Geometric Structure in the Representation Theory of Reductive p-adic Groups

Paul Baum
Penn State

University of Luxembourg

June 14, 2011

GEOMETRIC STRUCTURE IN THE REPRESENTATION THEORY OF REDUCTIVE P-ADIC GROUPS Let G be a reductive p-adic group. Examples are GL(n, F) SL(n, F) where n can be any positive integer and F can be any finite extension of the field Q_p of p-adic numbers. The smooth (or admissible) dual of G is the set of equivalence classes of smooth irreducible representations of G. The representations are on vector spaces over the complex numbers. The smooth dual has one point for each distinct smooth irreducible representation of G. Within the smooth dual there are subsets known as the Bernstein components, and the smooth dual is the disjoint union of the Bernstein components. This talk will explain a conjecture due to Aubert-Baum-Plymen (ABP) which says that each Bernstein component is a complex affine variety. These affine varieties are explicitly identified as certain extended quotients. The infinitesimal character of Bernstein and the L-packets which appear in the local Langlands conjecture are then described from this point of view. Recent results by a number of mathematicians (e.g. V. Heiermann, M. Solleveld) provide positive evidence for ABP.

Reference

The Hecke algebra of a reductive p-adic group: a geometric conjecture

by

Anne-Marie Aubert, Paul Baum. and Roger Plymen

in

book edited by Katia Consani and Matilde Marcolli based on meeting at MPI Bonn 2004

Title of book : **Non-Commutative Geometry and Number Theory**

Publisher: Vieweg Verlag (2006)

Reference

The Hecke algebra of a reductive p-adic group: a geometric conjecture

by

Anne-Marie Aubert, Paul Baum. and Roger Plymen

in

book edited by Katia Consani and Matilde Marcolli based on meeting at MPI Bonn 2004

Title of book : **Non-Commutative Geometry and Number Theory**

Publisher: Vieweg Verlag (2006)

Reference

Geometric structure in the representation theory of p-adic groups II

by

Anne-Marie Aubert, Paul Baum,
and Roger Plymen

Contemporary Mathematics
Proceedings of the AMS San Francisco Special Session on Representation Theory of Reductive p-adic Groups, eds. R. Doran, L. Spice

ABP Conjecture

ABP = Aubert-Baum-Plymen

The conjecture can be stated at four levels :

- K-theory
- Periodic cyclic homology
- Geometric equivalence of finite type algebras
- Representation theory

ABP Conjecture

ABP = Aubert-Baum-Plymen

The conjecture can be stated at four levels :

- K-theory
- Periodic cyclic homology
- Geometric equivalence of finite type algebras
- Representation theory $\longleftarrow$

Let G be a reductive p-adic group.

Examples are:

$\mathrm{GL}(n, F)$ $\mathrm{SL}(n, F)$

where F is any finite extension of the p-adic numbers $\mathbb{Q}_p$

Definition

A *representation* of G is a group homomorphism

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

The p-adic numbers $\mathbb{Q}_p$ in its natural topology is a locally compact and totally disconnected topological field. Hence G is a locally compact and totally disconnected topological group.

Definition

A representation

$$\phi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

of G is *smooth* if for every $v \in V$,

$$G_v = \{g \in G \mid \phi(g)v = v\}$$

is an open subgroup of G.

The smooth (or admissible) dual of G, denoted $\hat{G}$, is the set of equivalence classes of smooth irreducible representations of G.

$$\widehat{G} = \{\text{Smooth irreducible representations of } G\}/\sim$$

Problem: Describe $\widehat{G}$.

Since G is locally compact we may fix a (left-invariant) Haar measure dg for G.

The Hecke algebra of G, denoted $\mathcal{H}G$, is then the convolution algebra of all locally-constant compactly-supported complex-valued functions $f : G \to \mathbb{C}$.

$$\begin{aligned} (f+h)(g) &= f(g) + h(g) \\ (f*h)(g_0) &= \int_G f(g)h(g^{-1}g_0)dg \end{aligned} \quad \begin{cases} g \in G \\ g_0 \in G \\ f \in \mathcal{H}G \\ h \in \mathcal{H}G \end{cases}$$

Definition

A *representation* of the Hecke algebra $\mathcal{H}G$ is a homomorphism of $\mathbb{C}$ algebras

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

where V is a vector space over the complex numbers $\mathbb{C}$.

