Painlevé monodromy varieties: classical and quantum

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Based on Chekhov-Mazzocco-R. *arXiv:1511.03851*

Talk at **"Higher Geometry and Field Theory".**

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Plan:

- Painlevé equations, Isomonodromy and Affine cubics;
- Teichmüller theory;
- Geodesic length;
- Decorated character varieties;
- Quantisation
- Perspectives and output;

Painlevé equations

The Painlevé equations are non linear second order ODE of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right), \qquad z \in \mathbb{C},$$

where $F(z, w, y)$ is a rational function of z, w, y and the solutions $w(z; c_1, c_2)$ satisfy

1. Painlevé–Kowalevski property: $w(z; c_1, c_2)$ have no *critical points* that depend on c_1, c_2.
2. Otherwise, they are the only second order ODE without movable singularities (branching points).
3. For generic c_1, c_2, $w(z; c_1, c_2)$ are new functions, Painlevé Transcendents.

Painlevé property:

- **Example for 1-st ordre ODE**:

$$
\begin{array}{llll}
w' = w & \Longrightarrow & w = e^{z-z_0}, & \checkmark \\
w' = w^2 & \Longrightarrow & w = \dfrac{1}{z_0 - z}, & \checkmark \\
w' = w^3 & \Longrightarrow & w \sim \dfrac{1}{\sqrt{z - z_0}}. & \text{X}
\end{array}
$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z \qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{3w-1}{2w(w-1)}w_z^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\gamma w}{z} + \frac{(w-1)^2}{z^2}\frac{\alpha w^2 + \beta}{w} + \frac{\delta w(w+1)}{w-1}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w_z +$$
$$+\frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right]$$

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- All Painlevé (except for P_I) admit one-parameter family of solutions (in terms of special functions) and for some special values of parameteres they have particular rational solutions;
- Recently: P_{II} - has a genuine fully NC analogue (V. Retakh-V.R.)

All Painlevé equations are **isomonodromic deformation equations** (Jimbo-Miwa1980)

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda} - \frac{\mathrm{d}A}{\mathrm{d}z} = [A, B]$$

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The monodromy data are encoded in an affine cubic surface called *monodromy manifold*.

Monodromy manifolds for the Painlevé equations

$$PVI \quad x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

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$$PIV \quad x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_2 x_3 + 1 = \omega_4$$

$$PIII \quad x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 = \omega_1 - 1$$

$$PII \quad x_1x_2x_3 + x_1 + x_2 + x_3 = \omega_4$$

$$PI \quad x_1x_2x_3 + x_1 + x_2 + 1 = 0$$

Saito and van der Put

PVI as isomonodromic deformation

Painlevé sixth equation

- The Painlevé VI equation describes the isomonodromic deformations of the rank 2 meromorphic connections on $\mathbb{P}^1$ with simple poles.

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = \left(\frac{A_1(z)}{\lambda} + \frac{A_2(z)}{\lambda - t} + \frac{A_3(z)}{\lambda - 1},\right) Y,\ \lambda \in \mathbb{C} \setminus \{0, t, 1\} \quad (1)$$

where $A_1, A_2, A_3 \in \mathfrak{sl}_2(\mathbb{C})$, $A_1 + A_2 + A_3 = -A_\infty$, diagonal.

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- Monodromy matrices $\gamma_j(Y_\infty) = Y_\infty M_j$
- Describes by generators of the fundamental group under the anti-isomorphism

$$\rho : \pi_1 \left(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}, \lambda_1\right) \to \mathrm{SL}_2(\mathbb{C}).$$

PVI as isomonodromic deformation

- $\text{eigen}(M_j) = \text{eigen}(\exp(2\pi i A_j))$
- We fix the base point λ_1 at infinity and the generators of the fundamental group to be $\gamma_1, \gamma_2, \gamma_3$ such that γ_j encircles only the pole i once and are oriented in such a way that

$$M_1 M_2 M_3 M_\infty = \mathbb{I}, \qquad M_\infty = \exp(2\pi i A_\infty). \tag{2}$$

- Eigenvalues of A_j are $(\theta_j, -\theta_j)$, $j = 0, t, 1, \infty$.
- $$\alpha := (\theta_\infty - 1/2)^2; \qquad \beta := -\theta_0^2;$$
$$\gamma := \theta_1^2; \qquad \delta := (1/4 - \theta_t)^2.$$

PVI as isomonodromic deformation

Let:

$$G_j := \mathrm{Tr}(M_j) = 2\cos(\pi\theta_j), \quad j = 0, t, 1, \infty,$$

The Riemann-Hilbert correspondence

$$\mathcal{F}(\theta_0, \theta_t, \theta_1, \theta_\infty)/\mathcal{G} \to \mathcal{M}(G_1, G_2, G_3, G_\infty)/\mathrm{SL}_2(\mathbb{C}),$$

where $\mathcal{G}$ is the gauge group, is defined by associating to each Fuchsian system its monodromy representation class. The representation space $\mathcal{M}(G_1, G_2, G_3, G_\infty)$ is realised as an affine cubic surface (Jimbo)

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0, \qquad (3)$$

where:

PVI as isomonodromic deformation

$$x_1 = \mathrm{Tr}\,(M_2 M_3)\,, \quad x_2 = \mathrm{Tr}\,(M_1 M_3)\,, \quad x_3 = \mathrm{Tr}\,(M_1 M_2)\,.$$

and

$$-\omega_i := G_k G_j + G_i G_\infty, i \neq k, j,$$

$$\omega_\infty = G_1^2 + G_2^2 + G_3^2 + G_\infty^2 + G_1 G_2 G_3 G_\infty - 4.$$

Iwasaki proved that the triple (x_1, x_2, x_3) satisfying the cubic relation (3) provides a set of coordinates on a large open subset

$$S \subset \mathcal{M}(G_1, G_2, G_3, G_\infty).$$

In what follows, we restrict to such open set.

