Affine hyperspheres associated to special para-Kähler manifolds

Vicente Cortés\textsuperscript{1,a}, Marie-Amélie Lawn\textsuperscript{1,2,b} and Lars Schäfer\textsuperscript{1,2,c}

\textsuperscript{1} Institut Élie Cartan de Mathématiques, Université Henri Poincaré - Nancy 1, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France

\textsuperscript{2} Mathematisches Institut der Universität Bonn, Beringstraße 1, D-53115 Bonn, Germany

cortes@iecn.u-nancy.fr\textsuperscript{a}, lawn@iecn.u-nancy.fr\textsuperscript{b}, schafer@iecn.u-nancy.fr\textsuperscript{c}

September 5, 2005

Dedicated to Dmitri Vladimirovich Alekseevsky on the occasion of his sixty-fifth birthday.

Abstract

We prove that any special para-Kähler manifold is intrinsically an improper affine hypersphere. As a corollary, any para-holomorphic function $F$ of $n$ para-complex variables satisfying a non-degeneracy condition defines an improper affine hypersphere, which is the graph of a real function $f$ of $2n$ variables. We give an explicit formula for the function $f$ in terms of the para-holomorphic function $F$. Necessary and sufficient conditions for an affine hypersphere to admit the structure of a special para-Kähler manifold are given. Finally, it is shown that conical special para-Kähler manifolds are foliated by proper affine hyperspheres of constant mean curvature.

Research of the third author was supported by a grant of the ‘Studienstiftung des deutschen Volkes’.
1 Introduction

The notion of a special Kähler manifold was introduced by the physicists Bernard de Wit and Antoine Van Proeyen [DWVP] in the context of supersymmetry. In this framework, (affine) special Kähler geometry is precisely the target geometry for the scalar fields of $N = 2$ vector multiplets on four-dimensional Minkowski space. This geometry is related to a number of interesting mathematical themes ranging from moduli spaces to integrable systems, see [C2] for a survey. An intrinsic definition of special Kähler geometry was proposed by Dan Freed [F]:

Definition 1. A special Kähler manifold $(M, J, g, \nabla)$ is a Kähler manifold $(M, J, g)$ endowed with a flat torsion-free connection $\nabla$ such that $d\nabla J = 0$ and $\nabla \omega = 0$, where $\omega$ is the Kähler form.

That definition is equivalent to the original physical definition in terms of the so-called holomorphic prepotential, as follows from [ACD]. A related notion is that of a projective special Kähler manifold. Projective special Kähler manifolds arise as quotients of conical special Kähler manifolds by a holomorphic $\mathbb{C}^*$-action [ACD] and are the targets of $N = 2$ supergravity theories with vector multiplets as matter content in four dimensions, see [DWVP].

The systematic study of the special geometry of Euclidian (instead of Minkowskian) supersymmetry was recently started in [CMMS]. In that paper the allowed target manifolds for the scalar fields of vector multiplets on four-dimensional Euclidian space were determined. These manifolds were called special para-Kähler manifolds, since the main difference with the Minkowskian case is that the complex structure $J$ ($J^2 = -\text{Id}$) of the target manifold is replaced by a para-complex structure $J$ ($J^2 = \text{Id}$).

There is a close relation between special Kähler geometry and affine differential geometry, which was developed in a series of joint papers with Oliver Baues, which started with [BC1]. The central ingredient from affine differential geometry is the notion of an affine
hypersphere. These are hypersurfaces in affine space, governed by a certain non-linear partial differential equation of Monge-Ampére type, see the next section for a precise definition. The main result of [BC1] is that any simply connected special Kähler manifold admits a canonical realisation as an improper affine hypersphere. An explicit representation for the hypersphere in terms of the holomorphic prepotential was given in [C2]. Projective special Kähler manifolds are also related to affine hyperspheres. In fact, they admit a natural circle bundle which is intrinsically a proper affine hypersphere and is endowed with a compatible Sasakian structure [BC2]. This circle bundle is realized as the leaf of mean curvature 1 in a foliation of the underlying conical special Kähler manifold by proper affine hyperspheres of different positive constant mean curvature.

In this paper we show that the above results about special Kähler manifolds and affine hyperspheres carry over to special para-Kähler manifolds. We realise any simply connected special para-Kähler manifold as an improper affine hypersphere (theorem 3), and characterize the affine hyperspheres which arise by this construction (theorem 4). As an application, we prove that any para-holomorphic function $F$ of $n$ para-complex variables satisfying a non-degeneracy condition defines an improper affine hypersphere (corollary 3). The hypersphere is presented as the graph of a real-valued function $f$ for which an explicit formula is given (theorem 5). Finally, we show that conical special para-Kähler manifolds are foliated by proper affine hyperspheres of constant affine and pseudo-Riemannian mean curvature (theorem 6).

2 Affine geometry

In this section we recall several notions of affine differential geometry and some results, which are used in the later work. For more details we refer to [NS].

In the following discussion $\mathbb{R}^{n+1}$ is considered as affine space with its standard flat connection $\tilde{\nabla}$ and its standard volume form $\det$. Affine transformations preserving $\det$ are called equiaffine. We consider immersions of $n$-dimensional manifolds $M$ into $\mathbb{R}^{n+1}$ with $n > 1$ and such that $M$ admits a transversal vector field $\xi$. The transversal field $\xi$ induces on $M$ the volume form $\nu = \nu^\xi = \det(\xi, \ldots)$, a torsion-free connection $\nabla$, a quadratic covariant tensor field $g$, an endomorphism field $A$ (called the shape tensor) and a one-form $\theta$, such that we have the equations

$$\tilde{\nabla}_XY = \nabla_XY + g(X,Y)\xi, \quad (2.1)$$
$$\tilde{\nabla}_X\xi = -AX + \theta(X)\xi, \quad (2.2)$$

