Intertwining integrals on completely solvable Lie groups

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Let G be a locally compact topological group and H a closed subgroup of G. We note Δ_G, Δ_H the modular functions of G, H and put

$$\Delta_{H,G}(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$$

for $h \in H$. We denote by $\mathcal{K}(G, H)$ the space of all $\mathbb{C}$-valued continuous functions ϕ on G with compact support modulo H from right and satisfying the covariance relation

$$\phi(gh) = \Delta_{H,G}(h)\phi(g) \quad (g \in G,\ h \in H).$$

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Now, let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. We denote by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. G acts on $\mathfrak{g}^*$ by the coadjoint action and it is well known that the unitary dual $\widehat{G}$ is realized by the orbit space $\mathfrak{g}^*/G$. This orbit method due to Kirillov, Bernat and Pukanszky provides us a powerful tool to study various questions concerning unitary representations of G.

Let $H = \exp \mathfrak{h}$ be an analytic subgroup of G and χ a unitary character of H. We consider the monomial representation $\tau = \mathrm{ind}_H^G \chi$ of G induced up from χ. In order to study τ, we employ the orbit method. There exists a linear form f in $\mathfrak{g}^*$ such that

$$\chi(\exp X) = e^{if(X)} \ (X \in \mathfrak{h},\ i = \sqrt{-1}).$$

Then we write χ as χ_f.

Here we have some questions on τ: irreducible decomposition, Penney's Plancherel formula, explicit intertwining operator between τ and its irreducible decomposition and the algebra of invariant differential operators associated with τ. Our field of research on these questions is the affine space $\Gamma = f + \mathfrak{h}^\perp$ of $\mathfrak{g}^*$.

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At $\ell \in \mathfrak{g}^*$ we take a (real) polarization $\mathfrak{b}$ satisfying the Pukanszky condition and construct the monomial representation $\pi_\ell = \mathrm{ind}_B^G \chi_\ell$, where $B = \exp \mathfrak{b}$. The Pukanszky condition means $B \cdot \ell = \ell + \mathfrak{b}^{\perp}$ or equivalently that π_ℓ is the irreducible unitary representation of G corresponding to the coadjoint orbit $G \cdot \ell$. We denote by $\mathcal{H}_\ell$ the Hilbert space of π_ℓ.

Now we are interested in a formal intertwining integral from π_ℓ to τ given by

$$R(\tau, \pi_\ell)(\varphi)(g) = \oint_{H/(H\cap B)} \varphi(gh)\chi_f(h)\Delta_{H,G}^{-1/2}(h)d\nu_{H,H\cap B}(h)$$

for $\varphi \in \mathcal{H}_\ell$. This intertwining integral or its associated generalized vector

$$a_\ell(\varphi) = \overline{R(\tau, \pi_\ell)(\varphi)(e)} = \oint_{H/(H\cap B)} \overline{\varphi(h)\chi_f(h)}\Delta_{H,G}^{-1/2}(h)d\nu_{H,H\cap B}(h),$$

e being the unit element of G, comes in various aspects of our questions.

It was first suggested by Michèle Vergne and proved by Gérard Lion when G was nilpotent, $\ell = f$ and $\mathfrak{h}$ was a polarization. In that case, the space $\mathcal{H}_\ell^\infty$ of C^∞-vectors of π_ℓ coincides with the Schwartz space of the rapidly decreasing functions and we have no problem to see the convergence of the integral on this space.

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Once we leave the nilpotent case, it becomes difficult to analyze this object. Is it possible to apply the linear form, namely the function in question does it satisfy the required covariance relation? If this is the case, what can we say about the convergence of the integral? Today from now on we assume that $\mathfrak{h}$ is a polarization at f satisfying the Pukanszky condition. Hence, $\tau \simeq \pi_\ell$ and translating the structure by an element of H, we also assume that $f = \ell$.

Let us make our situation more precise. Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, $f \in \mathfrak{g}^*$ and $\mathfrak{h}_1,\ \mathfrak{h}_2$ two polarizations at f satisfying the Pukanszky condition. Put $H_j = \exp \mathfrak{h}_j\ (j = 1, 2)$ and induce two monomial representations $\pi_1 = \mathrm{ind}_{H_1}^G \chi_f$ and $\pi_2 = \mathrm{ind}_{H_2}^G \chi_f$ of G. Thus, $\pi_1 \simeq \pi_2$. We denote by $\mathcal{H}_j$ the Hilbert space of π_j for $j = 1,\ 2$.

