A bridge between numeration systems and graph directed iterated function systems

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Basic IFSs

\[ X_T = \text{usual triadic Cantor set.} \]

\[ \phi_0(x) = \frac{x}{3} \]

\[ \phi_1(x) = \frac{x + 2}{3} \]

\( X_T \) is the unique attractor of the homogeneous iterated function system \( \Phi = \{ \phi_0, \phi_1 \} \):

\[ X_T = \frac{X_T}{3} \cup \frac{X_T + 2}{3}. \]
Basic IFSs

Pascal’s triangle modulo 2

\[ \phi_a(x) = \frac{x + a}{2} \]

for \( a \in \{0, 1\}^2 \setminus \{(1, 0)\} \)

Menger’s sponge

\[ \phi_a(x) = \frac{x + a}{3} \]

for \( a \in \{0, 1, 2\}^3 \) s.t. \( |a|_1 \leq 1 \)
Initial question

Given a IFS $\Phi$, can we obtain its attractor $K_\Phi$ as the attractor of another IFS $\Psi$?
Feng and Wang answered for homogeneous IFS on $\mathbb{R}$

**Definition**
An IFS $\Phi = \{\phi_i\}_{i=1}^N$ satisfies the *Open Set Condition* (OSC) if there is a non-empty open set $V$ such that the sets $\phi_i(V)$ are disjoint subset of $V$.

**Theorem (Feng and Wang 2009)**
$\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ two homogeneous IFS on $\mathbb{R}$ and that satisfy OSC.
Suppose that $X = K_\Phi = K_\Psi$.

1. Suppose $\dim_H(X) = s < 1$. Then $\frac{\log |r_\Psi|}{\log |r_\Phi|} \in \mathbb{Q}$;

2. Suppose $\dim_H(X) = 1$. If $X$ is not a finite union of intervals, then $\frac{\log |r_\Psi|}{\log |r_\Phi|} \in \mathbb{Q}$. 
Feng and Wang's result looks like Cobham's theorem

**Definition**
A set $X \subseteq \mathbb{N}$ is $k$-recognizable if there is a finite automaton accepting exactly $\text{rep}_k(X)$.

**Example**
$X = 2\mathbb{N}$ is 2-recognizable:

```
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```

![Diagram of a finite automaton accepting $\text{rep}_2(X)$](image)
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```
1
\rightarrow
1
\rightarrow
0
\rightarrow
0
```

**Theorem (Cobham 1969)**

Let $k, \ell \geq 2$ be two integers such that $\frac{\log k}{\log \ell} \not\in \mathbb{Q}$.

A set $X \subset \mathbb{N}$ is simultaneously $k$- and $\ell$-recognizable if and only if it is a finite union of arithmetic progressions.
Two similar results appeared almost simultaneously

Let \( k \geq 2 \) be an integer.

**Definition**
A compact set \( X \subset [0, 1] \) is *\( k \)-self-similar* if its \( k \)-kernel is finite, where the \( k \)-kernel is the collection of sets

\[
N_{a,b}(X) = (k^aX - b) \cap [0, 1], \quad a, b \in \mathbb{N}, \ 0 \leq b < k^a.
\]

**Example**
The *Triadic Cantor set* \( X_T \) is 3-self-similar: its 3-kernel is

\[
\{ X_T, \{0\}, \{1\}, \{0, 1\} \}.
\]
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Let $k \geq 2$ be an integer.

**Definition**
A compact set $X \subset [0, 1]$ is \textit{k-self-similar} if its $k$-kernel is finite, where the $k$-kernel is the collection of sets

\[ N_{a,b}(X) = (k^a X - b) \cap [0, 1], \quad a, b \in \mathbb{N}, \ 0 \leq b < k^a. \]

**Theorem (Adamczewski and Bell 2011)**
Let $k, \ell \geq 2$ be two integers such that $\log k / \log \ell \notin \mathbb{Q}$.

A compact set $X \subset [0, 1]$ is \textit{simultaneously k- and \ell-self-similar} if and only if it is a finite union of intervals with rational endpoints.
Two similar results appeared almost simultaneously

Let \( k \geq 2 \) be an integer.

