University of Luxembourg
Mathematics Research Unit
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Exercises from Lecture 3

Riemann Surfaces

Exercise 12: Show that Riemann surfaces are path-connected.
Hint: *For a point x_0 of a Riemann surface X consider the set S of all points that can be connected with x_0 by a path. Show that S is non-empty, closed and open.*

Exercise 13: 1) Let a and b be two points in a topological space X. Check that the homotopy is an equivalence relation on the set of all curves from a to b.

2) Fill in the gaps and check the technical details in the definition of the fundamental group from the lecture.

Exercise 14: 1) Let a and b be two points in a topological space X. Assume there exists a path $\gamma : I \to X$ with $\gamma(0) = a$, $\gamma(1) = b$. Show that the map

$$\pi_1(X, a) \to \pi_1(X, b), \quad [\delta] \mapsto [\gamma^{-1} \cdot \delta \cdot \gamma] \tag{$*$}$$

is well defined and is an isomorphism of groups.

2) Check whether it is true that the isomorphism $(*)$ does not depend on the choice of γ if and only if $\pi_1(X, a)$ is abelian.

Exercise 15: Compute the fundamental groups of $\hat{\mathbb{C}}$ and of a complex torus $\mathbb{C}/\Gamma$.

Exercise 16: Prove that $0 \in \mathbb{C}$ is the only ramification point of the holomorphic map

$$\mathbb{C} \to \mathbb{C}, \quad z \mapsto z^k, \quad k \geqslant 2.$$

Exercise 17: Let $X \xrightarrow{f} Y$ be a covering and let y_1, y_2 be two points from Y. Let γ be a path connecting y_1 and y_2. In the lecture we have constructed a map $f^{-1}(y_1) \to f^{-1}(y_2)$ and claimed it to be a bijection. Construct the inverse map.