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- In algebraic geometry - projective completion:

$$\overline{M}_{\widetilde{\varphi}} := \{(u,v,w,t) \in \mathbb{P}^3 \, | x_1^2 t + x_2^2 t + x_3^2 t - x_1 x_2 x_3 +$$

$$+\omega_3 x_1 t^2 + \omega_2 x_2 t^2 + \omega_3 x_3 t^2 + \omega_4 t^3 = 0\}$$

is a del Pezzo surface of degree three and differs from it by three smooth lines at infinity forming a triangle [Oblomkov]
$t = 0, \qquad x_1 x_2 x_3 = 0.$

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- PVI ($\tilde{D}_4$) cubic with only $\omega_4 \neq 0$ admits the log-canonical symplectic structure $\bar{\vartheta} = \frac{du \wedge dv}{uv}$ under isomorphism $\mathbb{C}^* \times \mathbb{C}^*/\imath \to M_\varphi$ by

$$(u, v) \to (x_1 = -(u + \frac{1}{u}), x_2 = -(v + \frac{1}{v}), x_3 = -(uv + \frac{1}{uv})$$

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- The family (**??**) can be "uniformize" by some analogues of theta-functions related to toric mirror data on log-Calabi-Yau surfaces (M. Gross, P. Hacking and S.Keel (see Example 5.12 of "Mirror symmetry for log-Calabi-Yau varieties I, arXiv:1106.4977).

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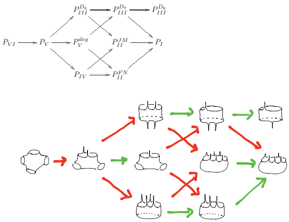
Use the confluence scheme of the Painlevé equations.



Basic ideas

- The character variety of a Riemann sphere with 4 holes $\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, \mathrm{t}, 1, \infty\}); \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is the monodromy cubic of the Painlevé VI $(\mathrm{Goldman\text{-}Toledo})$.

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- To visualize the confluence and the "decoration" we shall introduce two moves correspond to certain asymptotics in the (complexified) shear coordinates.
- Start from a sphere with 4 holes.

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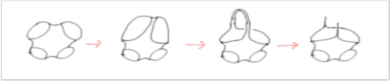

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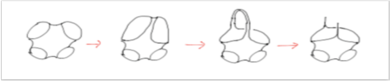

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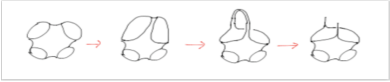

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This will provide cluster algebra of geometric type

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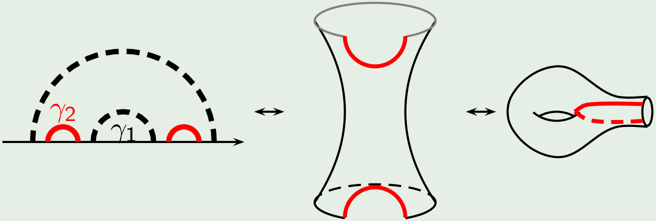

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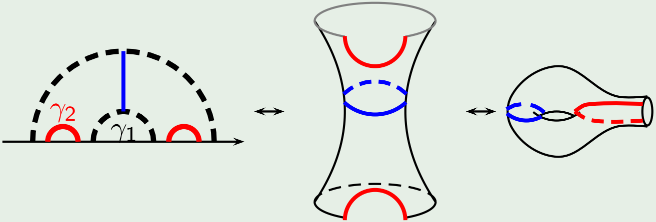

Theorem

Elements in $\pi_1(\Sigma_{g,s})$ are in 1-1 correspondence with conjugacy classes of closed geodesics.

Coordinates: geodesic lengths

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The geodesic length functions form an algebra with multiplication:

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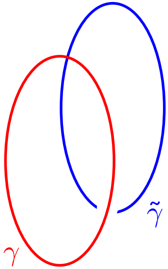

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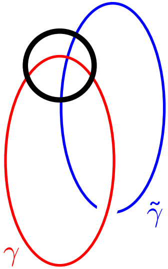

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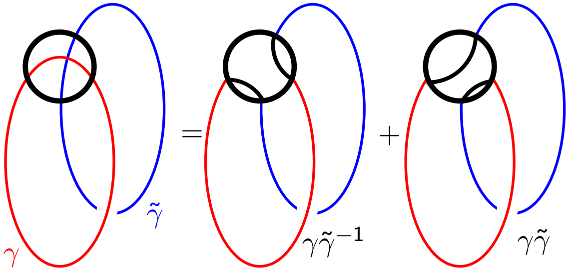
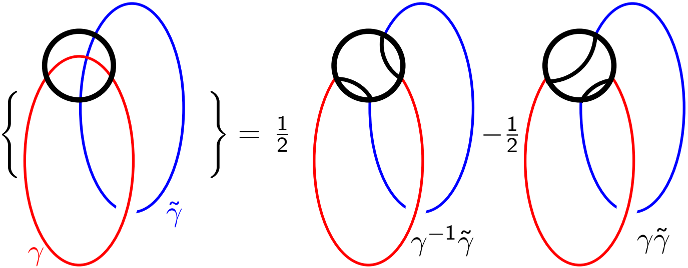
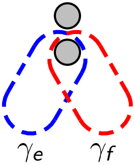