**Definition**
A compact set \( X \subset [0, 1]^d \) is *\( k \)-self-similar* if its \( k \)-kernel is finite, where the \( k \)-kernel is the collection of sets

\[
N_{a,b}(X) = (k^a X - b) \cap [0, 1]^d, \quad a \in \mathbb{N}, b \in \mathbb{N}^d, \ 0 \leq b_i < k^a.
\]

**Conjecture (Adamczewski and Bell 2011)**
Let \( k, \ell, d \geq 2 \) be two integers such that \( \frac{\log k}{\log \ell} \notin \mathbb{Q} \).

A compact set \( X \subset [0, 1]^d \) is simultaneously \( k \)- and \( \ell \)-self-similar if and only if it is a finite union of polyhedra with rational vertices.
Two similar results appeared almost simultaneously

Let $k \geq 2$ be an integer.

**Example**

Pascal’s triangle modulo 2 is 2-self-similar.
Two similar results appeared almost simultaneously

Let $k \geq 2$ be an integer.

**Definition**

A *Büchi automaton* is an automaton with a procedure of acceptance adapted to infinite words.

**Definition**

A set $X \subset \mathbb{R}^n$ is *weakly $k$-recognizable* if there is a weak Büchi automaton accepting exactly $\text{rep}_k(X)$

**Example**

The *Triadic Cantor set* $X_T$ is weakly 3-recognizable:
Two similar results appeared almost simultaneously

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A Büchi automaton is an automaton with a procedure of acceptance adapted to infinite words.

Definition
A set \( X \subset \mathbb{R}^n \) is weakly \( k \)-recognizable if there is a weak Büchi automaton accepting exactly \( \text{rep}_k(X) \)

Theorem (Boigelot, Brusten, Bruyère, Jodogne, Leroux and Wolper (2001 to 2009))

Let \( k, \ell \geq 2 \) be two integers such that \( \frac{\log k}{\log \ell} \notin \mathbb{Q} \).

A set \( X \subset \mathbb{R}^n \) is simultaneously weakly \( k \)- and \( \ell \)-recognizable if and only if it is definable by a first order formula in the structure \( \langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle \).
Two similar results appeared almost simultaneously

Let $k \geq 2$ be an integer.

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A *Büchi automaton* is an automaton with a procedure of acceptance adapted to infinite words.

**Definition**

A set $X \subset \mathbb{R}^n$ is *weakly $k$-recognizable* if there is a weak Büchi automaton accepting exactly $\text{rep}_k(X)$

**Remark**

Sets definable by a first order formula in the structure $\langle \mathbb{R}, \mathbb{Z}, +, \leq \rangle$ are exactly the periodic repetitions of finite unions of polyhedra with rational vertices.
All three notions are not equivalent

The following set is 2-recognizable but not obtained by an IFS.
The notion that unifies everything is GDIFS

Definition
A graph-directed iterated function system (GDIFS) is given by a 4-tuple

$$\mathcal{G} = (V, E, (X_v, \rho_v)_{v \in V}, (S_e)_{e \in E})$$

with $(V, E)$ is a directed graph, $(X_v, \rho_v)$ are metric spaces and $S_e$ are similarities.

An attractor for $\mathcal{G}$ is a collection of compact sets $(K_v \subset X_v)_{v \in V}$ such that for all $v$,

$$K_v = \bigcup_{e \in E_{v \rightarrow u}} S_e(K_u).$$
Closed Büchi automata and GDIFS

Theorem (Charlier, L., Rigo)

A set \( X \subset [0, 1]^d \) is \( k \)-recognizable by some closed Büchi automaton if and only if it is a union of sets of the attractor of a GDIFS whose similarities are of the form \( \frac{x + a}{k} \) for \( a \in A^d_k \).
Closed Büchi automata and GDIFS

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Sketch of proof

Let $A$ be a closed Büchi automaton accepting $\text{rep}_k(X)$.