Definition

A representation

$$\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$$

of the Hecke algebra $\mathcal{H}G$ is *irreducible* if $\psi : \mathcal{H}G \to \mathrm{End}_{\mathbb{C}}(V)$ is not the zero map and $\nexists$ a vector subspace W of V such that W is preserved by the action of $\mathcal{H}G$ and $\{0\} \neq W \neq V$.

Definition

A *primitive ideal* I in $\mathcal{H}G$ is the null space of an irreducible representation of $\mathcal{H}G$.

Thus

$$0 \longrightarrow I \hookrightarrow \mathcal{H}G \xrightarrow{\psi} \mathrm{End}_{\mathbb{C}}(V)$$

is exact where ψ is an irreducible representation of $\mathcal{H}G$.

There is a (canonical) bijection of sets

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

where $\mathrm{Prim}(\mathcal{H}G)$ is the set of primitive ideals in $\mathcal{H}G$.

Bijection (of sets)

$$\widehat{G} \longleftrightarrow \mathrm{Prim}(\mathcal{H}G)$$

What has been gained from this bijection?

On $\mathrm{Prim}(\mathcal{H}G)$ have a topology — the Jacobson topology.

If S is a subset of $\mathrm{Prim}(\mathcal{H}G)$ then the closure $\overline{S}$ (in the Jacobson toplogy) of S is

$$\overline{S} = \{J \in \mathrm{Prim}(\mathcal{H}G) \mid J \supset \bigcap_{I \in S} I\}$$

$\mathrm{Prim}(\mathcal{H}G)$ (with the Jacobson topology) is the disjoint union of its connected components.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ denotes the set of connected components of $\mathrm{Prim}(\mathcal{H}G)$.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is a countable set and has no further structure.

$\pi_o\mathrm{Prim}(\mathcal{H}G)$ is also known as the *Bernstein spectrum* of G.

$\pi_o\mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$ where (M, σ) can be any cuspidal pair i.e.
M is a Levi factor of a parabolic subgroup P of G
and σ is an irreducible super-cuspidal representation of M.
$\sim$ is the conjugation action of G, combined with tensoring by unramified characters of M.

$\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M,\sigma)\}/\sim$ where (M,σ) can be any cuspidal pair i.e. M is a Levi factor of a parabolic subgroup P of G
and σ is an irreducible super-cuspidal representation of M.
$\sim$ is the conjugation action of G, combined with tensoring by unramified characters of M.
Thus $(M,\sigma) \sim (M',\sigma')$ iff there exists an unramified character $\psi\colon M \to \mathbb{C} - \{0\}$ of M and an element g of G, $g \in G$, with

$$g(M, \psi \otimes \sigma) = (M', \sigma')$$

The meaning of this equality is:

- $gMg^{-1} = M'$
- $g_*(\psi \otimes \sigma)$ and σ' are equivalent smooth irreducible representations of M'.

For each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$, X_α denotes the connected component of $\mathrm{Prim}(\mathcal{H}G)$.

The subsets X_α of $\widehat{G}$ are known as the Bernstein components of $\widehat{G}$. The problem of describing $\widehat{G}$ now breaks up into two problems.

Problem 1 Describe the Bernstein spectrum $\pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$.

Problem 2 For each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G) = \{(M, \sigma)\}/\sim$, describe the Bernstein component X_α.

Problem 1 involves describing the irreducible super-cuspidal representations of Levi subgroups of G. The basic conjecture on this issue is that if M is a reductive p-adic group (e.g. M is a Levi factor of a parabolic subgroup of G) then any irreducible super-cuspidal representation of M is obtained by smooth induction from an irreducible representation of a subgroup of M which is compact modulo the center of M. This basic conjecture is now known to be true to a very great extent.

For Problem 2 the ABP conjecture proposes that each Bernstein component X_α has a very simple geometric structure.

Notation

$\mathbb{C}^\times$ denotes the (complex) affine variety $\mathbb{C} - \{0\}$.

Definition

A *complex torus* is a (complex) affine variety T such that there exists an isomorphism of affine varieties

$$T \cong \mathbb{C}^\times \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times.$$

Bernstein assigns to each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$ a complex torus T_α and a finite group Γ_α acting on T_α.

He then forms the quotient variety T_α/Γ_α and proves that there is a surjective map π_α mapping X_α onto T_α/Γ_α .

$$\begin{array}{c} X_\alpha \\ \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha \end{array}$$

This map π_α is referred to as the *infinitesimal character* or the *central character*.

In Bernstein's work X_α is a set (i.e. is only a set) so π_α

$$\begin{array}{c} X_\alpha \\ \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha \end{array}$$

is a map of sets.