for all tangent vector fields $X$ and $Y$. The data $(\nabla, g, A, \theta) = (\nabla^\xi, g^\xi, A^\xi, \theta^\xi)$ are called the Gauß-Weingarten data induced by the transversal vector field $\xi$. We now suppose, that $g$ is non-degenerate (this is independent of the chosen transversal field $\xi$) and consequently defines a pseudo-Riemannian metric on $M$. In this case there exists a canonical choice of the transversal field $\xi$, called the affine normal, determined up to sign by the conditions that $\nu^\xi$ is the metric volume $\text{vol}^g$ of $g$ and that $\nabla^\xi\nu^\xi = 0$. Such an immersion $\varphi$ with affine normal $\xi$ is called a Blaschke immersion, $g$ is called the Blaschke metric and $\nabla$ the induced connection.
The pair \((\nabla, g)\) induced by the immersion obeys some integrability constraints. Starting with an abstract pair \((\nabla, g)\) satisfying these constraints on a simply connected manifold \(M\), there exists, due to the fundamental theorem of affine differential geometry, a Blaschke immersion \(\varphi : M \to \mathbb{R}^{n+1}\) inducing \((\nabla, g)\).

In order to formulate the fundamental theorem, we need the notion of \(g\)-conjugate connection \(\nabla\) defined by the equation

\[
Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \text{with} \quad X, Y, Z \in \Gamma(TM).
\]  

(2.3)

**Theorem 1** (cf. [DNV], [NS] p. 21 and p. 75) Let \(M\) be a simply connected manifold with a torsion-free connection \(\nabla\) and a pseudo-Riemannian metric \(g\). Then there exists a Blaschke-immersion \(\varphi : M \to \mathbb{R}^{n+1}\), unique up to equiaffine transformations, with Blaschke metric \(g\) and induced connection \(\nabla\) if and only if the \(g\)-conjugate connection \(\nabla\) is torsion-free, projectively flat, and if the metric volume form \(\nu\) is \(\nabla\)-parallel.

Blaschke immersions with a shape tensor satisfying the equation

\[
A = \lambda Id, \quad \text{for some } \lambda \in \mathbb{R},
\]  

(2.4)

are called **affine hyperspheres**. If \(\lambda \neq 0\) the affine hypersphere is called **proper**. For \(\lambda = 0\) the affine hypersphere is called **parabolic** or **improper**. Parabolic hyperspheres can be characterized by

**Proposition 1** (see [NS] p. 42) A Blaschke immersion \(\varphi : M \to \mathbb{R}^{n+1}\) is a parabolic affine hypersphere if and only if the induced connection \(\nabla\) is flat.

We remark that for a Blaschke immersion with affine normal \(\xi\) the one-form \(\theta^\xi\) vanishes. Hence, one can compute \(A\) from the data \((g, \nabla)\) by means of the Gauß equation for the curvature, which shows that

\[
\tilde{\nabla}\xi = 0 \iff A = 0 \iff \nabla \text{ is flat}.
\]  

(2.5)

### 3 Special para-Kähler manifolds

Next we review the needed background material on special para-Kähler manifolds, see [CMMS].

**Definition 2** A **para-complex structure** on a real finite dimensional vector space \(V\) is an endomorphism \(J \in \text{End}(V)\), such that \(J^2 = Id\) and the two eigenspaces \(V^\pm := \ker(\text{Id} \mp J)\) of \(J\) have the same dimension. An **almost para-complex structure** on a smooth manifold \(M\) is an endomorphism field \(J \in \Gamma(\text{End}(TM))\) such that, for all \(p \in M\), \(J_p\) is a para-complex structure on \(T_p M\). It is called **integrable** if the distributions \(T^\pm M = \ker(\text{Id} \mp J)\) are integrable. An integrable almost para-complex structure on \(M\) is called a **para-complex structure** and a manifold \(M\) endowed with a para-complex structure is called a **para-complex manifold**.
Analogous to the complex case, the Nijenhuis tensor \( N_J \) of an almost para-complex structure \( J \) is defined by
\[
N_J(X, Y) := [X, Y] + [JX, JY] - J[X, JY] - J[JX, Y],
\] (3.1)
for all vector fields \( X \) and \( Y \) on \( M \). As shown in [CMMS] we have the

**Proposition 2** An almost para-complex structure \( J \) is integrable if and only if \( N_J = 0 \).

In the following we denote by \( C \) the real algebra of para-complex numbers, which is generated by \( 1 \) and the para-complex unit \( e \) with \( e^2 = 1 \). For all \( z = x + ey \in C \) we have the para-complex conjugation \( \bar{\cdot} : C \to C, x + ey \mapsto x - ey \), and the real and imaginary parts of \( z \)
\[
\Re(z) := \frac{z + \bar{z}}{2} = x, \quad \Im(z) := \frac{z - \bar{z}}{2} = y.
\]
The free \( C \)-module \( C^n \) is a para-complex vector space, where the para-complex structure is just the multiplication with \( e \). The para-complex conjugation extends componentwise to \( \bar{\cdot} : C^n \to C^n, v \mapsto \bar{v} \). The para-complex dimension of a para-complex manifold \( M \) is the integer \( n = \dim_C M = \frac{\dim M}{2} \).

**Definition 3** Let \( (M, J_M) \) and \( (N, J_N) \) be para-complex manifolds. A smooth map \( \varphi : (M, J_M) \to (N, J_N) \) is called para-holomorphic if \( d\varphi J_M = J_N d\varphi \). A para-holomorphic map \( f : (M, J) \to C \) is called a para-holomorphic function. A system of local para-holomorphic coordinates is a system of para-holomorphic functions \( z^i, i = 1, \ldots, n \), defined on an open subset \( U \subset M \) of a para-complex manifold such that
\[
(x^1 = \Re(z^1), \ldots, x^n = \Re(z^n), y^1 = \Im(z^1), \ldots, y^n = \Im(z^n))
\]
is a system of real local coordinates.

