Integration of Leibniz algebras

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Abstract

These are the lecture notes for the minicourse "Integration of Leibniz algebras" at the University of Luxembourg during the conference "Higher Lie theory" (09/12/2013 to 11/12/2013).

1   Lie racks

The first question which comes to mind when considering the integration problem for Leibniz algebras, is into which structure we want to integrate them. Let us argue here that this integrating structure should be related to *Lie racks*, following work of Kinyon [18]. Some mathematicians are convinced that one should refine the structure of a Lie rack (i.e. add more structure) in order to arrive at the correct notion integrating Leibniz algebras, but I will stay here with "pure" Lie racks.

Recall the notion of a rack: It comes from axiomatizing the notion of conjugation in a group.

Definition 1.1. *Let X be a set together with a binary operation denoted $(x, y) \mapsto x \triangleright y$ such that for all $x \in X$, the map $y \mapsto x \triangleright y$ is bijective and for all $x, y, z \in X$,*

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z).$$

Then we call X, or more precisely $(X, \triangleright)$, a (left) rack.

By construction, the most important example of a rack is the conjugation in a group G. The rack operation is in this case given by $(g, h) \mapsto ghg^{-1}$.

Definition 1.2. *Let R be a rack and X be a set. We say that R acts on X on the left in case for all $r \in R$, there are bijections $(r \cdot) : X \to X$ such that for all $x \in X$ and all $r, r' \in R$:*

$$r \cdot (r' \cdot x) = (r \triangleright r') \cdot (r \cdot x).$$

Clearly, the adjoint action $\mathrm{Ad}_r : R \to R$ defined by $\mathrm{Ad}_r(r') := r \triangleright r'$ in a left rack R is a left action of R on itself.

Definition 1.3. *A pointed rack $(X, \triangleright, 1)$ is a set X with a binary operation $\triangleright$ and an element $1 \in X$ such that the following axioms are satisfied:*

1. *$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$,*

2. *For each $a, b \in X$, there exists a unique $x \in X$ such that $a \triangleright x = b$,*

3. *$1 \triangleright x = x$ and $x \triangleright 1 = 1$ for all $x \in X$.*

Once again, the conjugation rack of a group is an example of a pointed rack.

Definition 1.4.   1. *A Lie rack X is a manifold and a pointed smooth rack, i.e. the structure maps are smooth.*

2. *A local Lie rack is a manifold X with an open subset $\Omega \subset X \times X$ where a Lie rack product $\triangleright$ is defined such that*

    (a) *If $(x, y), (x, z), (y, z), (x, y \triangleright z), (x \triangleright y, x \triangleright z) \in \Omega$, then $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$.*

    (b) *If $(x, y), (x, z) \in \Omega$ and $x \triangleright y = x \triangleright z$, then $y = z$.*

    (c) *For all $x \in X$, $(1, x), (x, 1) \in \Omega$ and as usual $1 \triangleright x = x$ and $x \triangleright 1 = 1$.*

Examples of Lie racks include obviously the conjugation racks associated to Lie groups. It will be important in the following to have a replacement for the semi-direct product in the context of racks. For this, let us first review the hemi-semi-direct product Leibniz algebra (this terminology follows [19], but is different from [18], where this structure is called *demi-semi-direct product*):

Lemma 1.5. *Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. The direct sum $V \oplus \mathfrak{g}$ together with the bracket*

$$[(v, X), (v', X')] = (X(v'), [X, X'])$$

becomes a Leibniz algebra, called the hemi-semi-direct product $V \times_{\mathrm{hs}} \mathfrak{g}$ of V and $\mathfrak{g}$.

Example (of a Lie rack): Let G be a Lie group and V be a (smooth) G-module. On $X := V \times G$, we define a binary operation $\triangleright$ by

$$(v, g) \triangleright (v', g') = (g(v'), gg'g^{-1})$$

for all $v, v' \in V$ and all $g, g' \in G$. X is a Lie rack with unit $1 := (0, 1)$ which is called a *linear Lie rack*. This is the "group-analog" of the hemi-semi-direct product of a Lie algebra with its representation, and we denote it by $V \times_{\mathrm{hs}} G$.

Let us define more generally this hemi-semi-direct product of racks:

Definition 1.6. *Let R be a rack and A be a rack module in the sense of Definition 1.2. The hemi-semi-direct product $A \times_{\mathrm{hs}} R$ of R with A is the following rack structure on the direct product set $A \times R$:*

$$(a, r) \triangleright (a', r') := (r(a'), r \triangleright r').$$

Now let us come to digroups which form a "stronger" structure in which one might want to integrate Leibniz algebras:

Definition 1.7. *A digroup* $(H, \vdash, \dashv)$ *is a set* H *together with two binary operations* $\vdash$ *and* $\dashv$ *satisfying the following axioms. For all* $x, y, z \in H$,

1. $(H, \vdash)$ *and* $(H, \dashv)$ *are semigroups,*

2. $x \vdash (y \dashv z) = (x \vdash y) \dashv z$,

3. $x \dashv (y \vdash z) = x \dashv (y \dashv z)$,

4. $(x \dashv y) \vdash z = (x \vdash y) \vdash z$,

5. *there exists* $1 \in H$ *such that* $1 \vdash x = x \dashv 1 = x$ *for all* $x \in H$,

6. *for all* $x \in H$, *there exists* $x^{-1} \in H$ *such that* $x \vdash x^{-1} = x^{-1} \dashv x = 1$.

An element $e \in H$ in a digroup H is called a *bar unit* in case $e \vdash x = x \dashv e = x$ for all $x \in H$. Bar units exist in a digroup, but are not necessarily unique. A digroup is a group if and only if $\vdash = \dashv$, and 1 is then the unique bar unit.

There is a digroup which resembles very much the linear Lie rack:

Remark 1.8. *Let* G *be a Lie group and* M *be a* G-*module (with underlying vector space). Define on* $H := M \times G$ *the structure of a digroup by*

$$(u, g) \vdash (v, h) := (g(v), gh)$$

and

$$(u, g) \dashv (v, h) := (u, gh)$$

for all $u, v \in M$ *and all* $g, h \in G$. *Then* $M \times G$ *is a Lie digroup with distinguished bar unit* $(0, 1)$. *The inverse of an element* (u, g) *is* $(0, g^{-1})$. *This Lie digroup is called the linear Lie digroup associated to* G *and* M.

Digroups give rise to racks in the following way (this is the meaning of the word "stronger" above):

Proposition 1.9. *Let* $(H, \vdash, \dashv)$ *be a digroup and put*

$$x \rhd y := x \vdash y \dashv x^{-1} \tag{1}$$

for all $x, y \in H$. *Then* $(H, \rhd)$ *is a rack, pointed in* 1. *Moreover, in case* $(H, \vdash, \dashv)$ *is a Lie digroup (i.e. all structures are smooth),* $(H, \rhd)$ *is a Lie rack.*

In the case of the example in Remark 1.8, the obtained Lie rack is the above described linear Lie rack $M \times_{\mathrm{hs}} G$. In this sense every *linear* Lie rack "comes from" a (linear) Lie digroup.