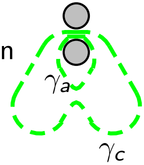
Poisson structure

$$\{G_\gamma, G_{\tilde\gamma}\} = \frac{1}{2} G_{\gamma\tilde\gamma} - \frac{1}{2} G_{\gamma\tilde\gamma^{-1}}.$$

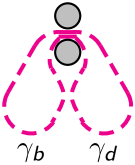




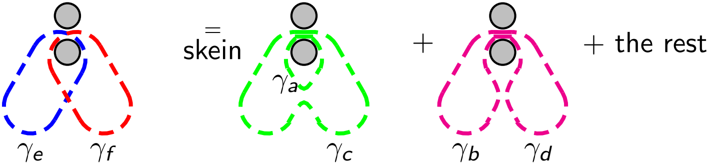
$\overset{=}{\text{skein}}$

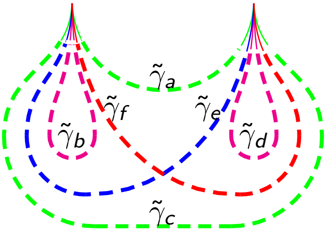


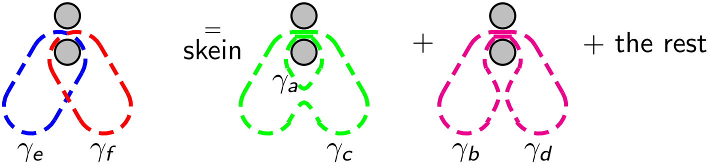
$+$

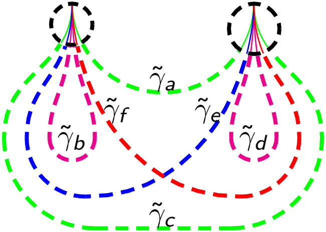


$+$ the rest


γ_e
γ_f
$\overset{=}{\text{skein}}$
γ_a
γ_c
$+$
γ_b
γ_d
$+$ the rest


$\tilde\gamma_a$
$\tilde\gamma_f$
$\tilde\gamma_e$
$\tilde\gamma_b$
$\tilde\gamma_d$
$\tilde\gamma_c$

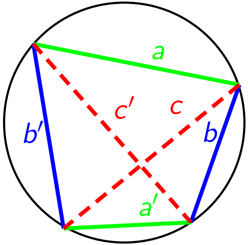




$$G_{\tilde\gamma_e} G_{\tilde\gamma_f} = G_{\tilde\gamma_a} G_{\tilde\gamma_c} + G_{\tilde\gamma_b} G_{\tilde\gamma_d}$$

Ptolemy Relation

$$aa' + bb' = cc'$$



Poisson structure

Theorem

The Poisson algebra of the λ-lengths of a complete cusped lamination is a Poisson cluster algebra [Chekhov-Mazzocco. *ArXiv:1509.07044*].

$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \mathcal{I}_{s_i,t_j,p_r,q_l}$$

$$\mathcal{I}_{s_i,t_j,p_r,q_l} = \frac{\epsilon_{i-r}\delta_{s,p}+\epsilon_{j-r}\delta_{t,p}+\epsilon_{i-l}\delta_{s,q}+\epsilon_{j-l}\delta_{t,q}}{4}$$

Decorated character variety

What is the *character variety* of a Riemann surface with cusps on its boundary?

Decorated character variety

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For Riemann surfaces with holes:

$$Hom\left(\pi_1(\Sigma), \mathbb{P}SL_2(\mathbb{C})\right)/GL_2(\mathbb{C}).$$

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- Replace tr by two characters: tr and tr_K.

Shear coordinates in the Teichmüller space

Fatgraph:

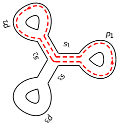

Figure: The fat graph of the 4 holed Riemann sphere. The dashed geodesic corresponds to x_3. The corresponding hyperbolic element $\gamma_{1;2} = \mathrm{Tr}(\mathrm{X}_{\mathrm{s}_1}\mathrm{L}\mathrm{X}_{\mathrm{p}_1}\mathrm{L}\mathrm{X}_{\mathrm{s}_1}\mathrm{R}\mathrm{X}_{\mathrm{s}_2}\mathrm{L}\mathrm{X}_{\mathrm{p}_2}\mathrm{L}\mathrm{X}_{\mathrm{s}_2}\mathrm{L})$

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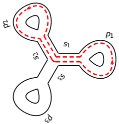

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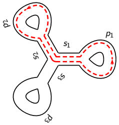
Decompose each hyperbolic element in Right, Left and Edge matrices Fock, Thurston

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$X_y := \begin{pmatrix} 0 & -\exp\left(\frac{y}{2}\right) \\ \exp\left(-\frac{y}{2}\right) & 0 \end{pmatrix}.$$



