Some examples of Hopf algebroids and generalizations

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While classical symmetries are usually described by groupoids and their infinitesimal and higher categorical analogues, quantum theory allows for noncommutative analogues: (weak, quasi-)Hopf algebras, Hopf algebroids and some higher versions.

Noncommutative geometry replaces a space by the collection (set, algebra, category, cohomology ring...) of objects (functions, modules, sheaves, over the would be space.

Sometimes a categorification as well: rings replaced by categories of representations (reconstruction theorems), strict maps by weak maps/HS/Morita.

Groupoid – a small category where all morphisms are invertible. Can be generalized internally (smooth groupoids as internal in the category of smooth manifolds etc.). Important in geometry, e.g. orbifolds correspond to Morita equivalence classes of proper etale groupoids.

Transformation (action) groupoids. M a topological space, G a topological group, and $G \times M \to M$ a continuous action given. Objects of the action groupoid are elements of M and

$$\mathrm{Mor}(m, m') = \{(m, g) \in M \times G, \mid gm = m'\}.$$

1993 Lu and Weinstein: what is the analogue of action groupoids in deformation quantization. 1994 J-H. Lu: *scalar extension Hopf algebroids*. G replaced by a Hopf algebra, M by a *braided commutative monoid in the category of Yetter-Drinfeld modules* over M. However, their examples finite dimensional.

Multiplication and comultiplication

Functions multiply and add **pointwise**, therefore functions on a space form a commutative algebra. An affine variety will be below replaced by the algebra of regular functions (affine Serre theorem). All morphisms dualize.

In general, if a space *M* in some category is replaced by a function algebra $\mathrm{Fun}(M)$; a function algebra on a group object $M = G$ has additional **comultiplication** Δ, dual multiplication: $\Delta(f)(x \otimes y) := f(x \cdot y)$ for $f \in \mathrm{Fun}(G \times G) \cong \mathrm{Fun}(G)\hat{\otimes}\mathrm{Fun}(G)$. When X affine algebraic group, the usual $\otimes$.

Reminder: bialgebras

Bialgebra B – associative algebra $(\mathrm{B}, \mathrm{m}, \eta)$ and a coalgebra: has comultiplication $\Delta : \mathrm{H} \to \mathrm{H} \otimes \mathrm{H}$ which is coassociative

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

and counital: $\exists \epsilon : \mathrm{H} \to \mathbb{C}$,

$$(\epsilon \otimes \mathrm{id}) \circ \Delta \cong \mathrm{id} \cong (\mathrm{id} \otimes \epsilon) \circ \Delta.$$

Compatibility: Δ, ϵ homomorphisms of algebras.
In the case of a group $\mathrm{X} = \mathrm{G}$, $\epsilon(\mathrm{f}) = \mathrm{f}(1_{\mathrm{G}})$, $\mathrm{f} \in \mathrm{Fun}(\mathrm{G})$.

Sweedler notation

Sweedler notation: $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

$$\sum \sum a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum \sum a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}$$

so we write simply

$$\sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

"only the order matters"

Hopf algebra of functions

A **Hopf algebra** is a bialgebra $(B, m, \eta, \Delta, \epsilon)$ with an antipode map $S : B \to B^{op}$,

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon = m \circ (id \otimes S) \circ \Delta.$$

For a group G, $(Sf)(g) = f(g^{-1})$, $g \in G$, $f \in Fun(G)$

(Co)modules

Algebras have actions, **modules**: $\nu : \mathrm{A} \otimes \mathrm{M} \to \mathrm{M}$.
Coalgebras have coactions, **comodules**: $\rho : \mathrm{M} \to \mathrm{M} \otimes \mathrm{C}$.
Extend Sweedler to $\rho(\mathrm{m}) = \sum \mathrm{m}_{(0)} \otimes \mathrm{m}_{(1)}$.
Modules over bialgebras have a **tensor product**: action given by $\nu(\mathrm{a}, \mathrm{m} \otimes \mathrm{n}) = \sum \nu_{\mathrm{M}}(\mathrm{a}_{(1)}, \mathrm{m}) \otimes \nu_{\mathrm{N}}(\mathrm{a}_{(2)}, \mathrm{n})$; dually comodules over bialgebras have a tensor product. Over Hopf algebras we also have duals (via antipode).
In physics, comultiplication needed so that the Hilbert space of **multiparticle states** inherits symmetry via tensor product of representations and quantum numbers appropriately “add”.

Comodule algebras as quantum spaces

- M space, G group, then an action $\mathrm{G} \times \mathrm{M} \to \mathrm{M}$ dualizes to a coaction $\rho : \mathrm{Fun}(\mathrm{M}) \to \mathrm{Fun}(\mathrm{G} \times \mathrm{M}) \cong \mathrm{Fun}(\mathrm{G}) \hat{\otimes} \mathrm{Fun}(\mathrm{M})$.
- Coaction is moreover here an algebra map

$$\rho(\mathrm{ab}) = \rho(\mathrm{a})\rho(\mathrm{b})$$

"$\mathrm{Fun}(\mathrm{X})$ is a left **comodule algebra** over $\mathrm{Fun}(\mathrm{G})$"

- More generally, we think of *noncommutative* left and right comodule algebras over Hopf algebras as **quantum** G**-spaces**.

Thus, one expects to make quantum transformation groupoids out of a Hopf algebra H and an arbitrary H-comodule algebra U. However, one needs some sort of a braiding present (discussion).