Kinyon shows in [18] the following theorem which is at the heart of all our attempts to integrate Leibniz algebras.

Theorem 1.10. *Let* $(X, \rhd, 1)$ *be a Lie rack, and let* $\mathfrak{h} := T_1 X$. *Then there exists a bilinear map* $[,] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ *such that*

1. $(\mathfrak{h}, [,])$ *is a (left) Leibniz algebra,*

2. *for each* $x \in X$, *the tangent map* $\Phi(x) := T_1 \phi(x)$ *of the left translation map* $\phi(x) : X \to X$, $y \mapsto x \rhd y$, *is an automorphism of* $(\mathfrak{h}, [,])$,

3. *if* $\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$ *is defined by* $Y \mapsto \mathrm{ad}_X(Y) := [X, Y]$, *then* $\mathrm{ad} = T_1\Phi$.

Proof. The main idea of the proof is to differentiate the smooth adjoint action of the Lie rack twice:

We have for all $x \in X$, $\phi(x)(1) = x \rhd 1 = 1$, thus $\Phi(x) := T_1\phi(x)$ is an endomorphism of $\mathfrak{h} := T_1X$. As each $\phi(x)$ is invertible, we have $\Phi(x) \in \mathrm{Gl}(\mathfrak{h})$. Now the map $\Phi : X \to \mathrm{Gl}(\mathfrak{h})$ satisfies $\Phi(1) = \mathrm{id}$, thus we may differentiate again in order to obtain $\mathrm{ad} : T_1X \to \mathfrak{gl}(\mathfrak{h})$. Now we set

$$[X, Y] := \mathrm{ad}_X(Y)$$

for all $X, Y \in \mathfrak{h} = T_1X$. In terms of the left translations $\phi(x)$, the rack identity can be expressed by the equation

$$\phi(x)(\phi(y)(z)) = \phi(\phi(x)(y))(\phi(x)(z)).$$

We differentiate this equation at $1 \in X$ first with respect z, then with respect to y to obtain

$$\Phi(x)\left([Y, Z]\right) = [\Phi(x)(Y), \Phi(x)(Z)]$$

for all $x \in X$ and all $Y, Z \in \mathfrak{h}$. This expresses the fact that for each $x \in X$, $\Phi(x) \in \mathrm{Aut}(T_1X, [,])$. Finally, we differentiate this last equation at 1 with respect to x to obtain

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

for all $X, Y, Z \in \mathfrak{h}$. This shows that $\mathfrak{h}$ is a left Leibniz algebra. $\qquad\square$

Example: In the special case of a linear Lie rack, we obtain the hemi-semi-direct product Leibniz algebra $\mathfrak{h} = V \times_{\mathrm{hs}} \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G, endowed with the bracket:

$$[(v, X), (v', X')] = (X(v'), [X, X']).$$

The G-module V is here seen as a $\mathfrak{g}$-module in the usual way.

Kinyon's main result in [18] is the integration of split Leibniz algebras. For a Leibniz algebra $\mathfrak{h}$, denote by $\mathfrak{s}$ the ideal generated by the *squares* $[X, X]$ for all $X \in \mathfrak{h}$.

Definition 1.11. *A Leibniz algebra $\mathfrak{h}$ is called split, or more precisely split over an ideal $\mathfrak{i}$ with $\mathfrak{s} \subset \mathfrak{i} \subset \ker(\mathrm{ad})$, in case there exists a subalgebra $\mathfrak{k} \subset \mathfrak{h}$ with $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{i}$ (as vector spaces).*

Actually, a Leibniz algebra is split (over $\mathfrak{i}$) if and only if it is a hemi-semi-direct product Leibniz algebra (of $\mathfrak{i}$ and some Lie algebra).

Theorem 1.12 (Kinyon). *Let $\mathfrak{h}$ be a split Leibniz algebra. Then there exists a linear Lie digroup with tangent Leibniz algebra isomorphic to $\mathfrak{h}$.*

The main idea is here to integrate the hemi-semi-direct product Leibniz algebra into a linear Lie digroup (integrating separately the Lie algebra and the module). The tangent Leibniz algebra to the corresponding Lie rack is again the hemi-semi-direct product we started with.

Remark 1.13. *In fact, Simon Covez showed in his (unpublished) master thesis that conversely, in case a Leibniz algebra integrates into a Lie digroup, it must be split over some ideal containing the ideal of squares (more precisely, it is split over the ideal $\ker(T_1i)$ where i is the inversion map of the digroup).*

2 Lie's third theorem

Here we review four different proofs of Lie's third theorem for Lie algebras before discussing how to generalize (some of) them to Leibniz algebras. Of course, we do not pretend that these are the only known proofs of Lie's third theorem.

Theorem 2.1 (Lie's third theorem). *Let $\mathfrak{g}$ be a real finite dimensional Lie algebra. Then there exists a connected Lie group G whose Lie algebra is isomorphic to $\mathfrak{g}$.*

1. Proof using Levi's theorem

Levi's theorem is the assertion that any finite dimensional real Lie algebra $\mathfrak{g}$ is the semi-direct product of its *radical*, i.e. the maximal solvable ideal of $\mathfrak{g}$, and a semi-simple subalgebra.

Now prove first Lie's theorem for a solvable $\mathfrak{g}$. This is done by induction on the dimension and trivial for dimension 1. Since $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, there is a subspace $\mathfrak{a} \subset \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a}$ and $\dim(\mathfrak{g}/\mathfrak{a}) = 1$. Observe that $\mathfrak{a}$ is in fact an ideal of $\mathfrak{g}$. Let $\mathfrak{b}$ be a subspace of $\mathfrak{g}$ of dimension 1 supplementary to $\mathfrak{a}$. By construction, this describes $\mathfrak{g}$ as the semi-direct product of $\mathfrak{a}$ and $\mathfrak{b}$: $\mathfrak{g} \cong \mathfrak{a} \rtimes \mathfrak{b}$.

The theorem is then true for $\mathfrak{a}$ and $\mathfrak{b}$ by induction hypothesis, and follows for $\mathfrak{g}$, because the semi-direct product of Lie groups is a Lie group whose tangent Lie algebra is the semi-direct product of Lie algebras.

On the other hand, in case $\mathfrak{g}$ is semi-simple, Lie's third theorem is seen as follows. The adjoint action $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a faithful representation of $\mathfrak{g}$. The Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ corresponding to $\mathfrak{g}$ can then be integrated into a Lie subgroup of $Gl(\mathfrak{g})$ whose Lie algebra is isomorphic to $\mathfrak{g}$, by Lie's first theorem.