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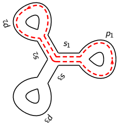
The three geodesic lengths: $x_i = \mathrm{Tr}(\gamma_{jk})$

$$x_1 = e^{s_2+s_3} + e^{-s_2-s_3} + e^{-s_2+s_3} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{s_3} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{-s_2}$$
$$x_2 = e^{s_3+s_1} + e^{-s_3-s_1} + e^{-s_3+s_1} + (e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}})e^{s_1} + (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{-s_3}$$
$$x_3 = e^{s_1+s_2} + e^{-s_1-s_2} + e^{-s_1+s_2} + (e^{\frac{p_1}{2}} + e^{-\frac{p_1}{2}})e^{s_2} + (e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}})e^{-s_1}$$



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$$\{x_1, x_2\} = x_1x_2 + 2x_3 + \omega_3, \quad \{x_2, x_3\} = x_2x_3 + 2x_1 + \omega_1, \quad \{x_3, x_1\} = x_3x_1 + 2x_2 + \omega_2.$$

The confluence from the cubic associated to PVI to the one associated to PV is realised by

$$p_3 \to p_3 - 2\log[\epsilon],$$

in the limit $\epsilon \to 0$. We obtain the following shear coordinate description for the *PV* cubic:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}}, \\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} - G \end{aligned}$$

where

$$G_i = e^{\frac{p_i}{2}}+e^{-\frac{p_i}{2}}, \quad i = 1,2, \quad G_3 = e^{\frac{p_3}{2}}, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1x_2x_3 + x_1^2 + x_2^2 - (G_1G_\infty + G_2G_3)x_1 - (G_2G_\infty + G_1G_3)x_2 - \\ -G_3G_\infty x_3 + G_\infty^2 + G_3^2 + G_1G_2G_3G_\infty = 0. \tag{5}$$

Note that the parameter p_3 is now redundant, we can eliminate it by rescaling. To obtain the correct PV- cubic, we need to pick $p_3 = -p_1 - p_2 - 2s_1 - 2s_2 - 2s_3$ so that $G_\infty = 1$.

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$\{x_1, x_2\} = x_1x_2 - G_3G_\infty, \quad \{x_2, x_3\} = x_2x_3 + 2x_1 - (G_1G_\infty + G_2G_3), \quad \{x_3, x_1\} = x_3x_1 + 2x_2 - (G_2G_\infty + G_1G_3).$

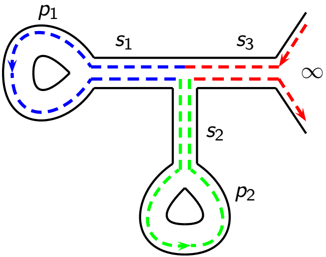

Figure: The fat graph corresponding to PV.

Geometrically speaking, sending the perimeter p_3 to infinity means that we are performing a chewing-gum move:
two holes, one of perimeter p_3 and the other of perimeter $s_1 + s_2 + s_3 + \frac{p_1}{2} + \frac{p_2}{2} + \frac{p_3}{2}$, become infinite, but the area between them remains finite.
This is leading to a Riemann sphere with three holes and two cusps on one of them. In terms of the fat-graph, this is represented by Figure 2.
The geodesic x_3 corresponds to the closed loop obtained going around p_1 and p_2 (green and red loops), while x_1 and x_2 are "asymptotic geodesics" starting at one cusp, going arond p_1 and p_2 respectively, and coming back to the other cusp.

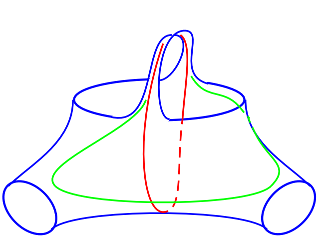

Figure: The process of confluence of two holes on the Riemann sphere with four holes. Chewing-gum move: hook two holes together and stretch to infinity by keeping the area between them finite (see Fig.). As a result we obtain a Riemann sphere with one less hole, but with two new cusps on the boundary of this hole. The red geodesic line which was initially closed becomes infinite, therefore two horocycles (the green dashed circles) must be introduced in order to measure its length.

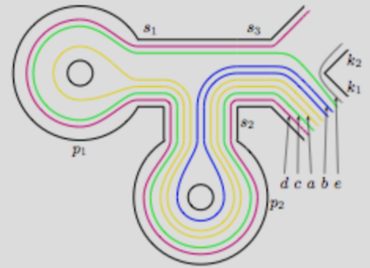

FIGURE 6. The system of arcs for PV.

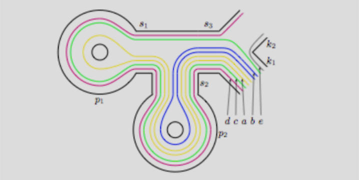

FIGURE 6. The system of arcs for PV.

$\gamma_b =$
$X(k_1)RX(s_3)RX(s_2)RX(p_2)RX(s_2)LX(s_3)LX(k_1)$- BUT its length is $b = \mathrm{tr}_K(\gamma_b) = \mathrm{tr}(bK)$, $K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

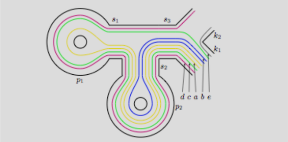

FIGURE 6. The system of arcs for PV.