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First we have:

Lemma

For any $X \in \mathfrak{h}_1 \cap \mathfrak{h}_2$,

$$Tr\ ad_{\mathfrak{h}_1/(\mathfrak{h}_1 \cap \mathfrak{h}_2)} X + Tr\ ad_{\mathfrak{h}_2/(\mathfrak{h}_1 \cap \mathfrak{h}_2)} X = 0.$$

Let $\varphi \in \mathcal{H}_1$ and $g \in G$. Using the above trace relation, we easily see that the function

$$\Phi_g : H_2 \ni h \mapsto \varphi(gh)\chi_f(h)\Delta_{H_2,G}^{-1/2}(h)$$

satisfies the required covariance relation

$$\Phi_g(hx) = \Delta_{(H_1 \cap H_2),H_2}(x)\Phi(h) \ (h \in H_2, \ x \in H_1 \cap H_2).$$

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Therefore, we can consider the formal intertwining operator

$$T_{21}(\varphi)(g) = \oint_{H_2/(H_1 \cap H_2)} \varphi(gh)\chi_f(h)\Delta_{H_2,G}^{-1/2}(h) d\nu_{H_2,H_1\cap H_2}(h)$$

for $\varphi \in \mathcal{H}_1$. The main problem is the convergence of this integral.

It is easy to see that H_2H_1 is locally closed in G so that the homogeneous space $H_2/(H_1 \cap H_2)$ is homeomorphic to the quotient space H_2H_1/H_1. So, if the simple product H_2H_1 is closed in G, then our integral would be convergent for $\varphi \in \mathcal{H}_1$ having compact support modulo H_1. Once we believed to have proved this property, but unfortunately the proof was wrong.

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The Pukanszky condition gives us

$$H_2H_1 = \{g \in G; g \cdot (f + \mathfrak{h}_1^{\perp}) \cap (f + \mathfrak{h}_2^{\perp}) \neq \emptyset\},$$

but the behavior of the affine space $g \cdot (f + \mathfrak{h}_1^{\perp})$ is difficult to see.

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(5) $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is abelian,

(6) $\mathfrak{h}_1$ or $\mathfrak{h}_2$ is a Vergne polarization. Namely we take a sequence of subalgebras

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g},$$
$$\dim(\mathfrak{g}_j) = j \ (0 \leq j \leq n)$$

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such that if $\mathfrak{g}_j$ is not an ideal then $\mathfrak{g}_{j\pm 1}$ are ideals and the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}_{j+1}/\mathfrak{g}_{j-1}$ is irreducible. Put $\ell_j = \ell|_{\mathfrak{g}_j}$. Then $\sum_{j=1}^{n} \mathfrak{g}_j(\ell_j)$ gives a polarization at ℓ satisfying the Pukanszky condition. Polarizations constructed in such a fashion are called Vergne polarizations.

When we apply the orbit method to exponential solvable Lie groups, we usually employ the induction on the dimension of G. If there exists a non-trivial ideal $\mathfrak{a}$ of $\mathfrak{g}$ on which f vanishes, we go down to the quotient algebra $\mathfrak{g}/\mathfrak{a}$.

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If there is no such ideal, then the center of $\mathfrak{g}$ is at most one-dimensional and we take a minimal non-central ideal $\mathfrak{a}$ of $\mathfrak{g}$. Then,

$$\mathfrak{a}^f = \{X \in \mathfrak{g}; f([X, \mathfrak{a}]) = \{0\}\}$$

is a proper subalgebra of $\mathfrak{g}$ and we try to go down to $G_0 = \exp \mathfrak{a}^f$ by using the modification

$$\mathfrak{h}_j \to \mathfrak{h}'_j = (\mathfrak{h}_j \cap \mathfrak{a}^f) + \mathfrak{a} \ (1 \leq j \leq 2)$$

if $\mathfrak{h}_j$ is not contained in $\mathfrak{a}^f$.

If we proceed by this process to prove the closeness of $H_2 H_1$, there is no problem when at least one of $\mathfrak{h}_j$ $(1 \leq j \leq 2)$ is contained in $\mathfrak{a}^f$. Suppose $\mathfrak{h}_j \not\subset \mathfrak{a}^f$ for $1 \leq j \leq 2$.

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Three cases are left to be examined. We can treat two cases: either

$$\mathfrak{h}_1 \cap \mathfrak{h}_2 \not\subset \mathfrak{a}^f,$$

or

$$\mathfrak{h}_1 \cap \mathfrak{h}_2 \subset \mathfrak{a}^f$$

and

$$\mathfrak{a} \not\subset \mathfrak{h}_1^0 + \mathfrak{h}_2^0,$$

where $\mathfrak{h}_j^0 = \mathfrak{h}_j \cap \mathfrak{a}^f$.

We finally come to the last case where we are blocked:

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In many cases listed above where $H_2 H_1$ turns out to be closed, the Pukanszky condition allowed us to choose convenient coexponential basis. But actually I have no idea to treat the last case in this line.

Looking for some ideas on what happens, we calculate un example. we assume that $\mathfrak{g}$ is completely solvable. Let $\mathfrak{n}$ be a nilpotent ideal of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$.