Say $\mathcal{G}$ is a groupoid. Inertia groupoid $\mathrm{I}\mathcal{G}$ has objects loops of $\mathcal{G}$, i.e. morphisms of the type $\mathrm{f} : \mathrm{a} \to \mathrm{a}$. Morphisms from $\mathrm{f} : \mathrm{a} \to \mathrm{a}$ to $\mathrm{g} : \mathrm{b} \to \mathrm{b}$ are the commuting squares of the form

$$\begin{tikzcd} \mathrm{a} \arrow[r, "\mathrm{f}"] \arrow[d, "\mathrm{u}"'] & \mathrm{a} \arrow[d, "\mathrm{u}"] \\ \mathrm{b} \arrow[r, "\mathrm{g}"'] & \mathrm{b} \end{tikzcd}$$

If $\mathcal{G}$ is a groupoid in geometry (say orbifold) then sheaves over $\mathcal{G}$ have a tensor product (push external tensor product of sheaves which lives over $\mathcal{G} \times \mathcal{G}$ along multiplication).

Sheaves over $\mathrm{I}\mathcal{G}$ are the *equivariant* sheaves for the adjoint action $(\mathrm{g}, \mathrm{h}) \mapsto \mathrm{ghg}^{-1}$ (group case).
Theorem (Conjecture ZŠ 2002, proved Hinich 2004, ZŠ 2004): (roughly) the Drinfeld **center** of the category of sheaves over the orbifold is equivalent to the category of equivariant sheaves over the inertia orbifold. Used by Ben Zvi, Nadler etc.

Why center $Z(\mathcal{C})$ of a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes)$?
It is *braided* monoidal ($\otimes$ almost commutes
$R_{V,W} : V \otimes W \to W \otimes V$ iso with coherences/QYBE).
Objects (X, ϕ), $\phi : X \otimes (-) \to (-) \otimes X$ natural iso,
$\phi_Z : X \otimes Z \to Z \otimes X$ such that $\phi_{Y \otimes Z}$ equals the composition

$$X \otimes Y \otimes Z \xrightarrow{\phi_Y \otimes \mathrm{id}_Z} Y \otimes X \otimes Z \xrightarrow{\mathrm{id}_Y \otimes \phi_Z} Y \otimes Z \otimes X$$

$$\mathrm{Hom}_{Z(\mathcal{C})}((X, \phi), (Y, \psi)) = \{X \xrightarrow{f} Y \mid \forall Z,\ (\mathrm{id}_Z \otimes f) \circ \phi_Z = \psi_Z \circ (f \otimes \mathrm{id}_Z)\}$$

$$(X, \phi) \otimes_{Z(\mathcal{C})} (Y, \psi) := (X \otimes Y, (\phi_Z \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \psi_Z))$$

For the category of Hopf modules ${}_{\mathrm{H}}\mathcal{M}^{\mathrm{H}}$ (compatible module + comodule = equivariant sheaf, cf. Lunts, ZŠ) the center is the category of *Yetter-Drinfeld modules* ${}_{\mathrm{H}}\mathcal{YD}^{\mathrm{H}}$. Left H-action $\blacktriangleright$, right H-coaction $\mathrm{X} \mapsto \mathrm{X}_{[0]} \otimes \mathrm{X}_{[1]}$ and

$$(\mathrm{h}_{(1)} \blacktriangleright \mathrm{X}_{[0]}) \otimes \mathrm{h}_{(2)}\mathrm{X}_{[1]} = (\mathrm{h}_{(2)} \blacktriangleright \mathrm{X})_{[0]} \otimes (\mathrm{h}_{(2)} \blacktriangleright \mathrm{X})_{[1]}\mathrm{h}_{(1)},$$

for all $\mathrm{h} \in \mathrm{H}$ and $\mathrm{X} \in \mathrm{M}$.

Yetter (knot theory, TFTs), Drinfeld (quantum groups). Later Radford-Towber (over bialgebras), Majid, Semikhatov etc. Self-dual anti-Yetter Drinfeld modules are the coefficients for Hopf-cyclic (co)homology (foliations Connes-Moscovici, Hajac-Rangipour etc., monadic ZŠ, Böhm-Stefan, Kaygun, Kaledin, Kowalzig).

It appears (J-H. Lu) that for quantum action groupoids we need braided commutative Yetter-Drinfeld module algebras!

H Hopf algebra, U an algebra. We say that the action $\blacktriangleright : \mathrm{H} \otimes \mathrm{U} \to \mathrm{U}$ is **Hopf** (or that U is a left H-**module algebra**) if

- $\mathrm{h} \blacktriangleright (\mathrm{uv}) = \sum (\mathrm{h}_{(1)} \blacktriangleright \mathrm{u})(\mathrm{h}_{(2)} \blacktriangleright \mathrm{v})$ (generalized Leibniz rule)
- $\mathrm{h} \blacktriangleright 1_{\mathrm{U}} = \epsilon(\mathrm{h}) 1_{\mathrm{U}}$ (unitality).

Semidirect (smash) product algebra $\mathrm{U}\sharp\mathrm{H}$ is the tensor product **k**-module $\mathrm{U} \otimes \mathrm{H}$ with the multiplication

$$(\mathrm{u}\sharp\mathrm{h})(\mathrm{v}\sharp\mathrm{k}) = \sum \mathrm{u}(\mathrm{h}_{(1)} \blacktriangleright \mathrm{v})\sharp\mathrm{h}_{(2)}\mathrm{k}$$

for all $\mathrm{h}, \mathrm{k} \in \mathrm{H}$, $\mathrm{u}, \mathrm{v} \in \mathrm{U}$.