π_α is surjective, finite-to-one and generically one-to-one.

The extended quotient

Let Γ be a finite group acting on an affine variety X.

$$\Gamma \times X \longrightarrow X$$

The quotient variety X/Γ is obtained by collapsing each orbit to a point.

For $x \in X$, Γ_x denotes the stabilizer group of x.

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$$

$c(\Gamma_x)$ denotes the set of conjugacy classes of Γ_x.

The extended quotient is obtained by replacing the orbit of x by $c(\Gamma_x)$.

This is done as follows:

Set $\tilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

$\tilde{X} \subset \Gamma \times X$

$\tilde{X}$ is an affine variety and is a sub-variety of $\Gamma \times X$.

Γ acts on $\tilde{X}$.

$\Gamma \times \tilde{X} \to \tilde{X}$

$g(\gamma, x) = (g\gamma g^{-1}, gx)$

The extended quotient, denoted $X//\Gamma$, is $\tilde{X}/\Gamma$.

i.e. The extended quotient $X//\Gamma$ is the ordinary quotient for the action of Γ on $\tilde{X}$.

The extended quotient is an affine variety.

$\tilde{X} = \{(\gamma, x) \in \Gamma \times X \mid \gamma x = x\}$

The projection $\tilde{X} \to X$

$$(\gamma, x) \mapsto x$$

Passes to quotient spaces to give a map

$$\rho : X//\Gamma \to X/\Gamma$$

Conjecture

There is a certain resemblance between

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & X_\alpha \\ \rho_\alpha \Big\downarrow & \text{and} & \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

Conjecture

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & & X_\alpha \\ \rho_\alpha \Big\downarrow & \text{and} & \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & & T_\alpha/\Gamma_\alpha \end{array}$$

are almost the same.

How can this conjecture be made precise?

The precise conjecture consists of two statements.

Conjecture

#1. The infinitesimal character

$$\pi_\alpha : X_\alpha \to T_\alpha/\Gamma_\alpha$$

is one-to-one if and only if the action of Γ_α on T_α is free.

#2. There exists a bijection

$$\nu_\alpha : T_\alpha//\Gamma_\alpha \longleftrightarrow X_\alpha$$

with the following properties:

$\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$
Within the admissible dual $\widehat{G}$ have the tempered dual $\widehat{G}_{tempered}$.
$\widehat{G}_{tempered} = \{$smooth tempered irreducible representations of $G\}/\sim$
$\widehat{G}_{tempered} =$ Support of the Plancherel measure
$K_\alpha =$ maximal compact subgroup of T_α.
K_α is a compact torus. The action of Γ_α on T_α preserves the maximal compact subgroup K_α , so can form the compact orbifold $K_\alpha//\Gamma_\alpha$.

Conjecture : Properties of the bijection ν_α

- The bijection $\nu_\alpha : T_\alpha//\Gamma_\alpha \longleftrightarrow X_\alpha$ maps
  $K_\alpha//\Gamma_\alpha$ onto $X_\alpha \cap \widehat{G}_{tempered}$
  $K_\alpha//\Gamma_\alpha \longleftrightarrow X_\alpha \cap \widehat{G}_{tempered}$

Conjecture : Properties of the bijection ν_α

- For many α the diagram

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & \xrightarrow{\nu_\alpha} & X_\alpha \\ \rho_\alpha \Big\downarrow & & \Big\downarrow \pi_\alpha \\ T_\alpha/\Gamma_\alpha & \xrightarrow[I]{} & T_\alpha/\Gamma_\alpha \end{array}$$

does not commute.
I $=$ the identity map of T_α/Γ_α.

Conjecture : Properties of the bijection ν_α

- In the possibly non-commutative diagram

$$\begin{array}{ccc} T_\alpha//\Gamma_\alpha & \xrightarrow{\nu_\alpha} & X_\alpha \\ \downarrow{\scriptstyle \rho_\alpha} & & \downarrow{\scriptstyle \pi_\alpha} \\ T_\alpha/\Gamma_\alpha & \xrightarrow[I]{} & T_\alpha/\Gamma_\alpha \end{array}$$

the bijection $\nu_\alpha : T_\alpha//\Gamma_\alpha \longrightarrow X_\alpha$ is continuous where $T_\alpha//\Gamma_\alpha$ has the Zariski topology and X_α has the Jacobson topology
AND the composition

$$\pi_\alpha \circ \nu_\alpha : T_\alpha//\Gamma_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$$

is a morphism of algebraic varieties.