Finally, thanks to Levi's theorem, a general Lie algebra is the semi-direct product of its radical and a semi-simple subalgebra, therefore Lie's third theorem follows from what we did before. $\square$

2. Proof using Ado's theorem

Ado's theorem is the assertion that any finite dimensional real Lie algebra $\mathfrak{g}$ possesses a finite dimensional, faithful representation V. This implies that $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(V)$. Then once again, this subalgebra integrates into a connected Lie subgroup of $Gl(V)$, whose tangent Lie algebra is isomorphic to $\mathfrak{g}$ by Lie's first theorem. $\square$

3. Homological proof

Any Lie algebra $\mathfrak{g}$ is a central extension of its center $Z(\mathfrak{g})$:

$$0 \to Z(\mathfrak{g}) \to \mathfrak{g} \overset{\mathrm{ad}}{\to} \mathrm{ad}_{\mathfrak{g}} \to 0,$$

where $\mathrm{ad}_{\mathfrak{g}} \subset \mathfrak{gl}(\mathfrak{g})$ is the image of $\mathfrak{g}$ under the adjoint map sending each element to the corresponding inner derivation, i.e.

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad X \mapsto \mathrm{ad}_X := [X, -].$$

Now we have already seen that $\mathrm{ad}_{\mathfrak{g}}$ integrates into a Lie subgroup G of $Gl(\mathfrak{g})$, and the trivial $\mathfrak{g}$-module $Z(\mathfrak{g})$ can be seen as a trivial G-module. It remains to integrate the Lie algebra 2-cocycle associated to this central extension into a group 2-cocycle. This is done using that G can be chosen 1-connected and also 2-connected, by interpreting the cocycle as a differential 2-form on G. This 2-form then leads to a group 2-cocycle (see [24]), which then gives rise to a central extension of groups whose tangent Lie algebra is isomorphic to $\mathfrak{g}$. $\qquad\square$

4. Infinite dimensional proof

The proof in [14] uses an infinite dimensional construction to provide a Lie group integrating $\mathfrak{g}$. On the path space $P(\mathfrak{g})$ (which is the Banach space of continuous maps from the interval $[0,1]$ to $\mathfrak{g}$ with uniform convergence), Duistermaat and Kolk consider the following product for all $\delta, \delta' \in P(\mathfrak{g})$:

$$(\delta \cdot \delta')(t) \;=\; \delta(t) + A_\delta(t)\delta'(t), \tag{2}$$

where $A_\delta \in \mathcal{C}^1([0,1], \mathfrak{gl}(\mathfrak{g}))$ is the solution of the differential equation

$$\frac{dA}{dt}(t) \;=\; \mathrm{ad}\,\delta(t) \circ A(t)$$

with initial condition $A(0) = \mathrm{id}_{\mathfrak{g}}$. It is interesting to observe that this product has the same form as the logarithmic derivative, sending differentiable curves in a Lie group to Lie algebra valued 1-forms. Equation (2) expresses the 1-cocycle identity of the logarithmic derivative in this context, see e.g. [22].

Duistermaat-Kolk show that the Banach space $P(\mathfrak{g})$ becomes an infinite dimensional Banach Lie group with this product, and that its Lie algebra is again $P(\mathfrak{g})$ with the bracket:

$$[X,Y](t) \;=\; \frac{d}{dt}\left[\int_0^t X(s)ds, \int_0^t Y(s)ds\right].$$

Furthermore, they show that the subgroup $P_0(\mathfrak{g})$ corresponding to the kernel of the (Lie algebra) averaging map

$$\mathrm{av} : P(\mathfrak{g}) \to \mathfrak{g}, \;\; X \mapsto \int_0^1 X(t)dt$$

is a closed connected normal Banach Lie subgroup, such that the quotient $P(\mathfrak{g})/P_0(\mathfrak{g})$ is a connected (finite dimensional) Lie group integrating $\mathfrak{g}$. $\qquad\square$

Now let us discuss (a little) these four proofs.

1. Barnes [3] shows that finite dimensional Leibniz algebras possess a Levi decomposition. It would be interesting to transpose the above proof to the setting of Leibniz algebras, where the right structure should be the hemi-semi-direct product.

2. Barnes [4] also shows that finite dimensional Leibniz algebras possess a faithful representation. We would need a version of Lie's first theorem for Leibniz algebras to transpose this proof to Leibniz algebras from there. We will take here a different point of view and embed Leibniz algebras into hemi-semi-direct products in the next section, following Kinyon-Weinstein [19].

3. Simon Covez will talk about how to transpose the homological proof of Lie's third theorem to Leibniz algebras. The Leibniz cocycle integrates only locally, thus leading to a *local* Lie rack. This is explained in [10].

4. It would be most interesting to transpose the infinite dimensional proof to the setting of Leibniz algebras, because it is this approach which generalizes to the integration of Lie algebroids by Crainic-Fernandes [11]. This approach could then lead directly to the integration of Leibniz algebroids into.....Lie rackoids.

3 Integration of Leibniz algebras

3.1 Leibniz algebras as subalgebras of a hemi-semi-direct product

Kinyon and Weinstein showed in [19] that every Leibniz algebra may be embedded into a hemi-semi-direct product Leibniz algebra.

Let a Leibniz algebra $\mathfrak{h}$ be given. Our most important example of a hemi-semi-direct product is to choose $\mathfrak{gl}(\mathfrak{h})$ as the Lie algebra and $\mathfrak{h}$ as the module in the construction of the hemi-semi-direct product. Kinyon and Weinstein noticed that every Leibniz algebra may be embedded in this type of hemi-semi-direct product $\mathfrak{h} \times_{\mathrm{hs}} \mathfrak{gl}(\mathfrak{h})$. The embedding map is simply $X \mapsto (X, \mathrm{ad}_X)$. In other words, the given Leibniz algebra $\mathfrak{h}$ is seen as a subalgebra of the hemi-semi-direct product $\mathfrak{h} \times_{\mathrm{hs}} \mathfrak{gl}(\mathfrak{h})$ by regarding it as the graph of the adjoint representation

$$\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}), \quad X \mapsto \mathrm{ad}_X,$$

where for each $Y \in \mathfrak{h}$, $\mathrm{ad}_X(Y) := [X, Y]$.

One can change this example somehow by considering the Lie algebra of derivations $\mathrm{der}(\mathfrak{h})$ instead of the Lie algebra $\mathfrak{gl}(\mathfrak{h})$. Notice that the derivations $\mathrm{der}(\mathfrak{h})$ of a Leibniz algebra $\mathfrak{h}$ form indeed a Lie algebra. The following proposition is due to Kinyon-Weinstein *loc. cit.*:

Proposition 3.1. *Every Leibniz algebra $\mathfrak{h}$ is embedded as a subalgebra of the hemi-semi-direct product $\mathfrak{h} \times_{\mathrm{hs}} \mathrm{der}(\mathfrak{h})$.*

3.2 Bass' approach to integration

This approach builds on a remark by H. Bass in the Lie algebra case, referred to in [15], and is already contained in [18] (end of Section 3), but Kinyon believed this integration to be too arbitrary, as it does not necessarily yield Lie groups in the case of Lie algebras.

Let $\mathfrak{h}$ be a finite-dimensional real Leibniz algebra.