If the antipode $S : H \to H^{op}$ is invertible, the YD condition is equivalent to

$$(f \blacktriangleright X)_{[0]} \otimes (f \blacktriangleright X)_{[1]} = (f_{(2)} \blacktriangleright X_{[0]}) \otimes f_{(3)} X_{[1]} S^{-1}(f_{(1)})$$

In the corresponding smash product $U \sharp H$, this is similar to a conjugation action.

Conceptually, an associative A-bialgebroid is a ring H such that $\mathrm{H}-\mathrm{Mod}$ has a structure of additive monoidal category with faithful exact strict monoidal functor to $\mathrm{A}\otimes\mathrm{A}^{\mathrm{op}}$-modules.

In other words, H can be obtained from (nonabelian but additive) Tannaka reconstruction theorem from A (this is a characterization theorem if we take a definition in terms of structural maps below).

Explicitly, comprises two algebras, the **base algebra** A and the **total algebra** $\mathrm{H} = (\mathrm{H}, \mu)$ which is an A-bimodule equipped with coassociative coproduct $\Delta : \mathrm{H} \to \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$ with a counit ϵ which are understood as maps in the category of bimodules – we say that H is an A-**coring** (cocategory).
A is a generalization of the field of units for H: equipped with a **source map** $\alpha : \mathrm{A} \to \mathrm{H}$ and a **target map** $\beta : \mathrm{A}^{\mathrm{op}} \to \mathrm{H}$ which are algebra maps with commuting images $[\alpha(\mathrm{a}), \beta(\mathrm{a}')] = 0$ that is $\mathrm{a}, \mathrm{a}' \in \mathrm{A}$; we sometimes say that H is an $\mathrm{A} \otimes \mathrm{A}^{\mathrm{op}}$-**ring**.

Definition of bialgebroid

An $A \otimes A^{op}$-ring (H, μ, α, β) and an A-coring (H, Δ, ϵ) on the same A-bimodule H form a **left** A**-bialgebroid** $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ if they satisfy the following compatibilies:

- (C1) the underlying A-bimodule structure of the A-coring structure is determined by the source and target map (part of the $A \otimes A^{op}$-ring structure): $r.a.r' = \alpha(r)\beta(r')a$.
- (C2) formula $\sum_\lambda h_\lambda \otimes f_\lambda \mapsto \epsilon(\sum_\lambda h_\lambda \alpha(f_\lambda))$ defines an action $\blacktriangleright: H \otimes A \to A$ which extends the left regular action $A \otimes A \to A$ along the inclusion $A \otimes A \xrightarrow{\alpha \otimes A} H \otimes A$.
- (C3) the linear map $h \otimes (g \otimes k) \mapsto \Delta(h)(g \otimes k)$, $H \otimes (H \otimes H) \to H \otimes H$, induces a well defined action $H \otimes (H \otimes_A H) \to H \otimes_A H$.

$\mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$ is not an algebra by componentwise product in general, hence Δ can not be an algebra map. Indeed the kernel I_{A} of the projection $\mathrm{H} \otimes \mathrm{H} \to \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$ of A-bimodules is only the *right* ideal generated by $\beta(\mathrm{a}) \otimes 1 - 1 \otimes \alpha(\mathrm{a})$, for $\mathrm{a} \in \mathrm{A}$.

(C3) is equivalent to: $\exists$ A-subbimodule $\mathrm{H} \times_{\mathrm{A}} \mathrm{H} \subset \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$, the **Takeuchi product** containing $\mathrm{Im}\Delta$, with factorwise multiplication, and the corestriction $\Delta| : \mathrm{H} \to \mathrm{H} \times_{\mathrm{A}} \mathrm{H}$ is a homomorphism of algebras.

Relevant for physics: Many known examples of Hopf algebroids come from

- Weak Hopf algebras (Mack-Schomerus 1989: true hidden symmetries in CFTs)
- Inclusions of von Neumann depth 2 subfactors (AQFT), e.g. BMW algebra
- Deformed Heisenberg algebras
- κ-deformed *phase* spaces of Planck scale physics. But Snyder space not yet an example.

Some Hopf algebroids give rise to Tamarkin-Tsygan noncommutative differential calculi (Kowalzig) and related issues in cyclic and Hochschild (co)homology.

Basic example over commutative base:

- $\mathrm{A} = \mathrm{C}^\infty(\mathrm{M})$ where M is a smooth manifold.
- $\mathrm{H} = \mathcal{D}$ is the algebra of **differential operators** with smooth coefficients.
- $\Delta(\mathrm{D})(\mathrm{f}, \mathrm{g}) = \mathrm{D}(\mathrm{f} \cdot \mathrm{g})$, $\mathrm{f} \mapsto \mathrm{f} \otimes 1$.
- $\alpha = \beta$ is the canonical embedding of functions into differential operators; the counit is taking the constant term.
- $\blacktriangleright$ is the usual action of differential operators on functions.

Deformation quantization: Ping Xu extends $\mathrm{C}^\infty(\mathrm{M})$ to $\mathrm{C}^\infty(\mathrm{M})[[\mathrm{h}]]$ where h is a formal variable. Then $\mathcal{D}[[\mathrm{h}]]$ is a left A-bialgebroid by extending the scalars; there he implicitly considers the completed tensor product.

