1) = b$. Show that the map

$$\pi_1(X, a) \to \pi_1(X, b), \quad [\delta] \mapsto [\gamma^{-1} \cdot \delta \cdot \gamma] \qquad (*)$$

is well defined and is an isomorphism of groups.

2) Check whether it is true that the isomorphism $(*)$ does not depend on the choice of γ if and only if $\pi_1(X, a)$ is abelian.

Exercise 15: Compute the fundamental groups of $\hat{\mathbb{C}}$ and of a complex torus $\mathbb{C}/\Gamma$.

Exercise 16: Prove that $0 \in \mathbb{C}$ is the only ramification point of the holomorphic map

$$\mathbb{C} \to \mathbb{C}, \quad z \mapsto z^k, \quad k \geqslant 2.$$

Exercise 17: Let $X \xrightarrow{f} Y$ be a covering and let y_1, y_2 be two points from Y. Let γ be a path connecting y_1 and y_2. In the lecture we have constructed a map $f^{-1}(y_1) \to f^{-1}(y_2)$ and claimed it to be a bijection. Construct the inverse map.

Exercise 18: Check that a universal covering is unique up to a homeomorphism.

Exercise 19: Find a universal covering $\widetilde{X}$ of a complex torus $X = \mathbb{C}/\Gamma$. Compute its group of deck transformations.

Exercise 20: Let $\Gamma = \mathbb{Z} + \mathbb{Z} \cdot \tau$, $\tau \in \mathbb{C}$, be a lattice in $\mathbb{C}$. Let n be a natural number and let $\Gamma' = \mathbb{Z} + \mathbb{Z} \cdot (n\tau)$. Put $X = \mathbb{C}/\Gamma$ and $X' = \mathbb{C}/\Gamma'$ and consider the map

$$X \to X', \quad [z] \mapsto [nz].$$

By Exercise 6 it is a holomorphic map of Riemann surfaces. Prove that it is a covering. What is the number of points in the fibres? Compute the group of deck transformations of this covering.

Exercise 21: Show that the map $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$ is a covering. Compute its group of deck transformations.

Exercise 22: Check that $\mathrm{PDiv}(X)$, $\mathrm{Div}^0(X)$, and $\mathrm{Pic}(X)$ are abelian groups and that the inclusions

$$\mathrm{PDiv}(X) \subset \mathrm{Div}^0(X) \subset \mathrm{Div}(X)$$

are inclusions of subgroups.

Exercise 23: We have proven that the Riemann-Roch space of a divisor is finite dimensional. Check in details that the kernel of the linear map $\xi : \mathcal{L}(D') \to \mathbb{C}$ used in the proof coincides with $\mathcal{L}(D)$.

Exercise 24: Let a be a point of a Riemann surface X. Show that the stalk $\mathcal{O}_{X,a}$ is a $\mathbb{C}$-algebra with the operations defined in the lecture:

$$f_a + g_a := (f+g)_a, \quad f_a \cdot g_a := (fg)_a, \quad \lambda \cdot f_a := (\lambda f)_a, \quad f_a, g_a \in O_{X,a}, \lambda \in \mathbb{C}.$$

In particular check that the definitions given in the lecture are well-defined, i. e., do not depend on the choice of representatives.

Exercise 25: Show that $\mathcal{O}_{\mathbb{C},a}$, $a \in \mathbb{C}$, is isomorphic to the ring $\mathbb{C}\{z\}$ of convergent power series in one variable.

Exercise 26: Check that the definition of a holomorphic differential form does not depend on the choice of a local coordinate.

Exercise 27: Let X be a Riemann surface and let $z : U \to V$ and $w : U \to W$ be two complex charts of X. Then $w = G \circ z$ for a holomorphic function $G : V \to W$. Show that

$$dw = G'(z) \cdot dz.$$

Exercise 28: Prove that $\Omega_{\hat{\mathbb{C}}}(\hat{\mathbb{C}}) = 0$.

Hint: *Consider an arbitrary* $\omega \in \Omega_{\hat{\mathbb{C}}}(\hat{\mathbb{C}})$ *in the standard charts of* $\hat{\mathbb{C}}$ *and apply Exercise 27 to the intersection of the standard chars.*

Exercise 29: Let ω be a meromorphic differential form on a Riemann surface X and let a be a point of X. Check that $\operatorname{ord}_a \omega$ defined in the lecture does not depend on the choice of a local coordinate.

In **Exercises 30–32** let $\Gamma = \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$ be a lattice in $\mathbb{C}$ and let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus.