Now we consider a (pseudo-Euclidean) scalar product \( g \) on a para-complex vector space \( (V, J) \). It is called para-hermitian if \( J \) is an anti-isometry for \( g \):
\[
J^* g := g(J\cdot, \cdot) = -g.
\]
The pair \((J, g)\) is called a para-hermitian structure and \((V, J, g)\) a para-hermitian vector space. This leads to the

**Definition 4** A para-hermitian manifold \( (M, J, g) \) is a para-complex manifold \( (M, J) \) endowed with a pseudo-Riemannian metric \( g \) such that \( J^* g = -g \). The two-form \( \omega := g(J\cdot, \cdot) = -g(\cdot, J\cdot) \) is called the fundamental two-form of \( (M, J, g) \). A para-Kähler manifold \((M, J, g)\) is a para-hermitian manifold \((M, J, g)\) such that \( J \) is parallel with respect to the Levi-Civita-connection \( D \) of \( g \), i.e \( DJ = 0 \). The fundamental two-form \( \omega \) will then be called the para-Kähler form.

We remark that \( DJ = 0 \) implies that the Nijenhuis tensor \( N_J \) vanishes and \( d\omega = 0 \), so that any para-Kähler manifold is a para-hermitian manifold with closed fundamental two-form. The converse statement is also true, see [CMMS] Theorem 1.
Definition 5 A special para-complex manifold \((M, J, \nabla)\) is a para-complex manifold \((M, J)\) endowed with a torsion-free flat connection \(\nabla\) such that \(\nabla J\) is a symmetric \((1, 2)\)-tensor-field, i.e. \(d^\nabla J = 0\). If, in addition, \((M, J)\) is endowed with a para-Kähler metric \(g\) such that the para-Kähler form is \(\nabla\)-parallel, then \((M, J, g, \nabla)\) is called a special para-Kähler manifold.

Given a connection \(\nabla\) on a manifold \(M\), we define for \(J \in \Gamma(\text{End}(T M))\) the connection
\[
\nabla^J X := J \nabla(J^{-1}X) = \nabla X + J(\nabla J^{-1})X, \quad X \in \Gamma(TM).
\]

As proven in [S],

Proposition 3 Let \((M, J, g)\) be a special para-Kähler manifold. Then \(\nabla^J\) is \(g\)-conjugate, i.e.
\[
Xg(Y, Z) = g(\nabla^J_X Y, Z) + g(Y, \nabla_X Z),
\]
for all \(X, Y, Z \in \Gamma(TM)\) and so is equal to the conjugate connection \(\nabla\).

Proof: The proof only uses the fact that \(\omega\) is \(\nabla\)-parallel and \(J\)-anti-invariant. With \(X, Y, Z \in \Gamma(TM)\) we compute
\[
Xg(Y, Z) = X\omega(JY, Z) = \omega(\nabla X(JY), Z) + \omega(JY, \nabla_X Z)
\]
\[
= -\omega(J\nabla_X(JY), JZ) + g(Y, \nabla_X Z)
\]
\[
= \omega(JZ, J\nabla_X(JY)) + g(Y, \nabla_X Z)
\]
\[
= g(Z, J\nabla_X(JY)) + g(Y, \nabla_X Z)
\]
\[
= g(\nabla^J_X Y, Z) + g(Y, \nabla_X Z).
\]

Proposition 4 [S] Let \((M, J)\) be a para-complex manifold with torsion-free flat connection \(\nabla\). Then \((M, J, \nabla)\) is a special para-complex manifold if and only if the conjugate flat connection \(\nabla^J\) is torsion-free.

Now let \(M\) be a para-complex manifold and \((z^1, \ldots, z^n)\) local para-holomorphic coordinates on \(U \subset M\). The cotangent bundle \(T^*M\) admits a canonical non-degenerate exact two-form \(\Omega\), which is of type \((2, 0)\) and para-holomorphic with respect to the natural para-complex structure on \(T^*M\). In fact, any point of \(T^*_p M, p \in M\), is of the form \(\sum w_i dz^i\big|_p\) and \(z^i, w_i\) can be considered as locally defined para-holomorphic functions on \(T^*M\). They define a local system of para-holomorphic coordinates \((z^i, w_j)\) on \(T^*_p M\big|_U\) and \(\sum w_i dz^i\) is the local expression of a globally defined para-holomorphic one-form on \(T^*M\), which does not depend on the choice of coordinates \((z^i)\). The two-form \(\Omega\) is given in these coordinates by
\[
\Omega = \sum dz^i \wedge dw_i = -d(\sum w_i dz^i).
\]

It is called the symplectic form of \(T^*M\). Obviously \(\Omega\) does not depend of the choice of coordinates.
We consider the para-holomorphic vector space \( V = T^*C^n \cong C^{2n} \) endowed with its standard para-complex structure \( J_V \), its para-complex conjugation and its symplectic form \( \Omega \) defined as above. We choose the standard linear para-holomorphic coordinates \( z_i \) on \( C^n \), which are real-valued on \( \mathbb{R}^n \subset C^n \). The corresponding para-holomorphic coordinates \( (z^i, w_j) \) are then linear and real-valued on \( T^*\mathbb{R}^n \subset V \). Let us define
\[
\gamma = \gamma_V = e^{\Omega(\cdot, \overline{\cdot})},
\]
which is a para-hermitian sesquilinear form, which means that it is \( C \)-sesquilinear and satisfies \( \gamma(w, v) = \overline{\gamma(v, w)} \) for all \( v, w \in V \). In addition, it is non-degenerate. Consequently, it induces a para-hermitian scalar product \( g_V \) on \( V \), given by
\[
g_V(v, w) := \Re(\gamma(v, w)), \quad v, w \in V.
\]
The fundamental two-form
\[
\omega_V(v, w) := g_V(J_V v, w) = \Im(\gamma(v, w)), \quad v, w \in V,
\]
is closed, so that \( (V, J_V, g_V) \) becomes a flat para-Kähler manifold with para-Kähler form \( \omega_V \).