Theorem. (Xu 2000) If M is Poisson manifold and the formal bidifferential operator $\mathcal{F} \in \mathcal{D}[[\mathrm{h}]]$ defines a deformation quantization of M with the (natural, differential) star product $\mu\mathcal{F}(\mathrm{f} \otimes \mathrm{g})$. Then $\mathcal{F}$ is a Drinfeld twist for the left $\mathrm{C}^\infty(\mathrm{M})[[\mathrm{h}]]$-bialgebroid of formal power series in regular differential operators $\mathcal{D}[[\mathrm{h}]]$. Consequently, each deformatiom quantization defines also a deformation of that bialgebroid.

$\mathcal{F} \in \mathrm{H} \otimes_{\mathrm{A}} \mathrm{H}$ is a Drinfeld twist for a left A-bialgebroid $(\mathrm{H}, \mu, \alpha, \beta, \Delta, \epsilon)$ if the 2-cocycle condition

$$(\Delta \otimes_{\mathrm{A}} \mathrm{id})(\mathcal{F})(\mathcal{F} \otimes_{\mathrm{A}} 1) = (\mathrm{id} \otimes_{\mathrm{A}} \Delta)(\mathcal{F})(1 \otimes_{\mathrm{A}} \mathcal{F}) \qquad (1)$$

and the counitality $(\epsilon \otimes_{\mathrm{A}} \mathrm{id})(\mathcal{F}) = 1_{\mathrm{H}} = (\mathrm{id} \otimes_{\mathrm{A}} \epsilon)(\mathcal{F})$ hold.
In terms of $\mathcal{F}^{-1}$ we can alternatively write the condition

$$(\mathcal{F}^{-1} \otimes_{\mathrm{A}_\star} 1)(\Delta \otimes_{\mathrm{A}_\star} \mathrm{id})(\mathcal{F}^{-1}) = (1 \otimes_{\mathrm{A}_\star} \mathcal{F}^{-1})(\mathrm{id} \otimes_{\mathrm{A}_\star} \Delta)(\mathcal{F}^{-1}).$$

Use the Sweedler-like notation for twist $\mathcal{F} = \mathrm{f}^1 \otimes \mathrm{f}_1$.

Theorem by Ping Xu. If H is a left A-bialgebroid then the formula

$$\mathrm{a} \star \mathrm{b} = \mu\mathcal{F}(\blacktriangleright \otimes \blacktriangleright)(\mathrm{f} \otimes \mathrm{g}) = (\mathrm{f}^1 \blacktriangleright \mathrm{a})(\mathrm{f}_1 \blacktriangleright \mathrm{b}) \qquad (2)$$

defines an associative algebra $\mathrm{A}_\star = (\mathrm{A}, \star)$ structure on A with the same unit; the formulas $\alpha_{\mathcal{F}}(\mathrm{a}) = \alpha(\mathrm{f}^1 \blacktriangleright \mathrm{a})\mathrm{f}_1$ and $\beta_{\mathcal{F}}(\mathrm{a}) = \beta(\mathrm{f}^1 \blacktriangleright \mathrm{a})\mathrm{f}_1$ define respectively an algebra homomorphism and antihomomorphism $\mathrm{A}_\star \to \mathrm{H}$ turning H into a $\mathrm{A}_\star$-ring; H has twisted coproduct

$$\Delta_{\mathcal{F}} : \mathrm{H} \to \mathrm{H} \otimes_{\mathrm{A}_\star} \mathrm{H}, \qquad \Delta_{\mathcal{F}}(\mathrm{h}) = \mathcal{F}^{-1}\Delta(\mathrm{h})\mathcal{F}$$

is coassociative and counital with the same counit.
$\mathrm{H}_{\mathcal{F}} = (\mathrm{H}, \mu, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, \Delta_{\mathcal{F}}, \epsilon)$ is a left $\mathrm{A}_\star$-bialgebroid.

$$(\mathrm{a}\star\mathrm{b})\star\mathrm{c} = \mu(\mu\otimes\mathrm{id})[(\Delta_0\otimes\mathrm{id})\mathcal{F}](\mathcal{F}\otimes\mathrm{id})(\blacktriangleright\otimes\blacktriangleright\otimes\blacktriangleright)(\mathrm{a}\otimes\mathrm{b}\otimes\mathrm{c})$$

$$\mathrm{a}\star(\mathrm{b}\star\mathrm{c}) = \mu(\mathrm{id}\otimes\mu)[(\mathrm{id}\otimes\Delta_0)\mathcal{F}](\mathrm{id}\otimes\mathcal{F})(\blacktriangleright\otimes\blacktriangleright\otimes\blacktriangleright)(\mathrm{a}\otimes\mathrm{b}\otimes\mathrm{c})$$

Cocycle condition implies associativity but for the converse we need that the kernel of $(\blacktriangleright\otimes\blacktriangleright\otimes\blacktriangleright)(\mathrm{a}\otimes\mathrm{b}\otimes\mathrm{c})$ for all $\mathrm{a},\mathrm{b},\mathrm{c}$ is not bigger than $\mathrm{I}_{\mathrm{A}}^{(2)}$.

J-H. Lu (1994) discovered a noncommutative analogue of transformation groupoids.