- $\forall q$, $W_q := \{\text{infinite words accepted by } A \text{ from } q\}$
  
  $$W_q = \bigcup_{(q, a, p)} aW_p$$
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- $w = au$ for $a \in A_k^d \Rightarrow \text{val}_k(w) = \frac{\text{val}_k(u) + a}{k}$. 

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\[ W_q = \bigcup_{(q,a,p)} aW_p \]

\( w = au \) for \( a \in A_k^d \) \( \Rightarrow \) \( \text{val}_k(w) = \frac{\text{val}_k(u) + a}{k} \).

Thus \( (K_q := \text{val}_k(W_q))_q \) are compact and satisfy:

\[ \forall q, K_q = \bigcup_{(q,a,p)} \frac{K_p + a}{k} \]
GDIFS and $k$-self-similar sets

Theorem (Charlier, L., Rigo)

A set $X \subset [0, 1]^d$ is $k$-self-similar if and only if it belongs to the attractor of a GDIFS whose similarities are of the form $S_a : x \mapsto \frac{x + a}{k}$ for $a \in A_k^d$. 
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Sketch of proof

$k$-self-similar $\Rightarrow$ GDIFS:

- $k$-kernel of $X$: $\{N_{a_i, b_i}\}_{i=1}^n$ 
- $N_{a, b} = \bigcup_{c \in A_k^d} \frac{N_{a+1, kb+c} + c}{k}$
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- Draw the following directed graph:
  
  $V = \{N_{a_i, b_i}\}_{i=1}^n$ $N_{a_i, b_i} \xrightarrow{c} N_{a_j, b_j}$ if $N_{a_j, b_j} = N_{a_i+1, kb_i+c}$. 
GDIFS and $k$-self-similar sets

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$k$-self-similar $\Rightarrow$ GDIFS:

- **$k$-kernel of $X$:** $\{N_{a_i,b_i}\}_{i=1}^n$
- $N_{a,b} = \bigcup_{c \in A_k^d} \frac{N_{a+1,kb+c} + c}{k}$
- **Draw the following directed graph:**
  $V = \{N_{a_i,b_i}\}_{i=1}^n \quad N_{a_i,b_i} \xrightarrow{c} N_{a_j,b_j}$ if $N_{a_j,b_j} = N_{a_i+1,kb_i+c}$.
- $X = N_{0,0}$. 
GDIFS and $k$-self-similar sets

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A set $X \subset [0, 1]^d$ is $k$-self-similar if and only if it belongs to the attractor of a GDIFS whose similarities are of the form $S_a : x \mapsto \frac{x + a}{k}$ for $a \in A_k^d$.

Sketch of proof

GDIFS $\Rightarrow$ $k$-self-similar:

- $X = K_1$ where $K_i = \bigcup_{i \to j} S_a(K_j)$ for all $i$
- for all $\ell$: $X = \bigcup_{1 \to \cdots \to j} S_{a_1} \circ \cdots \circ S_{a_\ell}(K_j)$
GDIFS and $k$-self-similar sets

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Sketch of proof

GDIFS $\Rightarrow$ $k$-self-similar:

- $X = K_1$ where $K_i = \bigcup_i S_a \circ \cdots \circ S_a(K_j)$ for all $i$
- for all $\ell$: $X = \bigcup_{1 \rightarrow \cdots \rightarrow j} S_{a_1} \circ \cdots \circ S_{a_\ell}(K_j)$
- $N_{\ell, b} = (k^\ell X - b) \cap [0, 1]^d = \left( \bigcup_i S_{a_1} \circ \cdots \circ S_{a_\ell} K_j + c \right) \cap [0, 1]^d$

with $c \in \mathbb{N}^d$
Advantages and disadvantages of the methods

Automata results are better for:
- multidimensional setting
- GDIFS constructions
- logical characterization

IFSs results are better for:
- larger class of contraction ratios
- stability under affine maps
- possibility to consider non-homogeneous IFSs (Elekes, Keleti, Máthé)
Further work

Try to handle other sets such as Rauzy fractals.
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Thank you