Conjecture : Properties of the bijection ν_α

- For each $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$ there is an algebraic family

$$\theta_t : T_\alpha//\Gamma_\alpha \longrightarrow T_\alpha/\Gamma_\alpha$$

of morphisms of algebraic varieties, with $t \in \mathbb{C}^\times$, such that

$$\theta_1 = \rho_\alpha \qquad \text{and} \quad \theta_{\sqrt{q}} = \pi_\alpha \circ \nu_\alpha$$

$\mathbb{C}^\times = \mathbb{C} - \{0\}$
q = order of the residue field of the p-adic field F over which G is defined
π_α = infinitesimal character of Bernstein

Conjecture : Properties of the bijection ν_α

- Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$ For each irreducible component $\mathbf{c} \subset T_\alpha // \Gamma_\alpha$ there is a cocharacter

$$h_c : \mathbb{C}^\times \longrightarrow T_\alpha$$

such that

$$\theta_t(x) = \lambda(h_c(t) \cdot x)$$

for all $x \in \mathbf{c}$.

cocharacter = homomorphism of algebraic groups $\mathbb{C}^\times \longrightarrow T_\alpha$
$\lambda : T_\alpha \longrightarrow T_\alpha / \Gamma_\alpha$ is the usual quotient map from T_α to T_α / Γ_α.

Question

Where are these correcting co-characters coming from?

Answer

The correcting co-characters are produced by the $SL(2,\mathbb{C})$ part of the Langlands parameters.

$$W \times SL(2,\mathbb{C}) \longrightarrow {}^{L}G$$

Example

$G = GL(2, F)$
F can be any finite extension of the p-adic numbers $\mathbb{Q}_p$.
q denotes the order of the residue field of F.
$X_\alpha = \{$ Smooth irreducible representations of $GL(2, F)$ having a non-zero Iwahori fixed vector$\}$

$$\begin{aligned} T_\alpha &= \{\text{unramified characters of the maximal torus of } GL(2,F)\} \\ &= \mathbb{C}^\times \times \mathbb{C}^\times \end{aligned}$$

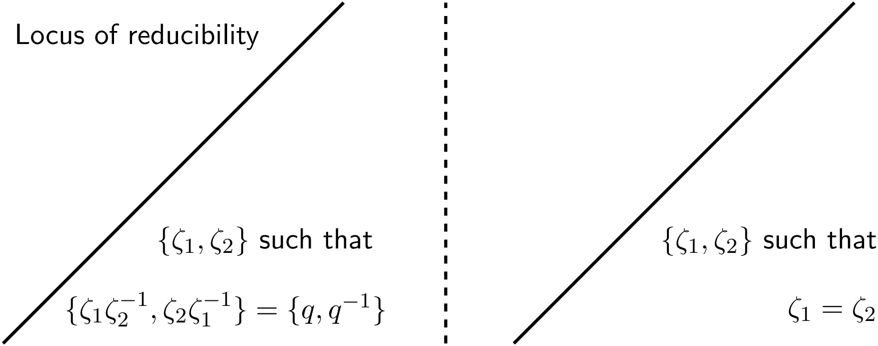
$$\Gamma_\alpha = \text{the Weyl group of } GL(2,F) = \mathbb{Z}/2\mathbb{Z}$$

$$0 \neq \gamma \in \mathbb{Z}/2\mathbb{Z} \qquad \gamma(\zeta_1, \zeta_2) = (\zeta_2, \zeta_1) \qquad (\zeta_1, \zeta_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\mathbb{C}^\times \times \mathbb{C}^\times // (\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}^\times \times \mathbb{C}^\times / (\mathbb{Z}/2\mathbb{Z}) \bigsqcup \mathbb{C}^\times$$

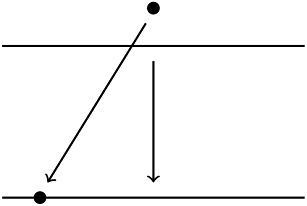
$$\mathbb{C}^\times \times \mathbb{C}^\times // (\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}^\times \times \mathbb{C}^\times / (\mathbb{Z}/2\mathbb{Z}) \bigsqcup \mathbb{C}^\times$$

$$\mathbb{C}^\times \times \mathbb{C}^\times / (\mathbb{Z}/2\mathbb{Z})$$

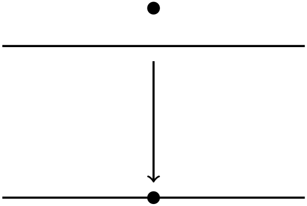


correcting cocharacter $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times$ is $t \mapsto (t, t^{-1})$

Infinitesimal character



Projection of the extended quotient on the ordinary quotient



QUESTION. In the ABP view of $\widehat{G}$, what are the L-packets?