- INPUT (essentially): (maybe finite dimensional) Hopf algebra H and braided commutative monoid U in ${}_{\mathrm{H}}\mathcal{YD}^{\mathrm{H}}$ (in dual sense, using two actions)
- OUTPUT: a Hopf algebroid with total algebra $\mathrm{U}\sharp\mathrm{H}$ and noncommutative base U

Improved by Brzeziński-Militaru (true YD, not dual, not finite dimensional), Böhm (symmetric version), Stojić (antipode condition, gap in BM proof for S bijective). Completed $\hat{\otimes}$ version, Stojić using ind-pro-objects (in progress).

From now on, $\mathfrak{g}$ a fin dim Lie **k**-algebra, $\mathrm{char}\,\mathbf{k} = 0$.
We would like to immitate the Heisenberg double for $\mathrm{U}(\mathfrak{g})$ with completed tensor products or find smaller variants without completion. Finite (Hopf/reduced) dual seem not to work, need some analytic functions $\mathcal{O}^\alpha_\beta$! However, those are regular functions on the automorphism group! We use this fact below, but first let us connect to Xu's story.
$\mathrm{e}_1, \ldots, \mathrm{e}_\mathrm{n}$ a basis, with structure constants $\mathrm{C}^\gamma_{\alpha\beta}$ given by

$$[\mathrm{e}_\alpha, \mathrm{e}_\beta] = \mathrm{C}^\gamma_{\alpha\beta}\mathrm{e}_\gamma, \quad \alpha, \beta, \gamma = 1, \ldots, \mathrm{n},$$

Let $\partial^1, \ldots, \partial^n$ be the dual basis of $\mathfrak{g}^*$, which are also (commuting) generators of $S(\mathfrak{g}^*)$. Let $\hat{S}(\mathfrak{g}^*)$ be the formal completion of $S(\mathfrak{g})$. We introduce an auxiliary matrix $\mathcal{C} \in M_n(\hat{S}(\mathfrak{g}^*))$ with entries

$$\mathcal{C}^\alpha_\beta := C^\alpha_{\beta\gamma}\partial^\gamma \ \in \hat{S}(\mathfrak{g}^*), \tag{3}$$

where we adopted the Einstein convention of understood summation over repeated indices. In this notation introduce the matrices $\mathcal{O} := \exp(\mathcal{C}) \in M_n(\hat{S}(\mathfrak{g}^*))$ and

$$\phi := \frac{-\mathcal{C}}{e^{-\mathcal{C}} - 1} = \sum_{N=0}^\infty \frac{(-1)^N B_N}{N!}\mathcal{C}^N, \qquad \tilde{\phi} := \frac{\mathcal{C}}{e^{\mathcal{C}} - 1} = \sum_{N=0}^\infty \frac{B_N}{N!}\mathcal{C}^N, \tag{4}$$

where B_N are the Bernoulli numbers.

By a simple comparison of the expressions (4) we obtain

$$\tilde{\phi}_{\alpha}^{\beta} = \phi_{\rho}^{\alpha} \mathcal{O}_{\beta}^{\rho}. \quad (5)$$

By $\hat{\mathrm{A}}_{\mathrm{n}}$ denote the completion by the degree of a differential operator of the n-th Weyl algebra A_{n} with generators $\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}}, \partial^1, \ldots, \partial^{\mathrm{n}}$. The underlying vector space of $\hat{\mathrm{A}}_{\mathrm{n}}$ is thus a completion of $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$.
Now define the elements $\hat{\mathrm{x}}^{\phi}, \hat{\mathrm{y}}^{\phi} \in \hat{\mathrm{A}}_{\mathrm{n}}$

$$\hat{\mathrm{x}}_{\rho}^{\phi} := \sum_{\tau} \mathrm{x}_{\tau} \phi_{\rho}^{\tau}, \quad \hat{\mathrm{y}}_{\rho}^{\phi} := \sum_{\tau} \mathrm{x}_{\tau} \tilde{\phi}_{\rho}^{\tau}. \quad (6)$$

Then $\hat{x}_\rho \mapsto \hat{x}^\phi_\rho$ extends to a unique algebra map $\alpha : U(\mathfrak{g}) \to \hat{A}_n$ and $\hat{x}_\rho \mapsto \hat{y}^\phi_\rho$ to a unique algebra map $\beta : U(\mathfrak{g})^{op} \to \hat{A}_n$. This *realization* map is related to the symmetrization (PBW) isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$; for other coalgebra isomorphisms we have different choice of ϕ (or different ordering). Our ϕ corresponds to symmetric ordering (Gutt star product).
From (5) it follows immediately that

$$\hat{y}^\phi_\alpha = \hat{x}^\phi_\beta \mathcal{O}^\beta_\alpha, \tag{7}$$

and one can also prove

$$[\hat{x}^\phi_\alpha, \hat{y}^\phi_\beta] = 0. \tag{8}$$

Instead of ϕ above we could take any homomorphism $\boldsymbol{\phi} : \mathfrak{g} \to \mathrm{Der}\, \mathrm{S}(\mathfrak{g})$ (bold!) and take $\phi^\alpha_\beta = \boldsymbol{\phi}(-\hat{\mathrm{x}}_\beta)(\partial^\alpha)$. They are in bijection with coalgebra isomorphisms between $\mathrm{U}(\mathfrak{g})$ and $\mathrm{S}(\mathfrak{g})$. Equivalently,

$$\boxed{(\delta_\rho \phi^\gamma_\mu)\phi^\rho_\nu - (\delta_\rho \phi^\gamma_\nu)\phi^\rho_\mu = \mathrm{C}^\sigma_{\mu\nu}\phi^\gamma_\sigma}$$

or the matrix of the inverses $1/\phi^\alpha_\beta$ satisfies the standard Maurer-Cartan equation. The connection with Hausdorff series and this fact suggest genralizations.