In the following let \( M \) be a connected para-complex manifold of para-complex dimension \( n \).

**Definition 6** A para-holomorphic immersion \( \varphi : M \to V \) is called para-Kählerian if \( g := \varphi^*g_V \) is non-degenerate and Lagrangian if \( \varphi^*\Omega = 0 \).

and we recall [CMMS] the

**Proposition 5** Let \( \varphi : M \to V \) be a para-Kählerian immersion. Then it induces on \( M \) the structure of a para-Kählerian manifold \( (M, J, g) \) with para-Kähler-form \( \omega = g(J\cdot, \cdot) \). Moreover, if \( \varphi \) is also Lagrangian, then
\[
\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i,
\]
where \( \tilde{x}^i = \Re(\varphi^*z^i) \) and \( \tilde{y}_i = \Re(\varphi^*w_i) \).

This proposition implies that such an immersion defines for all \( p \in M \) a system of local coordinates \((\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}^1, \ldots, \tilde{y}^n)\) on some open neighborhood \( U \subset M \) of \( p \) and so a flat torsion-free connection on \( U \). These local connections fit together to a connection \( \nabla \) on \( M \):

**Corollary 1** Let \( \varphi : M \to V \) be a para-Kählerian Lagrangian immersion. Then there exists a canonical flat torsion-free connection \( \nabla \) on \( M \) characterized by the condition \( \nabla(\Re(\varphi^*df)) = 0 \) for all para-complex affine functions \( f \) on \( V \).

Moreover, in [CMMS] the following theorem was proven
Theorem 2 Any para-Kählerian Lagrangian immersion $\varphi : M \to V$ induces on $M$ the structure of a special para-Kähler manifold $(M, J, g, \nabla)$. Conversely, if $(M, J, g, \nabla)$ is a simply connected special para-Kähler manifold, it admits a para-Kählerian Lagrangian immersion $\varphi : M \to V$ inducing the geometric data $(J, g, \nabla)$ and which is unique up to an affine transformation of $V$ with linear part in $\text{Sp}(\mathbb{R}^{2n})$.

We now consider a para-holomorphic function $F : U \to C$ where $U \subset C^n$ is an open subset. Its para-holomorphic Hessian is defined by

$$\partial^2 F := \left( \frac{\partial^2 F}{\partial z^i \partial z^j} \right).$$

$F$ is called non-degenerate if the real matrix $\Im(\partial^2 F)$ is invertible. Theorem 2 now leads to the

Corollary 2 Let $F : U \to C$ be a non-degenerate para-holomorphic function on an open subset $U \subset C^n$. Then $\varphi_F := dF : U \to T^*C^n = V$ is a para-Kählerian Lagrangian immersion and hence induces on $U$ the structure of a special para-Kähler manifold $(U, J, g, \nabla) =: M(F)$. Conversely, any special para-Kähler manifold is locally isomorphic to a manifold of the form $M(F)$.

$F$ is known as the para-holomorphic prepotential.

Remark: With respect to the standard para-holomorphic coordinates $(z^i)$ on $C^n \supset U$ and the corresponding coordinates $(z^i, w_j)$ on $T^*C^n$ we have:

$$\varphi_F^* z^i = z^i, \quad \varphi_F^* w_j = \frac{\partial F}{\partial z^i}. \quad (3.2)$$

4 Special para-Kähler manifolds as parabolic hyperspheres

Now we shall prove that any simply connected special para-Kähler manifold $(M, J, g, \nabla)$ is an intrinsic parabolic hypersphere, i.e. admits a Blaschke immersion as a parabolic hypersphere. Moreover, we shall characterize those parabolic hyperspheres which arise from special para-Kähler manifolds.

Theorem 3 Let $(M, J, g, \nabla)$ be a simply connected special para-Kähler manifold of dimension $m=2n$. Then there exists a Blaschke immersion $\varphi : M \to \mathbb{R}^{m+1}$ with induced connection $\nabla$ and Blaschke metric $g$. Moreover, $\varphi$ is a parabolic affine hypersphere and is uniquely determined up to equiaffine transformations.

Proof: We check the integrability conditions of theorem 1 for the pair $(\nabla, g)$. Since the metric volume form $\nu$ equals $\omega^n$, up to a constant factor, and the para-Kähler form $\omega$ is $\nabla$-parallel, $\nu$ is $\nabla$-parallel too. It remains to prove the vanishing of the torsion of the
g-conjugate connection $\nabla$ and its flatness, which implies the projective flatness. In view of the results of section 3, $\nabla = \nabla^J = J \circ \nabla \circ J$, so the flatness of $\nabla$ yields the flatness of $\nabla$ and the symmetry of $\nabla^J = 0$ yields the vanishing of the torsion of $\nabla$. This proves the existence of a Blaschke immersion $\varphi : M \to \mathbb{R}^{m+1}$, unique up to equiaffine transformations, with induced data $(g, \nabla)$. Since $\nabla$ is flat, $\varphi$ defines a parabolic hypersphere, thanks to proposition 1.

In order to characterize those parabolic hyperspheres coming from Blaschke immersions of special para-Kähler manifolds, we introduce the notion of a para-special parabolic hypersphere:

**Definition 7** An intrinsic parabolic hypersphere $(M, \nabla, g)$ is para-special if it admits an almost para-complex structure $J$ such that $\omega := g(J \cdot, \cdot)$ is skew-symmetric and $\nabla$-parallel.

**Lemma 1** Let $(M, \nabla, g)$ be a para-special intrinsic parabolic hypersphere. Then $J$ is a para-complex structure on $M$ and the connection $\nabla^J = J \circ \nabla \circ J$ is torsion-free.

