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Mathematics Research Unit
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Exercises from Lecture 12

Riemann Surfaces

Exercise 46: Let $X = \mathbb{C}/\Gamma$ be a complex torus. Show that the Abel's theorem can be formulated in this case as follows.

Theorem. *Let $D \in \mathrm{Div}^0(X)$, $D = \sum_{i=1}^{r} a_i \cdot x_i$, $a_i \in \mathbb{Z}$, $x_i \in X$. Then D is a principal divisor, i. e., $D = (f)$ for some meromorphic function $f \in \mathcal{M}_X(X)$, if and only if*

$$\sum_{i=1}^{r} a_i \cdot x_i = 0$$

as an element of $X = \mathbb{C}/\Gamma$.

Exercise 47: Let $D = \sum_{i=1}^{r} a_i \cdot x_i$ be a principal divisor, i. e., $D = (f)$ for some meromorphic function $f \in \mathcal{M}_X(X)$. Show that

$$\sum_{i=1}^{r} a_i \cdot x_i = 0$$

as an element of $X = \mathbb{C}/\Gamma$.

Hint: Let $\pi : \mathbb{C} \to X$ be the canonical projection. Consider $F(z) = f \circ \pi(z)$. Chose a fundamental parallelogram V in $\mathbb{C}$ such that there is no poles or zeros of F on its boundary ∂V. Consider the integral

$$\int_{\partial V} z \frac{F'(z)}{F(z)}\, dz$$

and apply the standard residue theorem.

Theorem. *For a meromorphic function g on V which possesses a continuous extension to the closure of V one has*

$$\frac{1}{2\pi i} \int_{\partial V} g(z) dz = \sum_{a \in V} \mathrm{res}_a\, g.$$