# Kac–Moody symmetric spaces

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# Overview

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Since $H$ is a Baire space, $f(K_n)$ has non-empty interior for some $n \in \mathbb{N}$, and thus $H$ is locally compact. Moreover, $f|_{K_n}\colon K_n \to f(K_n)$ is a quotient map.

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Let $q\colon G \to G/\ker(f)$ be the quotient homomorphism and $\phi\colon G/\ker(f) \to H$ be the bijective continuous homomorphism induced by $f$.
Then $\phi^{-1} \circ f|_{K_n} = q|_{K_n}$ is continuous, whence $\phi^{-1}|_{f(K_n)}$ is continuous, $\phi^{-1}$ is a continuous homomorphism, and $\phi$ is a topological isomorphism. □

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One can think of a cone as the object $G \in \mathrm{ob}(\mathbb{A})$, together with the family $(\phi(i))_{i\in\mathrm{ob}(\mathbb{I})}$ of morphisms $\phi(i)\colon \delta(i) \to G$, such that

$$\phi(j) \circ \delta(\alpha) = \phi(i) \text{ for all } i, j \in \mathrm{ob}(\mathbb{I}) \text{ and } \alpha \in \mathrm{Mor}(i, j)\ .$$

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A cone $(G, (\phi_i)_{i\in\mathrm{ob}(\mathbb{I})})$ is called a *colimit* of $\delta$ if, for each cone $(H, (\psi_i)_{i\in\mathrm{ob}(\mathbb{I})})$, there is a unique morphism $\psi\colon G \to H$ such that $\psi \circ \phi_i = \psi_i$ for all $i \in \mathrm{ob}(\mathbb{I})$.

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Examples: direct limits, free products, amalgamated products $A *_C B$

# Amalgams

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A cone $(G, (\phi_i)_{i \in \mathrm{ob}(\mathbb{I})})$ over $\delta$ in the category of abstract groups $\mathbb{G}$ is called an *enveloping group* of the amalgam and its colimit a *universal enveloping group*.

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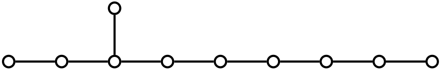

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# Topologies on colimits

Let $\delta\colon \mathbb{I} \to \mathbb{LCG}$ be a diagram of

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- Lie groups (if $G_i$, $(G, \mathcal{O})$ are $\sigma$-compact Lie groups).

# Lie groups as colimits

**Theorem 2 (Glöckner, Hartnick, K. 2010)**

*Let $G$ be a simply connected compact/split real semisimple Lie group with Lie group topology $\mathcal{O}$, let $T$ be a maximal torus of $G$, let $\Sigma = \Sigma(G_{\mathbb{C}}, T_{\mathbb{C}})$ be its root system, and let $\Pi$ be a system of fundamental roots of $\Sigma$.*

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The fact that $(G, (\iota_i)_{i \in \binom{\Pi}{1} \cup \binom{\Pi}{2}})$ is a colimit of $\delta$ in the category of abstract groups goes back to Tits (1974).

# Overview

# Definition via colimits

## Definition 3

Let

- $\Delta$ be an arbitrary Dynkin diagram without label $\infty$,
- $(G_\alpha)_{\alpha \in \Delta}$ be a family of copies of $\mathrm{SL}_2(\mathbb{R})$,
- $(G_{\alpha\beta})_{\{\alpha,\beta\} \in \binom{\Delta}{2}}$ be a family of appropriate simply connected split real Lie groups, and
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The topological version is established by Hartnick–K.–Mars (2013) based on ideas by Kac–Peterson (1980s).

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**Goal:** Construct a symmetric space for a topological Kac–Moody group.

# Overview

# A theorem by Loos

**Theorem 4 (Loos 1967)**

*Let $X$ be an affine symmetric space, and given $x, y \in X$ denote by $x \cdot y$ the point reflection of $y$ at $x$. Then*

$$\mu : X \times X \to X \quad : \quad (x, y) \mapsto x \cdot y$$

*is a $C^1$-map satisfying the following axioms:*

1. *for any $x \in X$ one has $x \cdot x = x$,*
2. *for any pair of points $x, y \in X$ one has $x \cdot (x \cdot y) = y$,*
3. *for any triple of points $x, y, z \in X$ one has*

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z),$$

4. *every $x \in X$ has a neighbourhood $U$ such that $x \cdot y = y$ implies $y = x$ for all $y \in U$.*

# A theorem by Loos

**Theorem 4 (continued)**

*Conversely, if $X$ is a smooth manifold and $\mu : X \times X \to X$ is a $C^1$-map subject to the axioms above, then $X$ is an affine symmetric space, and $\mu(x, y)$ is the point reflection of $y$ at $x$.*

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*If $X$ is a Riemannian symmetric space, then the isometries of $X$ are exactly the $C^1$-maps $\alpha : X \to X$ satisfying $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$.*

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*If $X$ is moreover of the non-compact type, then instead of the local condition it satisfies the global condition*

4. $x \cdot y = y$ *implies* $y = x$ *for all* $y \in X$.

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**Theorem 4 (continued)**

*Conversely, if $X$ is a smooth manifold and $\mu : X \times X \to X$ is a $C^1$-map subject to the axioms above, then $X$ is an affine symmetric space, and $\mu(x, y)$ is the point reflection of $y$ at $x$.*

*If $X$ is a Riemannian symmetric space, then the isometries of $X$ are exactly the $C^1$-maps $\alpha : X \to X$ satisfying $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$.*

*If $X$ is moreover of the non-compact type, then instead of the local condition it satisfies the global condition*

4. *$x \cdot y = y$ implies $y = x$ for all $y \in X$.*

Example: For any topological group $G$ the assignment

$$G \times G \to G : (x, y) \mapsto xy^{-1}x$$

satisfies axioms 1, 2, 3.

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A *geodesic* $\gamma \subset X$ is defined to be the image of a bijection

$$\phi : \mathbb{R} \to \gamma$$

such that

$$\phi(2x - y) = \mu(\phi(x), \phi(y)) \quad \text{for all } x, y \in \mathbb{R}.$$

# Overview

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# Freyn, Hartnick, Horn, K. 2017

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- $G/K$ admits a partial order, if Kostant convexity holds for $G$. (E.g., for type $E_{10}$.)