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Mathematics Research Unit
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Exercises from Lecture 7

Riemann Surfaces

In this exercise sheet let $\Gamma = \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$ be a lattice in $\mathbb{C}$ and let $X = \mathbb{C}/\Gamma$ be the corresponding complex torus.

Exercise 30: In the lecture we defined the Weierstraß elliptic function to be

$$\wp(z) = \wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{0\neq\gamma\in\Gamma}\left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right). \qquad (\star)$$

Following the following steps prove that it is well-defined.

(1) Understand what it means for $(\star)$ to be summable. You could consult the following note (in German): ftp://www.mathematik.uni-kl.de/pub/scripts/wirthm/Ft/Summbarkeit.pdf

(2) Let N be a positive integer. Consider the finite subsets $\Gamma_N \subset \Gamma$ defined as follows:

$$\Gamma_N := \{N\omega_1 + n\omega_2 \mid -N \leqslant n \leqslant N\} \cup \{-N\omega_1 + n\omega_2 \mid -N \leqslant n \leqslant N\} \cup \{m\omega_1 + N\omega_2 \mid -N \leqslant m \leqslant N\} \cup \{m\omega_1 - N\omega_2 \mid -N \leqslant m \leqslant N\}.$$

Draw a picture to visualize Γ_N. Estimate the sums

$$\sum_{\gamma\in\Gamma_N}\left(\frac{1}{(z-\gamma)^2} - \frac{1}{\gamma^2}\right)$$

and show that $(\star)$ is absolutely summable for $z \notin \Gamma$.

(3) Show that $\wp(z)$ is holomorphic on $\mathbb{C}\setminus\Gamma$ and has poles of order 2 at the lattice points. Notice that $\wp(-z) = \wp(z)$.

(4) Compute the derivative $\wp'(z)$ and show that it is an elliptic function.

(5) Show that for $\gamma \in \Gamma$ the function $h(z) := \wp(z+\gamma) - \wp(z)$ has zero derivative and hence is constant. Show that this constant equals zero: assume γ to be a generator of Γ and evaluate h at $z = -\gamma/2$ (notice that in this case $-\gamma/2$ can not belong to Γ).

Exercise 31: We identify the elliptic functions on $\mathbb{C}$ with the corresponding meromorphic functions on X. Prove the following equalities:

$$\mathcal{L}(2\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z),$$
$$\mathcal{L}(3\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z),$$
$$\mathcal{L}(4\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z) + \mathbb{C}\cdot\wp^2(z),$$
$$\mathcal{L}(5\cdot[0]) = \mathbb{C}\cdot 1 + \mathbb{C}\cdot\wp(z) + \mathbb{C}\cdot\wp'(z) + \mathbb{C}\cdot\wp^2(z) + \mathbb{C}\cdot\wp(z)\wp'(z).$$

Hint: *Since the left hand sides clearly contain the corresponding right hand sides and since by the Riemann-Roch theorem the dimension of $\mathcal{L}(n\cdot[n])$ is n it is enough to prove that the generators of the right hand sides are liner independent.*

Exercise 32: Notice that $\mathcal{L}(6 \cdot [0])$ contains the subspace

$$\mathbb{C} \cdot 1 + \mathbb{C} \cdot \wp(z) + \mathbb{C} \cdot \wp'(z) + \mathbb{C} \cdot \wp^2(z) + \mathbb{C} \cdot \wp(z)\wp'(z) + \mathbb{C} \cdot \wp'^2(z) + \mathbb{C} \cdot \wp^3(z). \qquad (*)$$

Conclude using the Riemann-Roch theorem that there is a linear relation between the generators of $(*)$.

Prove that

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3,$$

where

$$g_2 = 60 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^4}, \quad g_3 = 140 \sum_{0 \neq \gamma \in \Gamma} \frac{1}{\gamma^6}.$$

Hint: *Show that the elliptic function* $h(z) = (\wp'(z))^2 - (4\wp^3(z) - g_2\wp(z))$ *does not have any poles and conclude that it must be holomorphic and hence constant. You could use the Laurent expansions of* $\wp(z)$ *and* $\wp'(z)$ *at* 0.