Theorem 3.2. *On the vector space $\mathfrak{h}$, there exists a Lie rack structure which is given by*

$$(X, Y) \mapsto \exp(\mathrm{ad}_X)(Y) =: X \rhd Y$$

for all $X, Y \in \mathfrak{h}$. This global Lie rack structure has the following properties:

1. *In case $\mathfrak{h}$ is a Lie algebra, the corresponding Lie rack structure is locally the conjugation rack structure w.r.t. to a Lie group structure.*

2. The Lie rack structure is globally described by a Baker-Campbell-Hausdorff-formula (BCH-formula for short).

Proof. Note that $X \mapsto \exp(\mathrm{ad}_X)$ is an automorphism of $\mathfrak{h}$. The fact that the binary operation

$$(X, Y) \mapsto X \rhd Y = \exp(\mathrm{ad}_X)(Y)$$

is a rack product follows from the formula

$$\alpha \exp(\mathrm{ad}_X)\alpha^{-1} = \exp(\mathrm{ad}_{\alpha(X)})$$

for any automorphism $\alpha \in \mathrm{Aut}(\mathfrak{h})$. Indeed, the RHS of self-distributivity

$$X \rhd (Y \rhd Z) = (X \rhd Y) \rhd (X \rhd Z)$$

reads thanks to this formula

$$e^{\mathrm{ad}_{e^{\mathrm{ad}_X(Y)}}}\left(e^{\mathrm{ad}_X(Z)}\right) = e^{\mathrm{ad}_X} \circ e^{\mathrm{ad}_Y} \circ e^{-\mathrm{ad}_X}\left(e^{\mathrm{ad}_X(Z)}\right) = e^{\mathrm{ad}_X}\left(e^{\mathrm{ad}_Y(Z)}\right).$$

The BCH-formula which is referrred to in the statement is, for a Lie algebra $\mathfrak{h}$, the formula for the conjugation conj_* associated to the BCH-product:

$$\begin{aligned} \mathrm{conj}_*(X, Y) &= \exp(\mathrm{ad}_X)(Y) \\ &= Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \frac{1}{6}[X, [X, [X, Y]]] + \dots \end{aligned}$$

Observe that while the BCH group product is in general only locally defined, its associated conjugation is always globally defined.

For a general Leibniz algebra, we interpret this same formula as a BCH-formula for the rack product. □

One drawback of this Lie rack structure is that the underlying space is necessarily contractible. This will be different with the following approach. Another drawback is that in the case of a Lie algebra, the space is only *locally* a Lie group, but not necessarily globally. We will not be able to overcome this drawback.

3.3 hs-approach to integration

The hs-approach (approach using **h**emi-**s**emi-direct products) can be seen as modeled on the proof of Lie's third Theorem using Ado's Theorem. Here we embed Leibniz algebras as subalgebras of hemi-semi-direct products (taking the place of general linear Lie algebras), integrate these to linear Lie racks and identify then the subrack associated to the given Leibniz algebra.

Consider a Leibniz algebra $\mathfrak{h}$. Then the hemi-semi-direct products $\mathfrak{h} \times_{\mathrm{hs}} \mathfrak{gl}(\mathfrak{h})$ and $\mathfrak{h} \times_{\mathrm{hs}} \mathrm{der}(\mathfrak{h})$ integrate into the linear Lie racks $\mathfrak{h} \times_{\mathrm{hs}} Gl(\mathfrak{h})$ and $\mathfrak{h} \times_{\mathrm{hs}} \mathrm{Aut}(\mathfrak{h})$ respectively.

Now it remains to identify the subrack $R_{\mathfrak{h}} \subset \mathfrak{h} \times_{\mathrm{hs}} \mathrm{Aut}(\mathfrak{h})$ associated to the Leibniz subalgebra $\{(X, \mathrm{ad}_X) : X \in \mathfrak{h}\}$ of the hemi-semi-direct product Leibniz algebra $\mathfrak{h} \times_{\mathrm{hs}} \mathrm{der}(\mathfrak{h})$.

Proposition 3.3. *The subrack $R_\mathfrak{h} \subset \mathfrak{h} \times_{\mathrm{hs}} \mathrm{Aut}(\mathfrak{h})$ is explicitely described as*

$$R_\mathfrak{h} = \{(X, \exp(\mathrm{ad}_X)) : X \in \mathfrak{h}\}.$$

It is a closed subset of the direct product of the vector space $\mathfrak{h}$ and the exponential image $\exp(\mathrm{ad}(\mathfrak{h}))$ of the adjoint image of $\mathfrak{h}$ in the Lie group $Gl(\mathfrak{h})$. It acquires a manifold structure on some open subset.

Proof. From the algebraic point of view, all we have to show is that the set

$$R_\mathfrak{h} := \{(X, \exp(\mathrm{ad}_X)) : X \in \mathfrak{h}\}$$

is a subrack of the hemi-semi-direct product rack $\mathfrak{h} \times_{\mathrm{hs}} \mathrm{Aut}(\mathfrak{h})$. This is clear in the first variable, and follows again from the formula

$$\alpha \exp(\mathrm{ad}_X)\alpha^{-1} = \exp(\mathrm{ad}_{\alpha(X)})$$

for any automorphism $\alpha \in \mathrm{Aut}(\mathfrak{h})$ in the second variable.

The fact that the exponential image contains an open set where it has a manifold structure follows from the fact that the vanishing of the derivative of the exponential function defines a closed subset C (even an analytic subset). Call the complementary $U := \mathrm{ad}_\mathfrak{h} \setminus C$. The restriction of exp to U is a local diffeomorphism and therefore an open map, thus $\exp(U) =: O$ is an open subset of $G \subset \mathrm{Aut}(\mathfrak{h})$, the Lie group generated by $\exp(\mathrm{ad}_\mathfrak{h})$. O therefore inherits a manifold structure.

Now $\{(X, \mathrm{ad}_X) : X \in \mathfrak{h}\}$ is a linear subspace of $\mathfrak{h} \times \mathrm{ad}_\mathfrak{h}$, and therefore $\{(X, \mathrm{ad}_X) : \mathrm{ad}_X \in U\}$ is a submanifold of $\mathfrak{h} \times \mathrm{ad}_\mathfrak{h}$, which is sent to a submanifold $R_\mathfrak{h} \cap O$ by the local diffeomorphism exp. This submanifold is the "global" object integrating our Leibniz algebra. $\square$

Remark 3.4. The manifold structure on the open set O is a manifold structure on some 1-neighborhood in $\mathrm{Aut}(\mathfrak{h})$, and seems therefore no gain with respect to Covez' integration procedure. But the 1-neighborhood in Covez' procedure is a neighborhood where exp is a diffeomorphism, while here our O is "almost dense": In case $\exp(\mathfrak{g})$ is a manifold, then O is a dense open subset, because O contains $\exp(\mathfrak{g}) \setminus V$ where V is the set of critical values of exp. V is a measure zero subset of empty interior by (some variant of) Sard's theorem.