Exercise 30: In the lecture we defined the Weierstraß elliptic function to be

$$\wp(z) = \wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{0\neq\gamma\in\Gamma}\left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right). \qquad (\star)$$

Following the following steps prove that it is well-defined.

(1) Understand what it means for $(\star)$ to be summable. You could consult the following note (in German): ftp://www.mathematik.uni-kl.de/pub/scripts/wirthm/Ft/Summbarkeit.pdf

(2) Let N be a positive integer. Consider the finite subsets $\Gamma_N \subset \Gamma$ defined as follows:

$$\Gamma_N := \{N\omega_1 + n\omega_2 \mid -N \leqslant n \leqslant N\} \cup \{-N\omega_1 + n\omega_2 \mid -N \leqslant n \leqslant N\} \cup \{m\omega_1 + N\omega_2 \mid -N \leqslant m \leqslant N\} \cup \{m\omega_1 - N\omega_2 \mid -N \leqslant m \leqslant N\}.$$

Draw a picture to visualize Γ_N. Estimate the sums

$$\sum_{\gamma\in\Gamma_N}\left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right)$$

and show that $(\star)$ is absolutely summable for $z \notin \Gamma$.

(3) Show that $\wp(z)$ is holomorphic on $\mathbb{C}\setminus\Gamma$ and has poles of order 2 at the lattice points. Notice that $\wp(-z) = \wp(z)$.

(4) Compute the derivative $\wp'(z)$ and show that it is an elliptic function.

(5) Show that for $\gamma \in \Gamma$ the function $h(z) := \wp(z+\gamma) - \wp(z)$ has zero derivative and hence is constant. Show that this constant equals zero: assume γ to be a generator of Γ and evaluate h at $z = -\gamma/2$ (notice that in this case $-\gamma/2$ can not belong to Γ).

Exercise 31: We identify the elliptic functions on $\mathbb{C}$ with the corresponding meromorphic functions on X. Prove the following equalities:

$$\begin{aligned}
\mathcal{L}(2\cdot[0]) &= \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z),\\
\mathcal{L}(3\cdot[0]) &= \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z),\\
\mathcal{L}(4\cdot[0]) &= \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z) + \mathbb{C}\cdot\wp^2(z),\\
\mathcal{L}(5\cdot[0]) &= \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z) + \mathbb{C}\cdot\wp^2(z) + \mathbb{C}\cdot\wp(z)\wp'(z).
\end{aligned}$$

Hint: *Since the left hand sides clearly contain the corresponding right hand sides and since by the Riemann-Roch theorem the dimension of* $\mathcal{L}(n\cdot[n])$ *is* n *it is enough to prove that the generators of the right hand sides are liner independent.*

Exercise 32: Notice that $\mathcal{L}(6\cdot[0])$ contains the subspace

$$\mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z) + \mathbb{C}\cdot\wp^2(z) + \mathbb{C}\cdot\wp(z)\wp'(z) + \mathbb{C}\cdot\wp'^2(z) + \mathbb{C}\cdot\wp^3(z). \qquad (*)$$

Conclude using the Riemann-Roch theorem that there is a linear relation between the generators of $(*)$.

Prove that

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3,$$

where

$$g_2 = 60 \sum_{0\neq\gamma\in\Gamma} \frac{1}{\gamma^4}, \quad g_3 = 140 \sum_{0\neq\gamma\in\Gamma} \frac{1}{\gamma^6}.$$

Hint: *Show that the elliptic function $h(z) = (\wp'(z))^2 - (4\wp^3(z) - g_2\wp(z))$ does not have any poles and conclude that it must be holomorphic and hence constant. You could use the Laurent expansions of $\wp(z)$ and $\wp'(z)$ at* 0.

In the **Exercises 33–37** let $\Gamma = \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$ be a lattice in $\mathbb{C}$ and let $\wp(z)$ be the corresponding Weierstraß elliptic function.

Exercise 33: When can $\wp(z)$ have a zero of order 2?

Exercise 34: Show that the constant term of the Laurent expansion of $\wp$ at 0 vanishes.

Exercise 35: Let $\Gamma = \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$ be a lattice in $\mathbb{C}$. For integers $n \geqslant 3$ the families $\{\frac{1}{\gamma^n}\}_{0\neq\gamma\in\Gamma}$ are summable and the sums

$$G_n = \sum_{0\neq\gamma\in\Gamma} \frac{1}{\gamma^n}, \quad n \geqslant 3,$$

are called Eisenstein series. Show that the Laurent expansion of $\wp(z)$ at 0 is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2(n+1)}z^{2n}.$$

Show that $G_n = 0$ for odd n.