Proof: Since $\nabla \omega = 0$ and $J^* g = -g$ (by the skew-symmetry of $\omega$), the same calculation as in proposition 3 shows that $\nabla = \nabla^J$. By theorem 1, $\nabla$ and hence $\nabla^J$ is torsion-free. For parabolic hyperspheres, the induced connection $\nabla$ is flat. Hence, the connections $\nabla$ and $\nabla^J$ are both torsion-free and flat.

Next we calculate the Nijenhuis-tensor $N_J$, see (3.1). Let $X$ and $Y$ be $\nabla$-parallel vector fields, then $JX$ and $JY$ are $\nabla^J$-parallel and

$$N_J(X, Y) = -J[X, JY] - J[JX, Y]$$
$$= -J[(\nabla_X JY - \nabla_{JY} X) + (\nabla_{JX} Y - \nabla_Y JX)]$$
$$= \nabla^J_Y X - \nabla^J_X Y = [Y, X] = 0.$$  

This shows, that $J$ is integrable.

**Theorem 4** Any Blaschke immersion of a special para-Kähler manifold is a para-special parabolic affine hypersphere. A parabolic affine hypersphere $\varphi : M \to \mathbb{R}^{m+1}$ is a Blaschke immersion of a special para-Kähler manifold if and only if it is para-special.

Proof: Any Blaschke immersion of a special para-Kähler manifold $(M, J, g, \nabla)$ is a parabolic affine hypersphere, simply because $\nabla$ is flat, and it is obviously para-special. Conversely, we have to show that a para-special parabolic affine hypersphere $(M, J, g, \nabla)$ is special para-Kähler. We know that $\omega$ is skew-symmetric and parallel with respect to the torsion-free connection $\nabla$. In particular, $d\omega = 0$. Moreover, $J$ is integrable by lemma 1. These conditions imply that $(M, J, g)$ is a para-Kähler manifold, see [CMMS] Theorem 1. It remains to show that $d\nabla^J = 0$. This follows from the vanishing of the torsion of $\nabla^J$, see prop. 4 and lemma 1.

From the local description of special para-Kähler manifolds given in corollary 2 we obtain:

**Corollary 3** Any non-degenerate para-holomorphic function $F$ of $n$ para-complex variables defines a para-special parabolic hypersphere of dimension $m = 2n$. Conversely, any para-special parabolic hypersphere arises locally in this way.
5 The para-holomorphic representation formula

In this section we give the explicit form of the affine immersion in terms of the prepotential $F$. Let $U \subset \mathbb{C}^n$ be an arbitrary open set. Since we do not want to restrict to simply connected domains, we consider functions defined on the universal covering $\pi : \tilde{U} \to U$. Any para-holomorphic function on $\tilde{U}$ induces a para-holomorphic Lagrangian immersion:

$$\varphi : \tilde{U} \to T^* U \subset T^* \mathbb{C}^n, \quad \varphi(p) := dF \circ (\pi_* T_p \tilde{U})^{-1} \in T^*_p U, \quad p \in \tilde{U}. \quad (5.1)$$

We denote by $J$ the para-complex structure of $\tilde{U}$ and pull back the canonical coordinates of $T^* \mathbb{C}^n$ to $\tilde{U}$. The resulting para-holomorphic functions are $	ilde{z}_i := \varphi^* z_i$ and $\tilde{w}_j := \varphi^* w_j$.

In the following text we use the notation

$$\tilde{z} := (\tilde{z}_1, \ldots, \tilde{z}_n), \quad F_{\tilde{z}} = (F_{\tilde{z}_1}, \ldots, F_{\tilde{z}_n}) = \left( \frac{\partial F}{\partial \tilde{z}_1}, \ldots, \frac{\partial F}{\partial \tilde{z}_n} \right), \quad F_{\tilde{z}} \tilde{z} = \sum F_{\tilde{z}_k} \tilde{z}_k, \quad \text{etc.}$$

Recall that $\partial^2 F$ is the Hessian of $F$ with respect to the flat para-holomorphic torsion-free connection given by the coordinates $\tilde{z}_i$. As above $g := \Re(\varphi^* \gamma)$ is a para-Kähler metric. Moreover, as in section 3, a flat torsion-free connection $\nabla$ is given by the condition that the functions

$$\tilde{x}_i = \Re(\tilde{z}_i) \quad \text{and} \quad \tilde{y}_j = \Re(\tilde{w}_j) \quad (5.2)$$

are $\nabla$-affine functions. Further we denote

$$\tilde{u}_i = \Im(\tilde{z}_i) \quad \text{and} \quad \tilde{v}_j = \Im(\tilde{w}_j). \quad (5.3)$$

We define now $M(F)$ as the data $(\tilde{U}, J, g, \nabla)$.

**Theorem 5** Let $F$ be a non-degenerate para-holomorphic function defined on $\tilde{U}$. Then $M(F) = (\tilde{U}, J, g, \nabla)$ is a special para-Kähler manifold with para-Kähler-form given by $\omega = g(J \cdot, \cdot) = 2 \sum d\tilde{x}_i \wedge d\tilde{y}_i$. A para-special parabolic hypersphere $\psi_F : \tilde{U} \to \mathbb{R}^{2n+1}$ with respect to the volume form $\text{vol} = 2^n \text{det}$ on $\mathbb{R}^{2n+1}$ and with affine normal $\xi = \partial_{2n+1}$, Blaschke metric $g$, induced connection $\nabla$ and compatible almost para-complex structure $J$ is given by the formula

$$\psi_F(\tilde{z}) := (\Re(\tilde{z}), \Re(F_{\tilde{z}}), 2\Re(F_{\tilde{z}})\Im(\tilde{z}) - 2\Im(F)) = \left( \tilde{x}_1, \ldots, \tilde{x}_n, \tilde{y}_1, \ldots, \tilde{y}_n, 2 \sum \tilde{y}_k \tilde{u}_k - 2 \Im(F) \right). \quad (5.4)$$

It is unique up to equiaffine transformations of $\mathbb{R}^{2n+1}$.

