On vector bundles on curves and 1-dimensional sheaves

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Plan

Question to study

- Reminder on Simpson moduli spaces
- 1-dimensional sheaves and line bundles on curves
- Singular sheaves
- Formulation of the problem

Some results

- Boundary of the Simpson moduli spaces
- Modification of the Simpson moduli spaces by line bundles

Reminder on Simpson moduli spaces

$\mathbb{k} = \bar{\mathbb{k}}$ of characteristic zero, (say $\mathbb{k} = \mathbb{C}$)

$\mathfrak{X} = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(1))$ smooth projective variety/$\mathbb{k}$

$\mathcal{F} \in \mathscr{C}oh\,\mathfrak{X}$ — coherent sheaf on $\mathfrak{X}$ $\implies$

- $P(\mathcal{F}, m) = P(m) = \sum_i (-1)^i \cdot \dim H^i(\mathfrak{X}, \mathcal{F} \otimes \mathcal{O}_{\mathfrak{X}}(m)) \in \mathbb{Q}[m]$ is the **Hilbert polynomial** of $\mathcal{F}$.
- For $d = \deg P(m) \implies P(\mathcal{F}, m) = \sum_{i=0}^{d} \alpha_i(\mathcal{F}) \cdot \frac{m^i}{i!}$, $\alpha_d(\mathcal{F}) \in \mathbb{Z}$, $\alpha_d(\mathcal{F}) > 0$.
- $\mathcal{F}$ is **(semi-)stable** $\overset{def}{\iff}$ for every proper subsheaf $\mathcal{E} \subseteq \mathcal{F}$

$$\alpha_d(\mathcal{F}) \cdot P(\mathcal{E}, m) < (\leqslant) \alpha_d(\mathcal{E}) \cdot P(\mathcal{F}, m), \quad \text{for } m \gg 0$$

Fix $P \in \mathbb{Q}[m]$ $\implies$

Theorem (Carlos Simpson)

There is a projective moduli space $M_P(\mathfrak{X})$ of semi-stable sheaves with Hilbert polynomial P.

1-dimensional sheaves are genericaly line bundles on curves

The degree of $P(\mathcal{F}, m)$ coincides with $\dim \operatorname{Supp} \mathcal{F} \implies$

1-**dimensional** semistable sheaves $\iff$ semistable with linear Hilbert polynomial

1-dimensional sheaves are supported on curves.

- Stability implies purity and hence torsion-freeness on support
- $\operatorname{Supp} \mathcal{F} = C$ smooth curve in $\mathfrak{X} \implies$ line bundle on C
- Map $\quad M_P(\mathfrak{X}) \ni \mathcal{F} \mapsto \operatorname{Supp} F \in \{$curves in $\mathfrak{X}\}$
- Fibres over smooth curves are Jacobians.
- Over singular curves — “compactified Jacobians”.
- Complicated over non-integral curves.

Subvariety of singular sheaves

- $M' \subseteq M = M_P(\mathfrak{X})$ closed subvariety of **singular sheaves**, i. e., sheaves not locally free on support.
- M' not empty in general! $\Longrightarrow$
- $M_B = M \setminus M'$ space of line bundles (on support), **non-compact**
- M is a compactification of M_B.
- $\operatorname{codim}_M M' > 1$ (not a divisor) $\Longrightarrow$ M is not "maximal"

Questions

Find another compactification of M_B

- *by vector bundles;*
- *a maximal one (divisor, preferably with nice properties, in the complement of M_B).*

Problems to solve

1. understand M;
2. understand the boundary M';
3. find (examples of) a modification construction.

Restrict to $\mathfrak{X} = \mathbb{P}_2$, $P(m) = cm + d$, $\gcd(c,d) = 1 \implies$

- M is smooth of dimension $c^2 + 1$ (Le Potier). In this case M consists of isomorphism classes.