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We do not enter into the details, but first we construct a certain sequence $\mathcal{S}$ of ideals of $\mathfrak{g}$:

$$\mathcal{S} : \{0\} = \mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \cdots \subset \mathfrak{n}_{m-1} \subset \mathfrak{n}_m = \mathfrak{n},$$
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For $1 \leq j \leq m$, we set $\mathfrak{m}_j = \mathfrak{n}_j + \mathfrak{h}_1, \ M_j = \exp \mathfrak{m}_j$ and

$$I^{\mathfrak{h}_2} = \{1 \leq j \leq m; \mathfrak{h}_2 \cap \mathfrak{m}_j \neq \mathfrak{h}_2 \cap \mathfrak{m}_{j-1}\}.$$

Now we propose to establish the following assertion:

Assertion Let $j \in I^{\mathfrak{h}_2}$. There exists X_j in $\mathfrak{h}_2 \cap \mathfrak{m}_j$ outside of $\mathfrak{h}_2 \cap \mathfrak{m}_{j-1}$ such that $[X_j, \mathfrak{n}_j] \subset \mathfrak{n}_{j-1}$.

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If we get this assertion, we can prove inductively on j that $H_2 H_1 \cap M_j$ is closed. However, we guess that this assertion is much stronger than the closeness of $H_2 H_1$. In any event, we get by this way the:

Theorem

Let $G = \exp \mathfrak{g}$ *be a completely solvable Lie group with Lie algebra* $\mathfrak{g}$, $f \in \mathfrak{g}^*$ *and* $\mathfrak{h}_j$ $(j = 1, 2)$ *two polarizations of* $\mathfrak{g}$ *at* f *satisfying the Pukanszky condition. Put* $H_j = \exp \mathfrak{h}_j$ *for* $j = 1, 2$. *Let* $\mathfrak{n}$ *be a nilpotent ideal of* $\mathfrak{g}$ *containing* $[\mathfrak{g}, \mathfrak{g}]$. *If one of the following three conditions is satisfied, then the simple product set* $H_2 H_1$ *is closed in* G*:*

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(1) $[\mathfrak{h}_1, \mathfrak{h}_1] \subset \mathfrak{h}_2$ *or* $[\mathfrak{h}_2, \mathfrak{h}_2] \subset \mathfrak{h}_1$*;*

(2) $[\mathfrak{h}_i, \mathfrak{h}_i] \subset \mathfrak{n}(f|_{\mathfrak{n}})$ *for* $i = 1$ *or* $i = 2$*, more precisely*

$$[(\mathfrak{h}_i + \mathfrak{n}) \cap \mathfrak{h}_j, \mathfrak{h}_j] \subset \mathfrak{n}^f + (\mathfrak{h}_1 \cap \mathfrak{h}_2)$$

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Finally, without the closeness of H_2H_1, what can we say on the convergence of our integral? For instance, we recently got almost everywhere on Γ the convergence of our integral $R(\tau, \pi_\ell)(\varphi)$ for exponential groups where $\varphi \in \mathcal{H}_\ell$ has compact support modulo B, $\mathfrak{b}$ is a Vergne polarization and $\tau = \mathrm{ind}_H^G \chi_f$ has multiplicities of discrete type.

In the preceding situation, we put

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Then, concerning the multiplicity $m(\pi)$, do we have μ-almost everywhere the equality

$$m(\pi) = \dim((\mathcal{H}_\ell^{-\infty})^{H,\chi_f \Delta_{H,G}^{1/2}})?$$

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Moreover, we denote by $D_\tau(G/H)$ the algebra of G-invariant differential operators on the line bundle over G/H associated with the data (H, χ_f). When τ has finite multiplicities, a_ℓ turns out to be an eigen-distribution for all the elements of $D_\tau(G/H)$. I believe that this fact is useful in order to attack the

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Polynomial conjecture Assume that G is nilpotent and τ has finite multiplicities. The algebra $D_\tau(G/H)$ is isomorphic to the algebra $\mathbb{C}[\Gamma]^H$ of H-invariant polynomial functions on Γ.

We can also consider a polynomial conjecture about the restrictions $\pi|_K$ of irreducible unitary representation π of G on an analytic subgroup $K = \exp \mathfrak{k}$ of G. We denote by Ω the coadjoint G-orbit corresponding to π and by $\mathcal{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

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Let

$$\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}} = \{X \in \mathcal{U}(\mathfrak{g}); [\mathfrak{k}, X] \subset \ker(\pi)\}$$

and $D_\pi(G)^K = \pi(\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}})$.

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and $D_\pi(G)^K = \pi(\mathcal{U}_\pi(\mathfrak{g})^{\mathfrak{k}})$.

Polynomial conjecture Assume that $\pi|_K$ has finite multiplicities in its canonical central decomposition. The algebra $\mathcal{D}_\pi(G)^K$ is isomorphic to the algebra $\mathbb{C}[\Omega]^K$ of K-invariant polynomial functions on Ω.

In conclusion, there are still many problems to be studied.

THANK YOU!