**Proof:** The first statement is a consequence of corollary 2. Existence and uniqueness, up to equiaffine transformations, of a para-special parabolic hypersphere $\psi : M(F) \to \mathbb{R}^{2n+1}$ follow from section 4. We only need to establish the explicit form $\psi_F$. Hence, it suffices to prove that $\psi_F$ defines a parabolic hypersphere such that $g$ is its Blaschke metric, $\nabla$.
its induced connection and $J$ is a compatible almost para-complex structure. In order to do this, we have to show that the Gauß-Weingarten data $(\nabla^v, g^v, A^v)$ with respect to the transversal vector field $v = \partial_{2n+1}$ (compare with (2.2)) coincide with $(\nabla, g)$.

Let $\nabla$ be the canonical connection of $\mathbb{R}^{2n+1}$, then $v$ is $\nabla$-parallel and, consequently, one has $A^v = 0$ and $\theta^v = 0$. We are going to express $\nabla^v$ and $g^v$ with respect to the coordinate vector fields

$$\partial_{\tilde{x}^i} = \partial_i + \frac{\partial f}{\partial \tilde{x}^i} \partial_{2n+1}, \quad \partial_{\tilde{y}^j} = \partial_{n+j} + \frac{\partial f}{\partial \tilde{y}^j} \partial_{2n+1},$$

where $f = 2\sum (\tilde{y}_k) \tilde{u}^k - 2\mathcal{H}(F)$ is the last component of $\psi_F$ and we have identified $(\psi_F)_i, \partial_{\tilde{x}^i}$, with $\partial_{\tilde{x}^i}$ and $(\psi_F)_i, \partial_{\tilde{y}^i}$ with $\partial_{\tilde{y}^i}$. In this frame, the connection $\nabla$ is given by

$$\nabla_{\partial_{\tilde{x}^i}} \partial_{\tilde{x}^j} = \frac{\partial^2 f}{\partial \tilde{x}^i \partial \tilde{x}^j} v, \quad \nabla_{\partial_{\tilde{x}^i}} \partial_{\tilde{y}^j} = \frac{\partial^2 f}{\partial \tilde{x}^i \partial \tilde{y}^j} v, \quad \nabla_{\partial_{\tilde{y}^i}} \partial_{\tilde{y}^j} = \frac{\partial^2 f}{\partial \tilde{y}^i \partial \tilde{y}^j} v. \quad (5.5)$$

This shows that the covariant derivatives have only normal part. Therefore, the vector fields $\partial_{\tilde{x}^i}$ and $\partial_{\tilde{y}^i}$ are parallel for the connection $\nabla^v$ and, thus, $\nabla^v = \nabla$. From $\theta^v = 0$ we obtain $\nabla \nu^v = \nabla^v \nu^v = 0$ for the induced volume form $\nu^v = \text{vol}(v, \ldots)$. Further we find $g^v = \text{Hess}^v(f) = \nabla^2 f$. The proof can now be finished by showing the identity

$$g^v = g. \quad (5.6)$$

In fact, suppose that this identity holds, i.e. $g = g^v$ is the Blaschke metric. The metric volume form $\text{vol}^g$ of the para-Kähler manifold $M(F)$ can be written in terms of its para-Kähler-form $\omega = g(J\cdot, \cdot) = 2\sum_{i=1}^n d\tilde{x}^i \wedge d\tilde{y}^i$ as

$$\text{vol}^g = (-1)^{n(n-1)/2} \omega^n = 2^n d\tilde{x}^1 \wedge \ldots \wedge d\tilde{x}^n \wedge d\tilde{y}^1 \wedge \ldots \wedge d\tilde{y}^n = 2^n \det(v, \ldots) = \nu^v,$$

when the chosen orientation is that of $\nu^v$. Since the metric volume form $\text{vol}^g$ coincides with the volume form $\nu^v$, the vector field $v$ is the affine normal and $\psi_F$ is a parabolic hypersphere; $g = g^v$ is its Blaschke metric and $\nabla = \nabla^v$ the induced connection. To prove the identity (5.6) we need the lemma

**Lemma 2** The partial derivatives of the functions $\tilde{u}^i$ and $\tilde{v}_j$ on $M(F)$ with respect to the $\nabla$-affine local coordinates $(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}_1, \ldots, \tilde{y}_n)$ satisfy the following equations:

$$\sum_k (\tilde{u}_k^{\tilde{y}^i} (\tilde{v}_k) \tilde{y}_j) - (\tilde{u}_k^{\tilde{x}^i} (\tilde{v}_k) \tilde{x}^j) = -\delta_{ij},$$

$$\sum_k \tilde{u}_k^{\tilde{x}^i} (\tilde{v}_k) \tilde{x}^j = \sum_k \tilde{u}_k^{\tilde{x}^j} (\tilde{v}_k) \tilde{x}^i,$$

$$\sum_k \tilde{u}_k^{\tilde{y}^j} (\tilde{v}_k) \tilde{y}_i = \sum_k \tilde{u}_k^{\tilde{y}^i} (\tilde{v}_k) \tilde{y}_j,$$

$$\tilde{u}^{\tilde{x}^i} = - (\tilde{v}_j), \quad \tilde{u}^{\tilde{y}^i} = \tilde{v}_j,$$

$$(\tilde{v}_i) \tilde{x}^j = (\tilde{v}_j) \tilde{x}^i.$$
Proof: Since $\varphi : M(F) \rightarrow T^*C^n$ is a Lagrangian immersion, the pull back of the symplectic form $\Omega$ of $T^*C^n$ vanishes. Splitting $\varphi^*\Omega = 0$ into its real and imaginary parts yields the lemma. □