With appropriate completions implicit, $\mathrm{H}_{\mathfrak{g}} := \hat{\mathrm{A}}_{\mathrm{n}}$ is a Hopf $\mathrm{U}(\mathfrak{g})$-algebroid with coproduct Δ which on $\hat{\mathrm{S}}(\mathfrak{g}^*) \cong \mathrm{U}(\mathfrak{g})^*$ (identified via PBW map) agrees with the transpose of the multiplication in $\mathrm{U}(\mathfrak{g})$ and $\Delta(\mathrm{u}) = \mathrm{u} \otimes 1$ for $\mathrm{u} \in \mathrm{U}(\mathfrak{g})$. The source and target map are α and β above!
Alternatively, the map $\mathrm{U}(\mathfrak{g}) \to \hat{\mathrm{A}}_{\mathrm{n}}$ sends an element in $\mathrm{U}(\mathfrak{g})$ to an operator on $\hat{\mathrm{S}}(\mathfrak{g})$; this action is a right Hopf action and the total algebra H is the smash product of $\mathrm{U}(\mathfrak{g})$ and $\hat{\mathrm{S}}(\mathfrak{g}^*)$. This is however isomorphic as an algebra to $\hat{\mathrm{A}}_{\mathrm{n}}$. We shall thus identify $\hat{\mathrm{x}}_\mu \in \mathrm{U}(\mathfrak{g})$ and $\hat{\mathrm{x}}^\phi_\mu \in \hat{\mathrm{A}}_{\mathrm{n}}$ etc.

This Hopf algebroid can be reobtained from the Heisenberg-Weyl algebra (basic example over commutative base above) by twisting with a twist, which is in bigger generality (S. Meljanac) equal to

$$\mathcal{F} =: \exp\left(- \sum_{\alpha=1}^{\mathrm{n}} \mathrm{x}_\alpha (\Delta - \Delta_0) \partial^\alpha \right) :$$

In symmetric ordering fr $\mathrm{U}(\mathfrak{g})$ this formula can be diagrammatically expanded to a new formula for the Hausdorff series.

In symmetric ordering, the deformed coproduct Δ on $\hat{\mathrm{S}}(\mathfrak{g}^*) \cong \mathrm{U}(\mathfrak{g})^*$ is given by

$$\Delta \partial^\mu = 1 \otimes \partial^\mu + \partial^\alpha \otimes [\partial^\mu, \hat{\mathrm{x}}_\alpha] + \frac{1}{2}\partial^\alpha \partial^\beta \otimes [[\partial^\mu, \hat{\mathrm{x}}_\alpha], \hat{\mathrm{x}}_\beta] + \ldots$$

or, in symbolic form,

$$\Delta \partial^\mu = \exp(\partial^\alpha \otimes \mathrm{ad}\,(-\hat{\mathrm{x}}_\alpha))(1 \otimes \partial^\mu) = \exp(\mathrm{ad}(-\partial^\alpha \otimes \hat{\mathrm{x}}_\alpha))(1 \otimes \partial^\mu).$$

The last equality follows by noting that $[\partial^\alpha, 1] = 0$.

Corollary. In symmetric ordering, the deformed coproduct Δ on $\hat{\mathrm{S}}(\mathfrak{g}^*) \cong \mathrm{U}(\mathfrak{g})^*$ is also given by

$$\Delta \partial^\mu = \exp(\mathrm{ad}(\hat{\mathrm{y}}_\alpha \otimes \partial^\alpha))(\partial^\mu \otimes 1)$$

Using the Hadamard's formula $\mathrm{Ad}(\exp(\mathrm{A}))(\mathrm{B}) = \exp(\mathrm{adA})(\mathrm{B})$ we can reexpress the above formulas by

$$\Delta\partial^\mu = \exp(-\partial^\rho \otimes \hat{\mathrm{x}}_\rho)(1 \otimes \partial^\mu)\exp(\partial^\sigma \otimes \hat{\mathrm{x}}_\sigma) \tag{9}$$

$$\Delta\partial^\mu = \exp(\hat{\mathrm{y}}_\rho \otimes \partial^\rho)(\partial^\mu \otimes 1)\exp(-\hat{\mathrm{y}}_\sigma \otimes \partial^\sigma) \tag{10}$$

In particular, in the undeformed case when $\mathrm{C}^\lambda_{\mu\nu} = 0$ and $\hat{\mathrm{x}}_\alpha$, x_α and $\hat{\mathrm{y}}_\alpha$ coincide we obtain

$$\Delta_0\partial^\mu = \exp(-\partial^\alpha \otimes \mathrm{x}_\alpha)(1 \otimes \partial^\mu)\exp(\partial^\alpha \otimes \mathrm{x}_\alpha) \tag{11}$$

$$\Delta_0\partial^\mu = \exp(\mathrm{x}_\alpha \otimes \partial^\alpha)(\partial^\mu \otimes 1)\exp(-\mathrm{x}_\alpha \otimes \partial^\alpha) \tag{12}$$