In general, the image of the exponential map is neither open nor closed. For $Sl(2, \mathbb{C})$, it contains -1 as a non-interior point (see [14, p. 26]), so there is no way to have a dense open subset.

4 Formal group approaches to integration

The formal group approach to the integration of Lie algebras is explained in detail in Serre's book [23]. The main scheme is to pass from a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $U\mathfrak{g}$ which is a cocommutative Hopf algebra, to pass then to some kind of dual $U\mathfrak{g}^*$ in order to get a commutative Hopf algebra, and then to extract from it a formal group law. Another variant extracts from $U\mathfrak{g}^*$ an algebraic group by taking characters. The first person to think about formal group laws for algebras over an operad (and thus in particular for Leibniz algebras) was to our knowledge Benoit Fresse [16].

4.1 Bertram-Didry's approach

Here we sketch Manon Didry's approach to the integration of Leibniz algebras. She did her thesis (cf [13]) under supervision of Wolfgang Bertram.

The main idea of their approach is to examine the structure which one inherits from a Lie algebra structure on a vector space $\mathfrak{g}$ on the *iterated dual numbers* $T^n\mathfrak{g}$. Here $T^1\mathfrak{g} = \mathfrak{g}\otimes_{\mathbb{K}}\mathbb{K}[\epsilon]/(\epsilon^2)$, and then one iterates the construction. Denoting the different ϵ by ϵ_i, $i = 1,\ldots,n$, one obtains

$$T^n\mathfrak{g} = \mathfrak{g} \oplus \bigoplus_{\alpha \in I_n} \epsilon^\alpha \mathfrak{g} =: \mathfrak{g} \oplus G_n(\mathfrak{g}),$$

where I_n is the set of non-zero multi-indices of length n with values in $\{0,1\}$. For a Lie algebra $\mathfrak{g}$, Didry obtains in this way groups $G_n(\mathfrak{g})$ where the group product is polynomial and expressed in terms of iterated brackets. She describes these groups in terms of generators and relations.

Let us introduce some notation in order to state (one of) her theorem(s) in more precise terms. Let $\alpha \in I_n$. For an integer $m \in \{2,\ldots,|\alpha|\}$, the set $P^m(\alpha)$ denotes the set of increasing partitions in m subsets of the multi-index α with respect to the lexicographic ordering:

$$P^m(\alpha) = \{(\lambda^1,\ldots,\lambda^m) \in I_n^m \,|\, \alpha = \sum_{i=1}^m \lambda^i, \ \lambda^1 < \ldots < \lambda^m\}.$$

Theorem 4.1. *The (above defined) set $G_n(\mathfrak{g})$ carries a group structure given by*

$$\sum_{\alpha \in I_n} \epsilon^\alpha x_\alpha \cdot \sum_{\alpha \in I_n} \epsilon^\alpha y_\alpha = \sum_{\alpha \in I_n} \epsilon^\alpha (x \cdot y)_\alpha,$$

where

$$(x \cdot y)_\alpha = x_\alpha + y_\alpha + \sum_{m=2}^{|\alpha|} \sum_{\lambda \in P^m(\alpha)} [\ldots[[x_{\lambda^m}, y_{\lambda^1}], y_{\lambda^2}],\ldots, y_{\lambda^{m-1}}].$$

The unit element of this group is 0 and the inverse of an element $x = \sum_{\alpha \in I_n} \epsilon^\alpha x_\alpha$ is $x^{-1} = \sum_{\alpha \in I_n} \epsilon^\alpha (x^{-1})_\alpha$, where

$$(x^{-1})_\alpha = -x_\alpha + \sum_{m=2}^{|\alpha|} \sum_{\lambda \in P^m(\alpha)} (-1)^m [[x_{\lambda^m}, x_{\lambda^{m-1}}],\ldots, x_{\lambda^1}].$$

For a Leibniz algebra $\mathfrak{h}$, Didry still obtains polynomial groups (!) $G_n(\mathfrak{h})$. It does not seem clear how to extract from the $G_n(\mathfrak{h})$ the original Leibniz algebra $\mathfrak{h}$, while for a Lie algebra $\mathfrak{g}$, the usual tangent Lie algebra of the polynomial group $G_n(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$.

4.2 Mostovoy's approach

Finally, we also sketch Mostovoy's approach [21]. Mostovoy works in Loday-Pirashvili's category of linear maps. In [20] Loday and Pirashvili introduce the infinitesimal tensor product for the category of linear maps $\mathcal{LM}$. Objects in $\mathcal{LM}$

are linear maps $f : V \to W$, which are thought to be vertical with V upstairs and W downstairs. Morphisms $(\phi_1, \phi_0) : (f : V \to W) \to (f' : V' \to W')$ in $\mathcal{LM}$ are commutative squares

$$
\begin{array}{ccc}
V & \xrightarrow{\phi_1} & V' \\
\downarrow{f} & & \downarrow{f'} \\
W & \xrightarrow{\phi_0} & W'
\end{array}
$$

The category $\mathcal{LM}$ becomes a (strictly) symmetric tensor category with the *infinitesimal tensor product* $(f : V \to W) \otimes (f' : V' \to W')$ which is given by

$$
\begin{array}{c}
(V \otimes W') \oplus (W \otimes V') \\
\downarrow{f \otimes \mathrm{id}_{W'} + \mathrm{id}_W \otimes f'} \\
W \otimes W'.
\end{array}
$$

The unit object is obviously $0 : \{0\} \to k$.

Loday and Pirashvili exhibit algebraic objects in the tensor category $\mathcal{LM}$. For this, they use that the inclusion functor $W \mapsto (0 : \{0\} \to W)$ and the projection functor $(f : V \to W) \mapsto W$ are tensor functors which compose to the identity. This shows that for each algebraic structure in $\mathcal{LM}$, the downstairs object has the corresponding structure in the category of vector spaces. Using this principle, they show that in $\mathcal{LM}$:

- an associative algebra object $f : M \to A$ is the data of an associative algebra A, an A-bimodule M and a bimodule map $f : M \to A$,

- a Lie algebra object $f : M \to \mathfrak{g}$ is the data of a Lie algebra $\mathfrak{g}$, a (right) Lie module M and an equivariant map $f : M \to \mathfrak{g}$,

- a bialgebra object $f : M \to H$ is the data of a bialgebra H, an H-bimodule and H-bicomodule M such that left and right comodule maps are H-bimodule maps, and $f : M \to H$ is a bimodule map and a coderivation (!).