$$\{g_{s_i,t_j}, g_{p_r,q_l}\} = g_{s_i,t_j} g_{p_r,q_l} \frac{\epsilon_{i-r}\delta_{s,p}+\epsilon_{j-r}\delta_{t,p}+\epsilon_{i-l}\delta_{s,q}+\epsilon_{j-l}\delta_{t,q}}{4}$$

$$\begin{aligned} \{b, d\} &= \{g_{1_3,1_4}, g_{2_1,1_8}\} \\ &= g_{1_3,1_4} g_{2_1,1_8} \frac{\epsilon_{3-1}\delta_{1,2}+\epsilon_{4-1}\delta_{1,2}+\epsilon_{3-8}\delta_{1,1}+\epsilon_{4-8}\delta_{1,1}}{4} \\ &= -bd\frac{1}{2} \end{aligned}$$

The character variety of a Riemann sphere with three holes and two cusps on one boundary is 7-dimensional (rather than 2-dimensional like in PVI case). The fat-graph admits a complete cusped lamination as displayed in the figure below. A full set of coordinates on the character variety is given by the five elements in the lamination and the two parameters G_1 and G_2 which determine the perimeter of the two non-cusped holes.

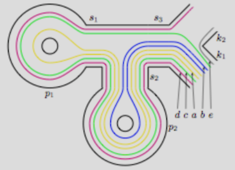

FIGURE 6. The system of arcs for PV.

Notice that there are two shear coordinates associated to the two cusps, they are denoted by k_1 and k_2, their sum corresponds to what we call p_3 above.
These shear coordinates do not commute with the other ones, they satisfy the following relations:

$$\{s_3, k_1\} = \{k_1, k_2\} = \{k_2, s_3\} = 1.$$

As a consequence in the character variety, the elements G_3 and G_∞ are not Casimirs.
In terms of shear coordinates, the elements in the lamination are expressed as follows:

$$\begin{aligned} a &= e^{k_1+s_1+2s_2+s_3+\frac{p_1}{2}+p_2}, \qquad b = e^{k_1+s_2+s_3+\frac{p_2}{2}}, \qquad e = e^{\frac{k_1}{2}+\frac{k_2}{2}}, \\ c &= e^{k_1+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}, \qquad d = e^{\frac{k_1}{2}+\frac{k_2}{2}+s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}}. \end{aligned} \tag{6}$$

They satisfy the following Poisson relations:

$$a \qquad \{a,b\}=ab, \quad \{a,c\}=0, \quad \{a,d\}=-\frac{1}{2}ad, \quad \{a,e\}=\frac{1}{2}ae, \tag{7}$$

$$\{b,c\}=0, \quad \{b,d\}=-\frac{1}{2}bd, \quad \{b,e\}=\frac{1}{2}be, \tag{8}$$

$$\{c,d\}=-\frac{1}{2}cd, \quad \{c,e\}=\frac{1}{2}ce, \quad \{d,e\}=0, \tag{9}$$

so that the element $G_3 G_\infty = de$ is a Casimir.

The symplectic leaves are determined by the values of the three Casimirs G_1, G_2 and $G_3 G_\infty$.
On each symplectic leaf, the PV monodromy manifold (5) is the subspace defined by those functions of a, b, c (and of the Casimir values $G_1,\ G_2,\ G_3 G_\infty$) which commute with $G_3 = e$. To see this, we can use relations (6) to determine the exponentiated shear coordinates in terms of a, b, c, d, e and then deduce he expressions of x_1, x_2, x_3 in terms of the lamination. We obtain the following expressions:

$$x_1 = -e\frac{a}{c} - d\frac{b}{c}, \qquad x_2 = -e\frac{b}{c} - G_1 d\frac{b}{a} - d\frac{b^2}{ac} - d\frac{c}{a}, \tag{10}$$

$$x_3 = -G_2\frac{c}{b} - G_1\frac{c}{a} - \frac{b}{a} - \frac{c^2}{ab} - \frac{a}{b}, \tag{11}$$

which automatically satisfy (5).

Due to the Poisson relations (7) the functions that commute with e are exactly the functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. Such functions may depend on the Casimir values G_1, G_2 and $G_3 G_\infty$ and e itself, so that $d = G_\infty$ becomes a parameter now. With this in mind, it is easy to prove that x_1, x_2, x_3 are algebraically independent functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ so that x_1, x_2, x_3 form a basis in the space of functions which commute with e.

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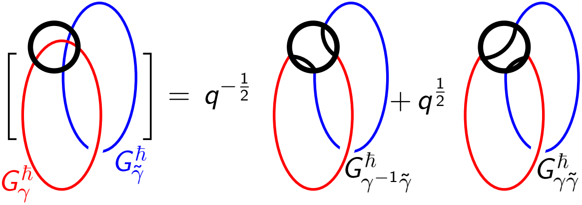
Quantisation

For standard geodesic lengths $G_\gamma \to G_\gamma^\hbar$ [Chekhov-Fock '99]:



$$[G_\gamma^\hbar, G_{\tilde\gamma}^\hbar] = q^{-\frac{1}{2}} G_{\gamma^{-1}\tilde\gamma}^\hbar + q^{\frac{1}{2}} G_{\gamma\tilde\gamma}^\hbar$$

Quantisation

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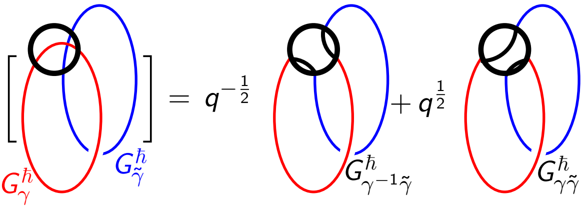

$$[G^\hbar_\gamma, G^\hbar_{\tilde\gamma}] = q^{-\frac{1}{2}}\, G^\hbar_{\gamma^{-1}\tilde\gamma} + q^{\frac{1}{2}}\, G^\hbar_{\gamma\tilde\gamma}$$