Comparing the formulas for the deformed and for the undeformed case we obtain new formulas relating Δ_0 to Δ. Indeed, comparing (9) and (11) we obtain

$$\Delta(\partial^\mu) = \mathcal{F}_{\mathrm{L}}^{-1}\Delta_0(\partial^\mu)\mathcal{F}_{\mathrm{L}} \tag{13}$$

where $\mathcal{F}_{\mathrm{L}}$ is the product of the two exponentials:

$$\mathcal{F}_{\mathrm{L}} = \exp(-\partial^\rho \otimes \mathrm{x}_\rho)\exp(\partial^\sigma \otimes \hat{\mathrm{x}}_\sigma) \tag{14}$$

and similarly comparing (10) to (12) we obtain

$$\Delta(\partial^\mu) = \mathcal{F}_{\mathrm{R}}^{-1}\Delta_0(\partial^\mu)\mathcal{F}_{\mathrm{R}} \tag{15}$$

where

$$\mathcal{F}_{\mathrm{R}} = \exp(\mathrm{x}_\rho \otimes \partial^\rho)\exp(-\hat{\mathrm{y}}_\sigma \otimes \partial^\sigma) \tag{16}$$

The relations (14) and (16) suggest that $\mathcal{F}_{\mathrm{L}}$ and $\mathcal{F}_{\mathrm{R}}$ might be Drinfeld twists which twists the undeformed Hopf algebroid (Heisenberg algebra) to the Hopf algebroid from the Section 3. But so far we have just shown that it gives the correct formulas for $\Delta(\partial^\mu)$.
To show that $\mathcal{F}_{\mathrm{L}}$ is in fact a twist we prove analogous formulas for the rest of generators, say $\Delta(\mathrm{x}_\mu) = \mathcal{F}_{\mathrm{L}}^{-1}(\mathrm{x}_\mu \otimes 1)\mathcal{F}_{\mathrm{L}}$.
Applying "inner" exponentials (6) and (7) we easily get

$$\begin{aligned} \exp(\partial^\rho \otimes \mathrm{x}_\rho)(\mathrm{x}_\mu \otimes 1)\exp(-\partial^\sigma \otimes \mathrm{x}_\sigma) &= \mathrm{x}_\mu \otimes 1 + 1 \otimes \mathrm{x}_\mu \\ &= \mathrm{x}_\mu \otimes 1 + 1 \otimes \hat{\mathrm{y}}_\tau \mathcal{O}^\tau_\sigma (\phi^{-1})^\sigma_\mu. \end{aligned} \tag{17}$$

Now we need to apply outer exponentials to each of the two summands on the right hand side.

By induction on $\mathrm{k} = 0, 1, 2, \ldots$ one checks that

$$\mathrm{ad}^{\mathrm{k}}(\partial^\rho \otimes \hat{\mathrm{x}}_\rho)(1 \otimes \hat{\mathrm{x}}_\mu) = [(-\mathcal{C})^{\mathrm{k}}]^\tau_\mu \otimes \hat{\mathrm{x}}_\tau \tag{18}$$

Hadamard's formula and Eq. (18) imply

$$\exp(-\partial^\sigma \otimes \hat{\mathrm{x}}_\sigma)(\mathrm{x}_\mu \otimes 1)\exp(\partial^\rho \otimes \hat{\mathrm{x}}_\rho) = \mathrm{x}_\mu \otimes 1 - (\tilde{\phi}^{-1})^\tau_\mu \otimes \hat{\mathrm{x}}_\tau. \tag{19}$$

For the second summand on the right hand side of (17), using $[\hat{\mathrm{x}}_\sigma, \hat{\mathrm{y}}_\tau] = 0$ and $1 \otimes \hat{\mathrm{y}}_\tau \mathcal{O}^\tau_\sigma(\phi^{-1})^\sigma_\mu =$
$(1 \otimes \hat{\mathrm{y}}_\tau)\exp(\partial^\nu \otimes \hat{\mathrm{x}}_\nu)\exp(-\partial^\lambda \otimes \hat{\mathrm{x}}_\lambda)(1 \otimes \mathcal{O}^\tau_\sigma(\phi^{-1})^\sigma_\mu)$

we conclude that

$$\exp(-\partial^\sigma \otimes \hat{x}_\sigma)(1 \otimes \hat{y}_\tau \mathcal{O}^\tau_\mu) \exp(\partial^\rho \otimes \hat{x}_\rho) = (1 \otimes \hat{y}_\tau)\Delta(\mathcal{O}^\tau_\sigma (\phi^{-1})^\sigma_\mu) = \Delta(x_\mu), \tag{20}$$

where we used $1 \otimes \hat{y}_\tau = \Delta(\hat{y}_\tau)$. We obtained the additional $x_\mu \otimes 1 - (\tilde{\phi}^{-1})^\tau_\mu \otimes \hat{x}_\tau$, but this can be shown to be in the ideal! Indeed, the formula (6) gives $x_\mu = \hat{x}_\sigma (\phi^{-1})^\sigma_\mu$ and (7) gives $\tilde{\phi}^{-1} = \mathcal{O}\phi^{-1}$, while the right ideal $I_{U(\mathfrak{g})}$ is generated by $\hat{x}_\rho \otimes 1 - \mathcal{O}^\tau_\rho \otimes \hat{x}_\tau = (\hat{x}_\beta \otimes 1 - \mathcal{O}^\tau_\beta \otimes \hat{x}_\tau)((\phi^{-1})^\beta_\alpha \otimes 1)$. It is clear here that for the twist to work it is essential that the base is larger than the field.