Loday and Pirashvili go on showing how to construct functors P (primitives) and U (universal enveloping algebra) associating to a Hopf algebra in $\mathcal{LM}$ a Lie algebra in $\mathcal{LM}$, and vice-versa. For a given Lie algebra $f : M \to \mathfrak{g}$, the enveloping algebra is $\phi : U\mathfrak{g} \otimes M \to U\mathfrak{g}$. Here the right $U\mathfrak{g}$-action on $U\mathfrak{g} \otimes M$ is induced by

$$(u \otimes m) \cdot x = ux \otimes m + u \otimes m \cdot x$$

for all $x \in \mathfrak{g}$, all $u \in U\mathfrak{g}$ and all $m \in M$, the left $U\mathfrak{g}$-action is given by multiplication on the first factor, and the map ϕ is induced by

$$1 \otimes m \mapsto f(m).$$

Leibniz algebras give rise to Lie algebra objects in $\mathcal{LM}$ by associating to $\mathfrak{g}$ the linear map $\pi : \mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$, i.e. the quotient map w.r.t. the ideal generated by the squares $[x, x]$ for $x \in \mathfrak{g}$ which leads to the quotient Lie algebra $\mathfrak{g}_{\mathrm{Lie}}$.

Mostovoy's idea is to integrate Leibniz algebras by looking at them as Lie algebra objects in $\mathcal{LM}$. As the formal integration procedure in Serre's book [23] works for any strict tensor category, he deduces that the integration problem is from this point of view mostly trivial.

Mostovoy defines a *formal group* in $\mathcal{LM}$ to be an object $\delta : V \to W$ together with a linear map

$$G \,:\, S\left((V \oplus V) \overset{\delta + \delta}{\to} (W \oplus W)\right) \to (V \overset{\delta}{\to} W)$$

such that the extension to a coalgebra morphism in $\mathcal{LM}$

$$S(V \overset{\delta}{\to} W) \otimes S(V \overset{\delta}{\to} W) \to S(V \overset{\delta}{\to} W)$$

is an associative algebra in $\mathcal{LM}$. Observe that there are no group objects in $\mathcal{LM}$. Mostovoy arrives at the following proposition:

Proposition 4.2. *The functor that assigns to a Lie algebra $(M \to \mathfrak{g})$ in $\mathcal{LM}$ the primitive part of the product in $U(M \to \mathfrak{g})$ is an equivalence of categories of Lie algebras in $\mathcal{LM}$ and of formal groups in $\mathcal{LM}$.*

Mostovoy then has a global interpretation of these formal group objects in terms of bundles over the Lie group G (which arises from integrating the Lie algebra $\mathfrak{g}$, i.e. the downstairs object of $M \to \mathfrak{g}$). This bundle $\xi \to G$ should have typical fiber M, an anchor map $p : \xi \to TG$ and a pair of actions. It would be interesting to establish links to Bertram-Didry's approach to integration and to the notion of rackoid of Laurent-Gengoux and Wagemann.

5 Deformation quantization of Leibniz algebras

The main idea here is that a (local) integration of a Lie or Leibniz bracket leads to a symplectic micromorphism which is readily quantizable by Fourier-Integral operators.

5.1 Symplectic micromorphisms

Let us recall the definition of a symplectic micromorphism (see [6], [7], [8], and [9] for more details) as well as some aspects of their quantization.

Definition 5.1. *A symplectic micromorphism $([L], \phi)$ from a symplectic micro-fold $[M, A]$ (i.e. a germ of a symplectic manifold around a Lagrangian subman-ifold $A \subset M$, called the core of the microfold) to a symplectic microfold $[N, B]$ is the data of a Lagrangian submanifold germ $[L]$ in $\overline{M} \times N$ around the graph $\mathrm{gr}(\phi)$ of a smooth map $\phi : A \to B$ such that the intersection $L \cap (A \times B) = \mathrm{gr}(\phi)$ is clean for a representative $L \in [L]$.*

The symplectic micromorphisms are the morphisms of a category, *the microsymplectic category*. We denote them by $([L], \phi) : [M, A] \to [N, B]$, and, when the symplectic microfold is $[T^*A, A]$, we simply write T^*A.

An important example of symplectic micromorphisms comes from cotangent lifts of smooth maps between manifolds. Namely, if $\phi : B \to A$ is a smooth map, then the conormal bundle $N^*\mathrm{gr}\phi$ of the graph of ϕ is a lagrangian submanifold

of $T^*(A \times B)$. Using the identification (Schwartz transform) between this last cotangent bundle and $\overline{T^*A} \times T^*B$, the conormal bundle to the graph yields a symplectic micromorphism, which we denote by $T^*\phi : T^*A \to T^*B$, by taking the germ of the resulting lagrangian submanifold

$$\left\{ \left((p_A, \phi(x_B)), ((T^*_{x_B}\phi)p_A, x_B) \right) : (p_A, x_B) \in \phi^*(T^*A) \right\}$$

around the graph of ϕ, and where (p_A, x_A) and (p_B, x_B) are the canonical coordinates on T^*A and T^*B respectively.

When the target and source symplectic microfold cores are *euclidean* (i.e. when $A = \mathbb{R}^k$ and $B = \mathbb{R}^l$ for some $k \geq 1$ and $l \geq 1$), a symplectic micromorphism from T^*A to T^*B can be associated with a family of formal Fourier Integral operators from $C^\infty(A)[[\hbar]]$ to $C^\infty(B)[[\hbar]]$ using the symplectic micromorphism generating function (see [9] for a general theory of symplectic micromorphism quantization).

Namely, as shown in [7], when the target and source symplectic microfold cores are euclidean, any symplectic micromorphism $([L], \phi)$ from T^*A to T^*B can be described by a generating function germ $[S_L] : \phi^*(T^*A) \to \mathbb{R}$ around the zero section of the pullback bundle $\phi^*(T^*A)$ as follows: There is a representative $L \in [L]$ such that

$$L = \left\{ \left((p_A, \frac{\partial S_L}{\partial p_A}(p_A, x_B)), (\frac{\partial S_L}{\partial x_B}(p_A, x_B), x_B) \right) : (p_A, x_B) \in W \right\},$$

where W is an appropriate neighborhood of the zero section in $\phi^*(T^*A)$. This generating function S_L is unique if one requires that it satisfies the property $S_L(0, x) = 0$. The geometric condition on the cleanness of the intersection in the definition above can be expressed in terms of the generating function as follows:

$$\frac{\partial S_L}{\partial p_A}(p_A, x_B) = \phi(x_B) \quad \text{and} \quad \frac{\partial S_L}{\partial x_B}(0, x_B) = 0. \tag{3}$$

In this light, one can see S_L as a deformation of the cotangent lift generating function, which is the first term of S_L in a Taylor expansion:

$$S_L(p_A, x_B) = \langle p_A, \phi(x_B) \rangle + \mathcal{O}(p_A^2).$$

Remark 5.2. *Conversely, any generating function germ $[S] : \phi^*(T^*A) \to \mathbb{R}$ satisfying conditions (3) defines uniquely a symplectic micromorphism $([L_S], \phi) : T^*A \to T^*B$.*