For arcs $g_{s_i,t_j} \to g^\hbar_{s_i,t_j}$:

$$q^{\mathcal{I}_{s_i,t_j,p_r,q_l}}\, g^\hbar_{s_i,t_j} g^\hbar_{p_r,q_l} = g^\hbar_{p_r,q_l} g^\hbar_{s_i,t_j}\, q^{\mathcal{I}_{p_r,q_l,s_i,t_j}}$$

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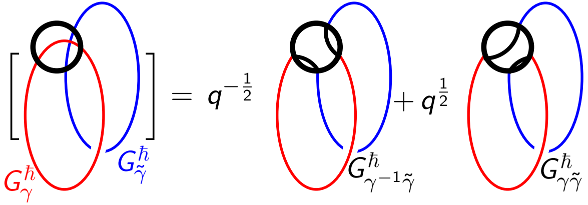

$$[G^\hbar_\gamma, G^\hbar_{\tilde\gamma}] = q^{-\frac{1}{2}} G^\hbar_{\gamma^{-1}\tilde\gamma} + q^{\frac{1}{2}} G^\hbar_{\gamma\tilde\gamma}$$

For arcs $g_{s_i,t_j} \to g^\hbar_{s_i,t_j}$:

$$q^{\mathcal{I}_{s_i,t_j,p_r,q_l}} g^\hbar_{s_i,t_j} g^\hbar_{p_r,q_l} = g^\hbar_{p_r,q_l} g^\hbar_{s_i,t_j} q^{\mathcal{I}_{p_r,q_l,s_i,t_j}}$$

This identifies the geometric basis of the quantum cluster algebras introduced by Berenstein - Zelevinsky**.**

Quantization-2

To produce the quantum Painlevé cubics, we introduce the Hermitian operators S_1, S_2, S_3 subject to the commutation inherited from the Poisson bracket of $\tilde{s}_i$:

$$[S_i, S_{i+1}] = i\pi\hbar\{\tilde{s}_i, \tilde{s}_{i+1}\} = i\pi\hbar, \quad i = 1, 2, 3, \ i + 3 \equiv i.$$

Observe that thanks to this fact, the commutators $[S_i, S_j]$ are always numbers and therefore we have

$$\exp(aS_i)\exp(bS_j) = \exp\left(aS_i + bS_i + \frac{ab}{2}[S_i, S_j]\right),$$

for any two constants a, b. Therefore we have the Weyl ordering:

$$e^{S_1+S_2} = q^{\frac{1}{2}} e^{S_1} e^{S_2} = q^{-\frac{1}{2}} e^{S_2} e^{S_1}, \quad q \equiv e^{-i\pi\hbar}.$$

Quantization-2

Theorem

(L. Chekhov-M. Mazzocco-V.R)
Denote by X_1, X_2, X_3 the quantum Hermitian operators corresponding to x_1, x_2, x_3 as above. The quantum commutation relations are:

$$q^{-\frac{1}{2}} X_i X_{i+1} - q^{\frac{1}{2}} X_{i+1} X_i = \left(\frac{1}{q} - q\right) \epsilon_k^{(d)} X_k - (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \omega_k^{(d)} \quad (12)$$

where $\epsilon_i^{(d)}$ and $\omega_i^{(d)}$ are the same as in the classical case. The quantum operators satisfy the following quantum cubic relations:

$$q^{\frac{1}{2}} X_3 X_1 X_2 - q \epsilon_3^{(d)} X_3^2 - q^{-1} \epsilon_1^{(d)} X_1^2 - q \epsilon_2^{(d)} X_2^2 +$$

$$q^{\frac{1}{2}} \epsilon_3^{(d)} \omega_3 X_3 + q^{-\frac{1}{2}} \omega_1^{(d)} X_1 + q^{\frac{1}{2}} \omega_2^{(d)} X_2 - \omega_4^{(d)} = 0.$$

Quantization-2

The Hermitian operators X_1, X_2, X_3 corresponding to x_1, x_2, x_3 are introduced as follows: consider the classical expressions for x_1, x_2, x_3 in terms of s_1, s_2, s_3 and p_1, p_2, p_3. Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version. For example (the case $\tilde{D}_5$): the classical x_1 is

$$x_1 = -e^{s_2+s_3} - e^{-(\tilde{s}_2+\tilde{s}_3)} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2},$$

and its quantum version is defined as

$$X_1 = -e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - e^{S_3-S_2} - e^{S_3+S_2} =$$

$$-e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - q^{-1/2}e^{-S_2}e^{S_3} - q^{1/2}e^{S_2}e^{S_3}.$$

Quantization-2

- Our theorem and close results of Marta Mazzocco show that we can interpret the Cherednik algebra and their close "relatives" as a quantisation of the (group algebra of the) monodromy group of the Painlevé equations. Here $q := e^{-i\pi\hbar}$ and $q^n \neq 1$.

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- The Askey-Wilson $AW(3)$ (or Zhedanov algebra) can be obtained from (12) for a special constant choice after a proper "rescaling".

"Physical Motivations"

- Standard Model$SU(3) \times SU(2) \times U(1)$ Gauge Theory

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- $D-$brane world: live on $D3-$brane $\perp$ $6D-$affine variety $\mathcal{M}$. $1+3D-$world-volume with SUSY YM and product gauge group.