- Resolutions

$$0 \to \bigoplus_{1}^{r} \mathcal{O}(b_i) \xrightarrow{A} \bigoplus_{1}^{r} \mathcal{O}(a_i) \to \mathcal{F} \to 0$$

$\implies$ tools for understanding M

Results on M and M'

- Understanding of $M_{dm+c}(\mathbb{P}_2)$ for small d (Drézet, Maican, Chung, Moon, Iena).
- $M = M_{dm\pm1}(\mathbb{P}_2)$ is up to codimension 2 a projective bundle over $H = \mathbb{P}_2^{[l]}$, $l = (d-1)(d-2)/2$, Hilbert scheme of l points. Generic sheaves are (twisted) ideals of l points on $C = \operatorname{Supp}\mathcal{F}$, a curve of degree d:

$$0 \to \mathcal{F} \to \mathcal{O}_C(d-3) \to \mathcal{O}_Z \to 0, \quad Z \in H.$$

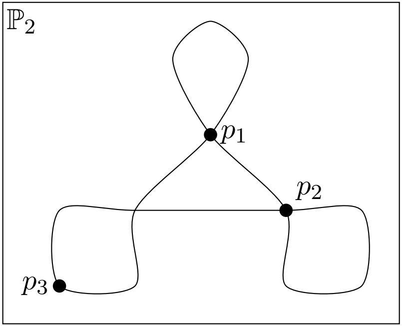
- Fibres of $M' \subseteq M \to H$ — arrangements of l linear subspaces of codimension 2 in the fibres of $M \to H \implies$

Theorem (Iena, Leytem)

M' is of codimension 2; singular for $d \geqslant 4$.

Geometric explanation (generic case)

- The fibre of $M \to H$ over $Z \in H \iff$ space of degree d curves through Z.



- The fibre of $M' \to H$ over $Z \in H \iff$ space of degree d curves $C \supseteq Z$ with $Z \cap \operatorname{Sing} C \neq \emptyset$.
- Space of curves singular at p_i is a linear subspace of codimension 2 in the fibre of $M \to H$.

Blowing up M along M'

For maximal boundary:
$\widetilde{M} = \mathrm{Bl}_{M'} M$ first candidate.

Need:

- Family of 1-dimensional sheaves over $\widetilde{M}$.
- New objects instead of singular sheaves.
- Are these new objects singular/non-singular?
- Are new objects classified by the exceptional divisor?

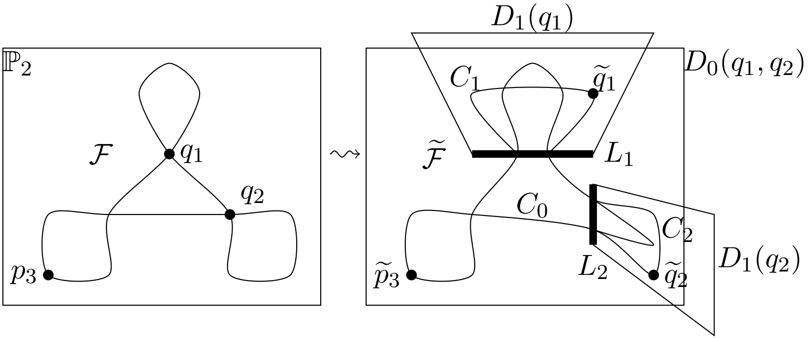
Construction of a family over $\mathrm{Bl}_{M'} M$

- Pull back the universal family over M to a family $\boldsymbol{\mathcal{U}}_{\widetilde{M}}$ over $\widetilde{M}$.
- $\widetilde{\widetilde{M} \times \mathbb{P}_2} \to \widetilde{M} \times \mathbb{P}_2$ the blow up along the subvariety where $\boldsymbol{\mathcal{U}}_{\widetilde{M}}$ is singular.
- Pull $\boldsymbol{\mathcal{U}}_{\widetilde{M}}$ back and consider the quotient $\widetilde{\boldsymbol{\mathcal{U}}}$ by the subsheaf annihilated by the ideal sheaf of the exceptional divisor of this blow-up.

Claim

The fibres of the new family are 1-dimensional sheaves.