Exercise 36: Show that $\wp(z) = \wp(w)$ if and only if either $z \equiv w \mod \Gamma$ or $z \equiv -w \mod \Gamma$.
Hint: *For a fixed w consider $h(z) = \wp(z) - \wp(w)$ and study its set of zeroes using that $\wp(z)$ is an even function.*

Exercise 37: Show that $\wp'(z) = 0$ if and only if $2z \in \Gamma$ and $z \notin \Gamma$.
Hint: *Using that $\wp'$ is an odd function show that $z \in \mathbb{C}$ with $2z \in \Gamma$ are zeroes of $\wp'$. Notice that modulo Γ there are 3 distinct z with $2z \in \Gamma$.*

Exercise 38: Let X be a Riemann surface. Let $\omega \in \Omega_X(X)$ be a holomorphic differential form on X, and let γ be a piece-wise smooth path in X. Show that the definition of

$$\int_\gamma \omega$$

given in the lecture does not depend on the choices of local charts and partitions of γ.

Exercise 39: Check the five properties of integrals given in the lecture.

Exercise 40: Write down the formal definitions of C^∞ differential forms on a Riemann surface.

Exercise 41: Refresh you knowledge about residues of meromorphic functions and about the theorem of Stokes (Green).

Exercise 42: Let f be a meromorphic function on a compact Riemann surface X. Apply the residue theorem to the differential form

$$\frac{df}{f}$$

and deduce the statement

$$\sum_{p\in X} \operatorname{ord}_p f = 0.$$

Exercise 43: Let X be a compact Riemann surface, let $a \in X$, let $D \in \operatorname{Div}(X)$. Put $D' = D + a$. Show that

$$\mathcal{K}(D') \subset \mathcal{K}(D), \quad \dim \mathcal{K}(D)/\mathcal{K}(D') \leqslant 1.$$

Exercise 44: Let Γ be a lattice in $\mathbb{C}$ and let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus. Fix some α_1 and β_1 as in the lecture, fix a basis of $\Omega_X(X)$ and compute the corresponding period matrix.

Exercise 45: Let X be a compact Riemann surface of genus g. Let $D \in \operatorname{Div}(X)$, $D > 0$. In the lecture we introduced the following $\mathbb{C}$-vector space.

$$I(D) = \left\{ \omega \in \mathcal{K}_X(X) \,\middle|\, \begin{array}{l} \text{for } a \in X \text{ with } D(a) > 0, \omega \text{ may have at } a \text{ a pole of order} \\ \text{at most } D(a)+1, \text{ moreover } \operatorname{res}_x \omega = 0 \text{ for all } x \in X \end{array} \right\}.$$

Prove that the dimension of $I(D)$ equals $\deg D + g$.

Exercise 46: Let $X = \mathbb{C}/\Gamma$ be a complex torus. Show that the Abel's theorem can be formulated in this case as follows.

Theorem. *Let $D \in \operatorname{Div}^0(X)$, $D = \sum_{i=1}^{r} a_i \cdot x_i$, $a_i \in \mathbb{Z}$, $x_i \in X$. Then D is a principal divisor, i. e., $D = (f)$ for some meromorphic function $f \in \mathcal{M}_X(X)$, if and only if*

$$\sum_{i=1}^{r} a_i \cdot x_i = 0$$

as an element of $X = \mathbb{C}/\Gamma$.

Exercise 47: Let $D = \sum_{i=1}^{r} a_i \cdot x_i$ be a principal divisor, i. e., $D = (f)$ for some meromorphic function $f \in \mathcal{M}_X(X)$. Show that

$$\sum_{i=1}^{r} a_i \cdot x_i = 0$$

as an element of $X = \mathbb{C}/\Gamma$.

Hint: Let $\pi : \mathbb{C} \to X$ be the canonical projection. Consider $F(z) = f \circ \pi(z)$. Choose a fundamental parallelogram V in $\mathbb{C}$ such that there are no poles or zeros of F on its boundary ∂V. Consider the integral

$$\int_{\partial V} z \frac{F'(z)}{F(z)}\, dz$$

and apply the standard residue theorem.

Theorem. *For a meromorphic function g on V which possesses a continuous extension to the closure of V one has*

$$\frac{1}{2\pi i}\int\limits_{\partial V} g(z)dz = \sum_{a\in V} \operatorname{res}_a g.$$