$D-$brane algebras and superpotentials. Basic principles:

- One can associate an algebra to the category of $D-$branes at a singular point p. In every known example, the collection of possible $D-$branes at p can be described as a collection of QFT with the same Lagrangian for each of the theories.

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- More precisely, one does specify the "matter representation" (as a collection of adjoint and bifundamental fields for the gauge groups G_i) and one specifies a superpotential $W-$ the trace of a polynomial in the matter fields.
- To such data one can assign a quiver whose vertices label the groups G_i and whose directed edges specify the bifundamental and adjoint fields in the matter representation.

Quiver Theory

- Action

$$\int d^4x[\int d^4\theta\Psi_i^\dagger e^V\Psi_i+(\frac{1}{4g^2}\int d^2\theta \mathrm{Tr}\mathcal{W}_\alpha\mathcal{W}^\alpha+\int \mathrm{d}^2\theta \mathrm{W}(\bar{\psi})+\mathrm{h.c.})]$$

W = superpotential;

$V(\varphi_i;\bar{\varphi}_i)=\sum_i\mid\frac{\partial W}{\partial\varphi_i}\mid^2+\frac{g^2}{4}(\sum_i q_i\mid\varphi_i\mid^2)^2$

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- Encode in a Quiver:
k nodes $\iff \mathcal{V}^{n_1}, \ldots \mathcal{V}^{n_k} \iff \prod_{j=1}^k U(n_j)$ gauge group;
Each arrow $i \to j \iff$ bifundamental fields X_{ij} of $U(n_i) \times U(n_j)$;
Each loop $i \to i \iff$ adjoint fields φ_i of $U(n_i)$;
Superpotential $W \iff$ linear combination of cycles: $\sum_i c_i$ gauge invariant operators;
Relations $\iff$ jacobian of $W(\varphi_i, X_{ij})$.
Vacuum:$\rightsquigarrow V(\varphi_i; \bar{\varphi}_i) = 0 \Rightarrow \frac{\partial W}{\partial \varphi_i} = 0; \sum_i q_i \mid \varphi_i \mid^2 = 0$.

Superpotential algebra

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- These are the algebra relations dictated by $\frac{\partial W}{\partial X_j}$. So, given a field theory description of the family of D-branes in the form above, the D-brane algebra is

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- This is called a superpotential algebra, which is a Calabi - Yau algebra.

Elementary example

- First example, we consider the case in which P is a smooth point. In physics language, the conformal fields theory is the $N = 4$ SUSY Yang-Mills theory, written in $N = 1$ language. The $N = 4$ gauge multiplet decomposes as an $N = 1$ gauge multiplet plus three complex scalar fields X, Y, Z each transforming in the adjoint representation of the group.

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- Thus, we find

$$\mathcal{A} = \mathbb{C}[X, Y, Z],$$

the (commutative) polynomial algebra in three variables.

Example 2. Sklyanin algebra-1

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- This algebra denotes by $Q_3(\mathcal{E}, a, b, c)$ where $(a, b, c) \in \mathbb{C}^3$ such that $Q_3(\mathcal{E}, a, b, c) = \mathbb{C} < X, Y, Z > / J_W$ with

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- The ideal J_W can be written as a non-commutative jacobian ideal $J_W = < \partial_X, \partial_Y, \partial_Z > \in \mathbb{C} < X, Y, Z >$ for superpotential

$$W = aXYZ + bYXZ + c(X^3 + Y^3 + Z^3)$$

Example 2. Sklyanin algebra-2

- Here we consider W as a cyclic word of three variables X, Y, Z, i.e. like an element of the quotient $A_\natural := \mathbb{C} < X, Y, Z > /[\mathbb{C} < X, Y, Z >, \mathbb{C} < X, Y, Z >]$ with

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- cyclic derivatives $\partial_X, \quad \partial_Y, \quad \partial_Z$ where

$$\partial_j : A_\natural \to \mathbb{C} < X, Y, Z >, j = X, Y, Z$$

defines for any cyclic word $\varphi \in A_\natural$ by

$$\partial_j \varphi := \sum_{k | i_k = j} X_{i_k+1} X_{i_k+2} \dots X_{i_N} \dots X_{i_1} X_{i_2} .. X_{i_k-1} \in \mathbb{C} < X, Y, Z >$$

Example 2. Sklyanin algebra-3

Etingof-Ginzburg:

- One can identify the Sklyanin algebra $Q_3(\mathcal{E}, 1, -q, \frac{c}{3})$ with the flat deformation of the Poisson algebra $(\mathbb{C}[x, y, z], \{-, -\}_\varphi)$ as above with $\varphi = \frac{1}{3}(x^3 + y^3 + z^3) + \tau xyz$ and $W = XYZ - qYXZ + \frac{c}{3}(X^3 + Y^3 + Z^3)$.

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- The coordinate ring $B_\varphi = \mathbb{C}[x, y, z]/\varphi\mathbb{C}[x, y, z]$ of the affine surface $\varphi = 0$ inherits a Poisson algebra structure.
- There is a degree 3 central element $\Phi \in Z(Q_3(\mathcal{E}, 1, -q, \frac{c}{3}))$ and the quotient of the Sklyanin 3-Calabi-Yau algebra by two-sided ideal $< \Phi >$ is a flat deformation of the Poisson algebra B_φ.