Now, using the generating function S_L of the symplectic micromorphism $([L], \phi)$ and a function germ $a : \phi^*(T^*A) \to \mathbb{R}$ around the zero section, one can construct a formal operator

$$\begin{array}{ccc} C^\infty(A)[[\hbar]] & \longrightarrow & C^\infty(B)[[\hbar]] \\ \psi & \mapsto & Q^a([L], \phi)\psi \end{array}$$

by taking the stationary phase expansion of the following oscillatory integral:

$$\int_{T^*A} \chi(p_A, x_A)\psi(x_A)a(p_A, x_B)e^{\frac{i}{\hbar}(S_L(p_A, x_B) - p_A x_A)} \frac{dx_A dp_A}{(2\pi\hbar)^n},$$

where χ is a cutoff function with compact support around the critical points of the phase $S_L(p_A, x_B) - p_A x_A$ (w.r.t. the integration variables) and with value 1 on this critical locus, which is nothing but the points in $\{(0, \phi(x_B)) : x_B \in B\}$. Since the critical locus is contained in the zero section, the asymptotic expansion does not depend on the cutoff functions and, hence, is well-defined. To simplify the notation, we will abuse it slightly, and write from now on:

$$(Q^a([L], \phi))\psi(x_B) = \int_{\mathbb{R}^k} \widehat{\psi}(p_A) a(p_A, x_B) e^{\frac{i}{\hbar} S_L(p_A, x_B)} \frac{dp_A}{(2\pi\hbar)^{k/2}},$$

to mean the asymptotic expansion above, and where $\widehat{\psi}$ is the asymptotic Fourier transform of ψ; namely,

$$\widehat{\psi}(p_A) = \int_{\mathbb{R}^k} \psi(x_A) e^{-\frac{i}{\hbar} p_A x_A} \frac{dx_A}{(2\pi\hbar)^{k/2}}.$$

5.2 Gutt star-product as the quantization of a symplectic micromorphism

Let us now apply the reasoning of the previous section to the quantization of the linear Poisson structure on the dual of a Lie algebra $\mathfrak{g}$. Consider first the integrating Lie group G. Taking the cotangent lift of the group operation $m : G \times G \to G$ yields a symplectic micromorphism

$$([T^*m], \triangle_{\mathfrak{g}^*}) : [T^*G, \mathfrak{g}^*] \otimes [T^*G, \mathfrak{g}^*] \to [T^*G, \mathfrak{g}^*],$$

where we take the core in the source and target symplectic microfolds to be not the cotangent bundle zero section G, but rather the fiber above the identity, i.e. the dual of the Lie algebra. Identifying $[T^*G, \mathfrak{g}^*]$ with $[T^*\mathfrak{g}^*, \mathfrak{g}^*]$ (which we will denote simply by $T^*\mathfrak{g}^*$) using the Lagrangian embedding germ

$$[T^*\mathfrak{g}^*, \mathfrak{g}^*] \to [T^*G, \mathfrak{g}^*], \quad (X, \xi) \mapsto (\exp(X), (T_1^* L_{\exp(X)})^{-1}\xi),$$

the Lagrangian germ $[T^*m]$ becomes the cotangent lift of the local group operation $BCH : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and $([T^*m], \triangle_{\mathfrak{g}^*})$ becomes a symplectic micromorphism from $T^*\mathfrak{g}^* \otimes T^*\mathfrak{g}^*$ to $T^*\mathfrak{g}^*$, whose underlying Lagrangian submanifold germ coincides with the multiplication of the local symplectic groupoid integrating the linear Poisson structure on $\mathfrak{g}^*$.

This local/formal symplectic groupoid is described in [5], where it is shown that T^*m can be described in term of the following generating function germ

$$S(X, Y, \xi) = \Big\langle \xi, BCH(X, Y) \Big\rangle$$

as follows:

$$T^*m = \left\{ \left((X, \frac{\partial S}{\partial X}), (Y, \frac{\partial S}{\partial Y}), (\frac{\partial S}{\partial \xi}, \xi) \right) : (X, Y, \xi) \in W \right\}$$

where W is an appropriate neighborhood of the zero section in $T^*\mathfrak{g}^* \oplus T^*\mathfrak{g}^*$.

Once the generating function of a symplectic micromorphism is computed, it is easy to obtain a family of (formal) FIOs quantizing it as explained in the

previous section. In the case at hand, we obtain the following family of formal operators

$$Q^a(T^*m) : \mathcal{C}^\infty(\mathfrak{g}^*)[[\epsilon]] \otimes \mathcal{C}^\infty(\mathfrak{g}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{g}^*)[[\epsilon]]$$

of the form (in the notation from the previous section and with $\epsilon := \frac{\hbar}{2i}$):

$$Q^a(T^*m)(f \otimes g)(\xi) = \int_{\mathfrak{g} \times \mathfrak{g}} \widehat{f}(X)\widehat{g}(Y)a(X,Y,\xi)e^{\frac{i}{\hbar}S(X,Y,\xi)}\frac{dXdY}{(2\pi\hbar)^n}, \quad (4)$$

where a is the germ of a smooth function on $T^*\mathfrak{g}^* \oplus T^*\mathfrak{g}^*$ around the zero section, called the *amplitude* of the FIO $Q^a(T^*m)$, and n is the dimension of $\mathfrak{g}$.

When $a = 1$ and S is the generating function of $([T^*m], \triangle_{\mathfrak{g}^*})$, we have that

$$f *_a g = Q^a(T^*m)(f \otimes g)$$

coincides with the Gutt star-product [1, 2, 17]. For other star-products in integral form on duals of Lie algebras as in (4), we refer the reader to the work of Ben Amar [1, 2].

Remark 5.3. *For a general amplitude a, $f *_a g$ is not necessarily associative.*

5.3   Deformation quantization of Leibniz algebras

Let $(\mathfrak{h}, [,])$ be a Leibniz algebra and $(R_\mathfrak{h}, \triangleright)$ its integrating Lie rack from Section 3. The idea is to proceed by analogy and to quantize the Lagrangian relation

$$T^*\triangleright : T^*R_\mathfrak{h} \times T^*R_\mathfrak{h} \to T^*R_\mathfrak{h}$$

as we did for the group operation in the case of a Lie algebra.

As we saw in the Lie case, it is better to consider the local model, i.e. the integrating rack

$$\triangleright : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}, \quad (X,Y) \mapsto e^{\mathrm{ad}_X}(Y) =: \mathrm{Ad}_X(Y)$$

defined on $\mathfrak{h}$. Denote by $T^*\mathfrak{h}^*$ the product $\mathfrak{h} \times \mathfrak{h}^*$. The symplectic form is then simply the canonical pairing. The first step is to take the cotangent lift of the rack operation and compute its generating function:

Proposition 5.4. *The cotangent lift of $\triangleright$ yields a symplectic micromorphism*

$$T^*\triangleright : T^*\mathfrak{h}^* \otimes T^*\mathfrak{h}^* \to T^*\mathfrak{h}^*$$

with generating function

$$S_\triangleright(X,Y,\xi) := \langle \xi, \mathrm{Ad}_X(Y)\rangle.$$

Proof. Consider the generating function

$$
\begin{aligned}
S_\triangleright(X,Y,\xi) &:= \langle \xi, \mathrm{Ad}_X(Y)\rangle \\
&= \langle \xi, Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \ldots \rangle
\end{aligned}
$$

We will denote the variables by $(X,Y) =: P$ and ξ, and write accordingly $S_\triangleright(X,Y,\xi) = S_\triangleright(P,\xi)$.