Modifying the boundary

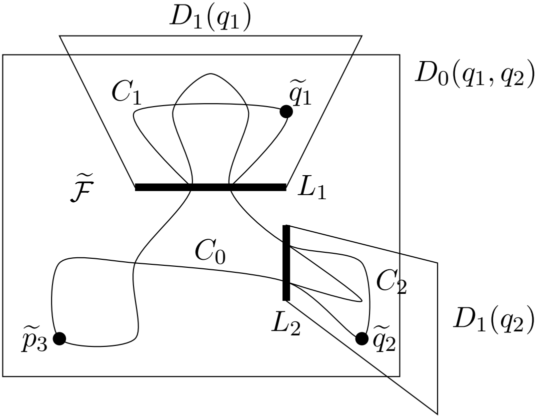


- Twisted ideal sheaf of l points $Z = \{q_1, \dots, q_r, p_{r+1}, \dots, p_l\}$ on a degree d curve C with $Z \cap \operatorname{Sing} C = \{q_1, \dots, q_r\}$ $\implies$
- Twisted ideal sheaves $\widetilde{\mathcal{F}}$ of l points $\{\widetilde{q}_1, \dots, \widetilde{q}_r, \widetilde{p}_{r+1}, \dots, \widetilde{p}_l\}$ in $C' = C_0 \cup C_1 \cup \cdots \cup C_r$ in a surface $D(q_1, \dots, q_r)$.

Equivalence classes of new objects

Definition

Two new sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ associated to $Z \subseteq C$ on $D(q_1, \ldots, q_r)$ are called equivalent if there is an automorphism ϕ of $D(q_1, \ldots, q_r)$ identical on $D_0(q_1, \ldots, q_r)$ such that $\phi^*(\mathcal{E}_1) \cong \mathcal{E}_2$.



New objects are (generically) line bundles on support

Proposition

The points of the exceptional divisor of $\widetilde{M}$ classify the equivalence classes of sheaves of new type.

Generic fibres over the exceptional divisor of $\widetilde{M}$ are line bundles on curves.

$\Longrightarrow$

- This solves the modification problem in an open subset of M.
- Complete solution only for $d = 3$.

Smoothness of the exceptional divisor of $\widetilde{M} \to M$

- $Z \in H \quad l$ points on $\mathbb{P}_2$.
- $F \quad$ fibre of $M \to H$ over Z, $F \cong \mathbb{P}_{3d-1}$.
- $F' \quad$ fibre of $M' \subseteq M \to H$.
- $F' = \bigcup_1^r F_i, \quad \operatorname{codim}_F F_i = 2, \quad F_i \subseteq F$ linear.
- If F_i intersect transversally at some point $\mathcal{F} \in F' \implies$ the exceptional divisor of $\widetilde{M} \to M$ is smooth over $\mathcal{F}$.
- In this case $\mathcal{F}$ is substituted by $\prod_1^r \mathbb{P}_1$ in $\widetilde{M}$.
- Always holds for $r \leqslant 3$, in particular for $d = 4 \implies$ true up to high codimension.

Summary

For $M = M_{dm\pm1}(\mathbb{P}_2)$.

- Family of 1-dimensional sheaves over $\widetilde{M} = \mathrm{Bl}_{M'} M$.
- Generic singular sheaf from M is substituted by non-singular sheaves.
- The exceptional divisor of $\widetilde{M} \to M$ classifies the equivalence classes of new objects.
- The exceptional divisor is smooth in high codimension.
- arXiv:1511.01847, 1305.2400, 1103.1485, 1012.5843

Work in progress

- The sheaves in the complement of the open subvariety in M parameterized by $Z \subseteq C$ as above are (expected to be) substituted by "less singular" sheaves.
- Iterating our construction: get a recompatification $\widetilde{M}$ of the Simpson moduli spaces by vector bundles (on 1-dimensional support).
- New functor $\widetilde{\mathcal{M}}$ together with a natural transformation to the Simpson functor $\mathcal{M}$ together with a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \longrightarrow & \mathrm{Hom}(\_\,, \widetilde{M}) \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathrm{Hom}(\_\,, M). \end{array}$$

The end.