Superpotentials of marginal and relevant deformations-1

- There is a "physical interpretation" of the Sklyanin superpotential (Berenstein-Leigh) as a marginal deformation of the superpotential from the Example 1:

$$W + W_{marg} =$$

$$= g\mathrm{tr}(\mathrm{X}[\mathrm{Y},\mathrm{Z}]) + \mathrm{tr}(\mathrm{aXYZ} + \mathrm{bYXZ} + \frac{\mathrm{c}}{3}(\mathrm{X}^3 + \mathrm{Y}^3 + \mathrm{Z}^3)) \in \mathrm{A}_\natural.$$

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- The structure of the vacua of D-brane gauge theories relates to the Non-Commutative Geometry also via another superpotentials (relevant deformations) having the form

$$W_{rel} = \mathrm{tr}(\frac{\mathrm{m}_1}{2}\mathrm{X}^2 + \frac{\mathrm{m}_2}{2}(\mathrm{Y}^2 + \mathrm{Z}^2) + \mathrm{e}_1\mathrm{X} + \mathrm{e}_2\mathrm{Y} + \mathrm{e}_3\mathrm{Z})$$

Superpotentials of marginal and relevant deformations-2

- The "vacua" of the theory with $W_{tot} = W + W_{marg} + W_{tel}$ superpotential corresponds to solutions of

$$\partial_i W_{tot} = 0, i = X, Y, Z.$$

Superpotentials of marginal and relevant deformations-2

- The "vacua" of the theory with $W_{tot} = W + W_{marg} + W_{tel}$ superpotential corresponds to solutions of

$$\partial_i W_{tot} = 0, i = X, Y, Z.$$

- The defining equations (for $a = 1, b = -q$):

$$\left\{ \begin{array}{rl} X_1X_2 - qX_2X_1 = & -cX_3^2 - m_2X_3 - e_3 \\ X_2X_3 - qX_3X_2 = & -cX_1^2 - m_1X_1 - e_1 \\ X_3X_1 - qX_1X_3 = & -cX_2^2 - m_2X_2 - e_2 \end{array} \right. \tag{13}$$

This relations contain our (12) (again, after a special constant choice and a "rescaling").

Etingof-Ginzburg ideology-1:

- Let $M = \mathbb{C}^3$ considering as the simplest Calabi-Yau manifold and $\varphi \in \mathcal{A} = \mathbb{C}[x_1, x_2, x_3]$ defines the Poisson bracket of jacobian type as above.

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- $M_\varphi : \varphi(x_1, x_2, x_3) = 0$ is an affine surface in M and the coordinate ring $\mathcal{B}_\varphi := \mathbb{C}[M_\varphi] = \mathcal{A}/(\varphi)$ is a commutative Poisson algebra with the structure induced by φ

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- Let $\varphi^{\tau,\nu} = \tau x_1 x_2 x_3 + \frac{\nu}{3}(x_1^3 + x_2^3 + x_3^3) + P(x_1) + Q(x_2) + R(x_3) = 0$ be the family of affine surfaces containing the E_6 del Pezzo. Here $\mathrm{degP}, \mathrm{degQ}$ and $\mathrm{degQ} < 3$.

Etingof-Ginzburg ideology-2:

- Let $A = \mathbb{C} < X_1, X_2, X_3 >$ and $A_\natural$ be defined as above and $\Phi^{q,\nu}_{P,Q,R} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) + P(X_1) + Q(X_2) + R(X_3) \in A_\natural$

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- $\mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})$ is a filtered algebra defined by three inhomogeneous "jacobian" relations:

$$X_iX_j - qX_jX_i = \nu X_k^2 + \frac{\mathrm{d}P(Q,R)}{\mathrm{d}X_k}, (i,j,k) = (1,2,3) \quad (14)$$

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- The superpotential $\Phi^{q,\nu}_{P,Q,R} = \Phi^{q,\nu} + \Phi_{P,Q,R}$ where $\Phi^{q,\nu} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) \in A_\natural^{(3)}$ and $\Phi_{P,Q,R} \in A_\natural^{(\leq 2)}$ is a 3-CY-superpotential (for generic parameters)

Etingof-Ginzburg ideology-3:

$$\begin{tikzcd} \mathcal{A}_\varphi \arrow[r, squiggly, "\text{fl. def.}"] \arrow[d] & \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R}) \arrow[d] \\ \mathcal{B}_\varphi \arrow[r, squiggly] & B(\Phi^{q,\nu}_{P,Q,R}, \Psi) = \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})/(\Psi). \end{tikzcd}$$

In our case $\Phi^{q,0}_{P,Q,R} := X_1X_2X_3 - qX_2X_1X_3$

$$\Psi^{q,\epsilon,\omega} = X_1X_2X_3 - q^2X_2X_1X_3 + \epsilon_1^{(d)}\frac{q-1}{\sqrt{q}}X_1^2 + \epsilon_2^{(d)}q^{3/2}(q-1)X_2^2 + \epsilon_3^{(d)}\frac{q^3-1}{\sqrt{q}}X_3^2 - -\omega_1^{(d)}(q-1)X_1 - \omega_2^{(d)}q(q-1)X_2 - \omega_3^{(d)}(q^2-1)X_3 \tag{15}$$

Conclusion

- A Riemann surface of genus g, n holes and k cusps on the boundary admits a complete cusped lamination of $6g-6+2n+2k$ arcs which triangulate it.

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Many thanks for your attention!!!