As shown in [7] Sections 3.1 and 3.2 (see also [5] Section 1.2), a generating function of the type

$$
\begin{aligned}
S_\triangleright(P, \xi) &= \langle \xi, Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \ldots \rangle \\
&= \langle \Phi(\xi), P \rangle + \mathcal{O}(P^2)
\end{aligned}
$$

where $\Phi : \mathfrak{h}^* \to \mathfrak{h}^* \times \mathfrak{h}^*$, $\Phi(\xi) = (0, \xi)$, yields a symplectic micromorphism

$$
([L_S], \Phi) : T^*\mathfrak{h}^* \otimes T^*\mathfrak{h}^* \to T^*\mathfrak{h}^*
$$

where

$$
\begin{aligned}
L_S &= \left\{ \left((X, \frac{\partial S_\triangleright}{\partial X}), (Y, \frac{\partial S_\triangleright}{\partial Y}), (\frac{\partial S_\triangleright}{\partial \xi}, \xi) \right) \mid \xi \in \mathfrak{h}^*, \ X, Y \in \mathfrak{h} \right\} \\
&= \{ ((X, \langle [X, Y], \xi \rangle), (Y, \mathrm{Ad}_X^*(\xi)), (\mathrm{Ad}_X(Y), \xi)) \mid \xi \in \mathfrak{h}^*, \ X, Y \in \mathfrak{h} \}
\end{aligned}
$$

which one recognizes to be the cotangent lift of the map $(X, Y) \mapsto \mathrm{Ad}_X(Y)$. $\quad\square$

We are now ready to quantize $T^*\triangleright : T^*\mathfrak{h} \otimes T^*\mathfrak{h} \to T^*\mathfrak{h}$. As before, the family of semi-classical FIO quantizing the symplectic micromorphism is given by

$$
Q^a(T^*\triangleright)(f \otimes g)(\xi) = \int_{\mathfrak{h} \times \mathfrak{h}} \widehat{f}(X) \widehat{g}(Y) a(X, Y, \xi) e^{\frac{i}{\hbar} S_\triangleright(X, Y, \xi)} \frac{dX\,dY}{(2\pi\hbar)^n},
$$

where a is the germ of an amplitude and $\widehat{f}$ and $\widehat{g}$ are the asymptotic Fourier transforms. Our main theorem in [12] reads now:

Theorem 5.5. *For $a = 1$, the operation*

$$
\triangleright_\hbar : \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \otimes \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]] \to \mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]
$$

defined by

$$
f \triangleright_\hbar g := Q^{a=1}(T^*\triangleright)(f \otimes g)
$$

is a quantum rack, i.e.

(1) $\triangleright_\hbar$ restricted to $U_\mathfrak{h} = \{ E_X := e^{\frac{i}{\hbar}\langle X, - \rangle} =: e^{\frac{i}{\hbar}X} \mid X \in \mathfrak{h} \}$ is a rack structure and

$$
e^{\frac{i}{\hbar}X} \triangleright_\hbar e^{\frac{i}{\hbar}Y} = e^{\frac{i}{\hbar}\mathrm{conj}_*(X, Y)},
$$

(2) $\triangleright_\hbar$ restricted to $\triangleright_\hbar : U_\mathfrak{h} \times \mathcal{C}^\infty(\mathfrak{h}^*) \to \mathcal{C}^\infty(\mathfrak{h}^*)$ is a rack action and

$$
(e^{\frac{i}{\hbar}X} \triangleright_\hbar f)(\xi) = (\mathrm{Ad}_{-X}^* f)(\xi).
$$

Moreover, $\triangleright_\hbar$ coincides with the Gutt quantum rack $f \triangleright_a g := f *_a g *_a \overline{f}$ on the restrictions in the Lie case (although it is different on the whole $\mathcal{C}^\infty(\mathfrak{h}^*)[[\epsilon]]$).

Remark 5.6. *Actually, Property (2) in the theorem above holds also for square integrable functions, and we even obtain a unitary rack action:*

$$
\triangleright_\hbar : U_\mathfrak{h} \times L^2(\mathfrak{h}^*) \to L^2(\mathfrak{h}^*).
$$

Proof of the theorem:

The first property follows from the fact that exponentials Fourier transform to delta functions:

$$
\begin{aligned}
\left(e^{\frac{i}{\hbar}\bar{X}} \rhd_\hbar e^{\frac{i}{\hbar}\bar{Y}}\right)(\xi) &= \int \widehat{e^{\frac{i}{\hbar}\bar{X}}}\, \widehat{e^{\frac{i}{\hbar}\bar{Y}}}\, e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX\,dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= (2\pi\hbar)^{\dim(\mathfrak{h})} \int \delta_{\bar{X}}(X)\delta_{\bar{Y}}(Y) e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX\,dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= e^{\frac{i}{\hbar}\langle \mathrm{Ad}_{\bar{X}}(\bar{Y}),\xi\rangle} = e^{\frac{i}{\hbar}\langle \mathrm{conj}_*(\bar{X},\bar{Y}),\xi\rangle}.
\end{aligned}
$$

Now $\rhd_\hbar$ satisfies the rack identity on $U_\mathfrak{h}$, because conj_* does. Furthermore,

$$
E_Y \mapsto E_X \rhd_\hbar E_Y = E_{\mathrm{conj}_*(X,Y)}
$$

is bijective for all $X \in \mathfrak{h}$, because $Y \mapsto \mathrm{conj}_*(X,Y)$ is. It is also clear from the formula above that this rack structure coincides with the Gutt rack structure in the case of a Lie algebra.

The second property also follows from the fact that exponentials Fourier-transform to delta functions:

$$
\begin{aligned}
\left(e^{\frac{i}{\hbar}\bar{X}} \rhd_\hbar f\right)(\xi) &= \int \widehat{e^{\frac{i}{\hbar}\bar{X}}}\, \widehat{f}(Y) e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX\,dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= (2\pi\hbar)^{(\dim(\mathfrak{h}))/2} \int \delta_{\bar{X}}(X)\widehat{f}(Y) e^{\frac{i}{\hbar}\langle \xi, \mathrm{Ad}_X(Y)\rangle} \frac{dX\,dY}{(2\pi\hbar)^{\dim(\mathfrak{h})}} \\
&= \frac{1}{(2\pi\hbar)^{(\dim(\mathfrak{h}))/2}} \int \widehat{f}(Y) e^{\frac{i}{\hbar}\langle \mathrm{Ad}^*_{-X}\xi,Y\rangle} dY \\
&= f(\mathrm{Ad}^*_{-X}\xi).
\end{aligned}
$$

One sees that this defines a rack action from the fact that the coadjoint action Ad^*_{-X} is a rack action. $\qquad\square$

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