CONJECTURAL ANSWER. Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$. In the list $h_1, h_2, \ldots, h_r$ of correcting cocharacters (one h_j for each irreducible component of the affine variety $T_\alpha // \Gamma_\alpha$) there may be repetitions — i.e. it may happen that for $i \neq j$, $h_i = h_j$. It is these repetitions that give rise to L-packets.

Fix $\alpha \in \pi_o \mathrm{Prim}(\mathcal{H}G)$. Let
$Z_1, Z_2, \ldots, Z_r$ be the irreducible components of the affine variety $T_\alpha /\!/ \Gamma_\alpha$.
Let $h_1, h_2, \ldots, h_r$ be the correcting cocharacters.
Let $\nu_\alpha : T_\alpha /\!/ \Gamma_\alpha \longrightarrow X_\alpha$ be the bijection of ABP.
CONJECTURE. Two points $[(\gamma, t)]$, $[(\gamma', t')]$ have

$$\nu_\alpha[(\gamma, t)] \text{ and } \nu_\alpha[(\gamma', t')] \text{ in the same L} - \text{packet}$$

if and only if

$$h_i = h_j \qquad \text{where } [(\gamma, t)] \in Z_i \text{ and } [(\gamma', t')] \in Z_j$$

and

$$\text{For all } \tau \in \mathbb{C}^\times, \;\; \theta_\tau[(\gamma, t)] = \theta_\tau[(\gamma', t')]$$

WARNING. An L-packet might have non-empty intersection with more than one Bernstein component. The conjecture does not address this issue. The statement of the conjecture begins

$$\text{Fix } \alpha \in \pi_o \text{Prim}(\mathcal{H}G).$$

So the conjecture assumes that a Bernstein component has been fixed — and then describes the intersections of L-packets with this Bernstein component.

Example

$G = SL(2, F)$
F can be any finite extension of the p-adic numbers $\mathbb{Q}_p$.
q denotes the order of the residue field of F.
$X_\alpha = \{$ Smooth irreducible representations of $GL(2, F)$ having a non-zero Iwahori fixed vector$\}$

$$T_\alpha = \{\text{unramified characters of the maximal torus of } SL(2,F)\} \\ = \mathbb{C}^\times$$

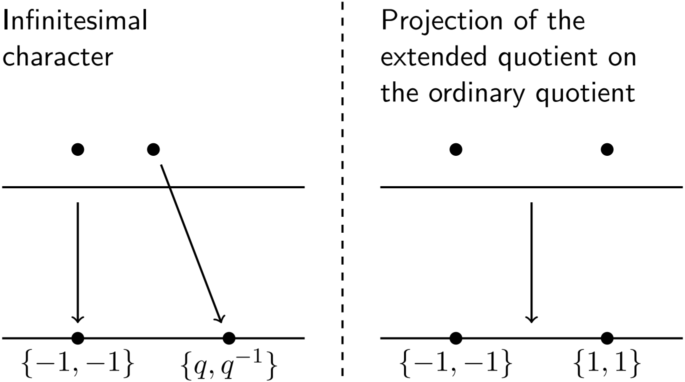
$\Gamma_\alpha =$ the Weyl group of $SL(2, F) = \mathbb{Z}/2\mathbb{Z}$

$0 \neq \gamma \in \mathbb{Z}/2\mathbb{Z} \qquad \gamma(\zeta) = \zeta^{-1} \qquad \zeta \in \mathbb{C}^\times$

$\mathbb{C}^\times//(\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}^\times/(\mathbb{Z}/2\mathbb{Z}) \bigsqcup \bullet \bigsqcup \bullet$

Infinitesimal character

Projection of the extended quotient on the ordinary quotient



Correcting cocharacter is $t \mapsto t^2$.

Preimage of $\{-1,-1\}$ is an L-packet.

Baum-Connes

$\updownarrow$

ABP

$\updownarrow$

Local Langlands

Theorem (V. Lafforgue)

Baum-Connes is valid for any reductive p-adic group G.

Theorem (Harris and Taylor)

Local Langlands is valid for GL(n, F).

Theorem (ABP)

ABP is valid for GL(n, F).