Using equations (3.2), (5.2) and (5.3) we have

$$\partial_{\tilde{x}i} \mathcal{Z}(F) = \mathcal{Z}(\partial_{\tilde{x}i} F) = \sum_j \mathcal{Z} \left( \frac{\partial \tilde{z}^j}{\partial \tilde{x}i} \right) \partial_{\tilde{x}z} F = \sum_j \mathcal{Z}((\delta_i^j + e\tilde{u}^j_i) F_{\tilde{x}i}) = \tilde{v}_i + \sum_j \tilde{u}^j_i \tilde{y}_j$$

and

$$\partial_{\tilde{y}j} \mathcal{Z}(F) = \mathcal{Z}(\partial_{\tilde{y}j} F) = \sum_k \mathcal{Z} \left( \frac{\partial \tilde{z}^k}{\partial \tilde{y}j} \right) \partial_{\tilde{x}z} F = \sum_k \mathcal{Z}(e \tilde{u}^k_j F_{\tilde{x}k}) = \sum_k \tilde{u}^k_j \tilde{y}_k.$$

Next we calculate the Hessian of $\mathcal{Z}(F)$, with the help of lemma 2:

$$\partial^2_{\tilde{x}i, \tilde{x}j} \mathcal{Z}(F) = (\tilde{v}_i)_{\tilde{x}j} + \sum_k \tilde{u}^k_{\tilde{x}i, \tilde{x}j} \tilde{y}_k, \quad \partial^2_{\tilde{x}i, \tilde{y}j} \mathcal{Z}(F) = \sum_k \tilde{u}^k_{\tilde{x}i, \tilde{y}j} \tilde{y}_k, \quad \partial^2_{\tilde{y}i, \tilde{y}j} \mathcal{Z}(F) = \tilde{u}^i_{\tilde{y}j} + \sum_k \tilde{u}^k_{\tilde{y}i, \tilde{y}j} \tilde{y}_k.$$

This yields the metric $g^\nu$:

$$g^\nu(\partial_{\tilde{x}i}, \partial_{\tilde{x}j}) = \partial^2_{\tilde{x}i, \tilde{x}j} f = 2 \sum_k \tilde{u}^k_{\tilde{x}i, \tilde{x}j} \tilde{y}_k - 2((\tilde{v}_i)_{\tilde{x}j} + \sum_k \tilde{u}^k_{\tilde{x}i, \tilde{x}j} \tilde{y}_k) = -2(\tilde{v}_i)_{\tilde{x}j},$$

$$g^\nu(\partial_{\tilde{x}i}, \partial_{\tilde{y}j}) = 2 \partial^2_{\tilde{x}i, \tilde{y}j} f = 2(\tilde{u}^i_{\tilde{x}j} + \sum_k \tilde{u}^k_{\tilde{x}i, \tilde{y}j} \tilde{y}_k) - 2 \sum_k \tilde{u}^k_{\tilde{x}i, \tilde{x}j} \tilde{y}_k = 2\tilde{u}^i_{\tilde{x}j}, \quad (5.7)$$

$$g^\nu(\partial_{\tilde{y}i}, \partial_{\tilde{y}j}) = \partial^2_{\tilde{y}i, \tilde{y}j} f = 2(\tilde{u}^i_{\tilde{y}j} + \sum_k \tilde{u}^k_{\tilde{y}i, \tilde{y}j} \tilde{y}_k) - 2(\tilde{u}^k_{\tilde{y}i} + \sum_k \tilde{u}^k_{\tilde{y}i, \tilde{y}j} \tilde{y}_k) = 2\tilde{u}^k_{\tilde{y}i}.$$

Remark, that from equation (5.7) and lemma 2 we obtain

$$g^\nu(\partial_{\tilde{y}i}, \partial_{\tilde{x}j}) = 2\tilde{u}^i_{\tilde{x}j} = -2(\tilde{v}_i)_{\tilde{x}j}. \quad (5.8)$$

Now we regard $g$ and the para-Kähler form $\omega = 2 \sum dx^i \wedge dy_i$, as isomorphisms $TM \rightarrow T^*M$, by inserting the first argument. In order to prove the identity (5.6), we check that $g^{-1}g^\nu = id$. The relation $g = \omega \circ J$, together with the skew-symmetry of $J$ with respect to $\omega$, yields $g^{-1} = J^{-1} \circ \omega^{-1} = J \circ \omega^{-1} = -\omega^{-1} \circ J^*$. More explicitly, one has

$$\omega^{-1} = \frac{1}{2} \sum \partial_{\tilde{y}i} \wedge \partial_{\tilde{x}i}, \quad J^* dx^i = d\tilde{u}^i, \quad J^* dy_j = d\tilde{v}_j,$$

and so with lemma 2

$$g^{-1}(dx^i, dx^j) = -\omega^{-1}(d\tilde{u}^i, dx^j) = -\frac{1}{2} \tilde{u}^i_{\tilde{y}j},$$

$$g^{-1}(dx^i, dy_j) = -\omega^{-1}(d\tilde{u}^i, d\tilde{y}_j) = \frac{1}{2} \tilde{u}^i_{\tilde{x}j},$$

$$g^{-1}(dy_i, dy_j) = -\omega^{-1}(d\tilde{v}_i, d\tilde{y}_j) = \frac{1}{2} (\tilde{v}_i)_{\tilde{x}j}.$$
6 Conical special para-Kähler manifolds and proper affine hyperspheres

Conical special para-Kähler manifolds arise in the study of supergravity on four-dimensional Riemannian (rather than Lorentzian) space-times with vector multiplets as matter. Any non-degenerate para-holomorphic function $F$ of $n$ para-complex variables which is homogeneous of degree two defines a conical affine special para-Kähler manifold [CMT].

**Definition 8** [CMT] A conical special para-Kähler manifold $(M, J, g, \nabla, \xi)$ is a special Kähler manifold $(M, J, g, \nabla)$ endowed with a vector field $\xi$ such that
\[
\nabla \xi = D \xi = \text{Id},
\]
where $D$ is the Levi-Civita connection. It is called regular if the function
\[
k := \frac{1}{2} g(\xi, \xi)
\]
has no zeros.

We shall always assume that $M$ is connected and, multiplying $g$ by $-1$, if necessary, that $k > 0$.

**Proposition 6** The level sets $M_c := \{ k = c \}$, $c \in \mathbb{R}$, are smooth hypersurfaces perpendicular to $\xi = \text{grad}(k)$, or empty.

**Proof:** We compute the differential:
\[
dk = g(D\xi, \xi) = g(\cdot, \xi) \neq 0,
\]
by the regularity assumption. This shows that the level sets are smooth and that $\xi$ is the gradient of $k$, which is perpendicular to the level sets of $k$.

Next we write the Gauß and Weingarten equations for the embedding $M_c \subset (M, \nabla)$ with respect to a transversal field $\tilde{\xi} = \lambda \xi$, where $\lambda \neq 0$ is an arbitrary constant.

**Proposition 7** For all vector fields $X, Y$ on $M_c$ we have:
\[
\nabla_X Y = \nabla_{\tilde{\xi}} Y - \frac{1}{2\lambda c} g(X, Y)\tilde{\xi}
\]
\[
\nabla_X \tilde{\xi} = \lambda \text{Id},
\]
where $\nabla_{\tilde{\xi}} Y := (\nabla_X Y)^\top$ is the induced connection on $M_c$. In particular, the Weingarten map $A$ and the affine metric $h$ with respect to $\tilde{\xi}$ are given by:
\[
A = -\lambda \text{Id} \quad \text{and} \quad h = -\frac{1}{2\lambda c} g|_{M_c}.
\]
Moreover, the induced volume $\nu^{\tilde{\xi}} = \nu(\tilde{\xi}, \ldots)$ is parallel with respect to the induced connection. Here $\nu = \text{vol}^g$ stands for the metric volume form of the oriented pseudo-Riemannian manifold $(M, J, g)$.
Proof: The second equation follows immediately from $\nabla \xi = \text{Id}$. Writing $\nabla = D + S$ and using the fact that $S_X$ is $g$-symmetric, see [S] Proposition 6 (iii), and $S_X \xi = 0$, we compute the scalar product of $\xi$ with the left-hand side of (6.3):

$$g(\nabla_X Y, \xi) = g(D_X Y, \xi) + g(S_X Y, \xi) = -g(Y, D_X \xi) + g(Y, S_X \xi) = -g(X, Y).$$

The scalar product with the right-hand side is

$$-\frac{1}{2\lambda c} g(X, Y) g(\tilde{\xi}, \xi) = -\frac{1}{2c} g(X, Y) g(\xi, \xi) = -g(X, Y).$$

This proves (6.3)–(6.5). The last statement is an easy consequence of (6.4).

Theorem 6. Let $(M, J, g, \nabla, \xi)$ be a regular connected conical special para-Kähler manifold. The level sets $M_c = \{ k = c \} \subset (M, \nabla, \nu = \text{vol}^g)$, $c > 0$, of the positive function $k = \frac{1}{2} g(\xi, \xi)$ are proper affine hyperspheres of affine mean curvature $\kappa = (2c)^{-\frac{2n}{2n+1}}$ with respect to the Blaschke normal $E = -(2c)^{-\frac{2n}{2n+1}} \xi$, or empty. Moreover, the pseudo-Riemannian mean curvature of $M_c$ equals $1/\sqrt{2c}$. In particular, $M_{1/2}$ has both mean curvatures equal to 1 with respect to the Blaschke normal $E = -\xi$, which is a unit normal for the ambient pseudo-Riemannian metric $g$.

Proof: We have to check that $\nu \tilde{\xi} = \text{vol}^h$ for a certain choice of $\lambda$. We will check the equality on a $g$-orthonormal local frame of $TM_c$. Let us extend the unit vector field

$$e_1 := \tilde{\xi} / |\tilde{\xi}|_g = \frac{\tilde{\xi}}{|\lambda|\sqrt{2c}}$$

to an oriented orthonormal local frame $(e_1, e_2, \ldots, e_{2n})$ of $TM$ along $M_c$. Then $(e_2, \ldots, e_{2n})$ is an oriented $g$-orthonormal local frame of $TM_c$ and we calculate

$$\nu \tilde{\xi}(e_2, \ldots, e_{2n}) = |\tilde{\xi}|_g \nu(e_1, e_2, \ldots, e_{2n}) = |\tilde{\xi}|_g = |\lambda|\sqrt{2c}$$

and

$$\text{vol}^h(e_2, \ldots, e_{2n}) = |e_2|^2|e_{2n}| = \left(\frac{1}{\sqrt{2|\lambda|c}}|e_2|_g\right)^{2n-1} = \frac{1}{(2|\lambda|c)^{n-\frac{1}{2}}}.$$ 

Therefore, $\nu \tilde{\xi} = \text{vol}^h$ holds if and only if

$$|\lambda| = (2c)^{-\frac{2n}{2n+1}}.$$ 

This proves that $M_c \subset (M, \nabla, \nu)$ is a proper affine hypersphere with Blaschke normal $E = \pm(2c)^{-\frac{2n}{2n+1}} \xi$ and mean curvature $\kappa = \mp(2c)^{-\frac{2n}{2n+1}}$. For the second choice of sign, i.e. $\lambda = -(2c)^{-\frac{2n}{2n+1}}$, we get positive affine and pseudo-Riemannian mean curvature.

References


[CMT] V. Cortés, Projective special (para-)Kähler manifolds, part of a joint paper under preparation with T. Mohaupt and U. Theis about supergravity on Riemannian space-times and the local c-map.