Regarding that the map $\mathrm{H} \otimes_{\mathbf{k}} \mathrm{H} \to \mathrm{H} \otimes_{\mathbf{k}} \mathrm{H}$, $\mathrm{w} \mapsto \mathcal{F}\mathrm{w}\mathcal{F}^{-1}$ is a homomorphism of algebras our check for generators ∂^μ and x_α implies

Proposition. For every $\mathrm{h} \in \mathrm{H}_{\mathfrak{g}}$,

$$\Delta(\mathrm{h}) = \mathcal{F}_{\mathrm{L}}\Delta_0(\mathrm{h})\mathcal{F}_{\mathrm{L}}^{-1} + \mathrm{I}_{\mathrm{U}(\mathfrak{g})} = \mathcal{F}_{\mathrm{R}}\Delta_0(\mathrm{h})\mathcal{F}_{\mathrm{R}}^{-1} + \mathrm{I}_{\mathrm{U}(\mathfrak{g})},$$

where $\mathrm{I}_{\mathrm{U}(\mathfrak{g})}$ is the right ideal generated by $\beta(\mathrm{u}) \otimes 1 - 1 \otimes \alpha(\mathrm{u})$ for $\mathrm{u} \in \mathrm{U}(\mathfrak{g})$.

$$\mathcal{F}_{\mathrm{L}}(\mathrm{x}_\mu \otimes 1 - 1 \otimes \mathrm{x}_\mu)\mathcal{F}_{\mathrm{L}}^{-1} \in \mathrm{I}_{\mathrm{U}(\mathfrak{g})}$$

I now explain (descriptively, without slides):
From this a theorem follows (using that the joint kernel above is small) that $\mathcal{F}_{\mathrm{L}}$ and $\mathcal{F}_{\mathrm{R}}$ are twists for obtaining our Hopf algebroid from the basic example of Heisenberg algebra! This has vast generalizations. The exponentials above generalize to the canonical elements in infinite dimensional Heisenberg doubles.
Xu's example can localize horizontally. Coring framework can be used to generalize the concept of Hopf algebroid to a system of compatible algebroids (some problems for nc base).

k-linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ **automorphism** of Lie algebra $\mathfrak{g}$ if $[\psi(\mathrm{x}), \psi(\mathrm{y})] = \psi([\mathrm{x}, \mathrm{y}])$ what for $\psi(\mathrm{e}_\alpha) = \mathrm{e}_\beta \mathrm{M}^\beta_\alpha$ takes the form

$$[\mathrm{e}_\alpha \mathrm{M}^\alpha_\mu, \mathrm{e}_\beta \mathrm{M}^\beta_\nu] = \mathrm{C}^\gamma_{\alpha\beta} \mathrm{e}_\gamma \mathrm{M}^\alpha_\mu \mathrm{M}^\beta_\nu,$$

$$C^{\sigma}_{\mu\nu} M^{\gamma}_{\sigma} = C^{\gamma}_{\alpha\beta} M^{\alpha}_{\mu} M^{\beta}_{\nu} \tag{21}$$

hence $\mathrm{Aut}\mathfrak{g}$ can be identified with the affine algebraic subgroup of the automorphism of the underlying vector subspace. Thus we can equivalently describe it with its function algebra which is in the basis identified with a Hopf quotient of the Hopf algebra of regular functions $\mathrm{Fun}(\mathrm{GL}(n))$.

Reminder on Hopf algebra of functions on $GL(n)$

$\mathrm{M}(\mathrm{n},\mathbf{k})$ of $\mathrm{n}\times\mathrm{n}$ matrices with (commutative) entries in a field **k** is isomorphic to $\mathbf{k}^{\mathrm{n}^2}$ as a **k**-vector space. This isomorphism induces a structure of affine **k**-variety on $\mathrm{M}(\mathrm{n},\mathbf{k})$. The regular functions on $\mathrm{M}(\mathrm{n},\mathbf{k})$ are polynomials in matrix entries. Introduce n^2 *regular* functions

$$\mathrm{G}^{\mathrm{i}}_{\mathrm{j}}:\mathrm{M}(\mathrm{n},\mathbf{k})\to\mathbf{k},\qquad \mathrm{G}^{\mathrm{i}}_{\mathrm{j}}(\mathrm{a})=\mathrm{a}^{\mathrm{i}}_{\mathrm{j}},\quad \mathrm{a}\in\mathrm{M}(\mathrm{n},\mathbf{k}),\ \ \mathrm{i},\mathrm{j}=1,\ldots\mathrm{n}.$$

Localizing at $\det\mathrm{G}$ get

$$\mathrm{Fun}(\mathrm{GL}(\mathrm{n},\mathbf{k}))\cong\mathbf{k}[\mathrm{G}^1_1,\mathrm{G}^1_2,\ldots,\mathrm{G}^{\mathrm{n}}_{\mathrm{n}},(\det\mathrm{G})^{-1}]/\langle\det\mathrm{G}\cdot(\det\mathrm{G})^{-1}-1\rangle$$

Instead of $\det^{-1}$, can use generators $\bar{\mathrm{G}}^{\alpha}_{\beta}$ with matrix identity $\mathrm{G}\bar{\mathrm{G}}=\bar{\mathrm{G}}\mathrm{G}=\mathrm{I}$.

Hopf algebra of functions on $Fun(Aut\,\mathfrak{g})$

Δ and ϵ are given by

$$\begin{array}{lll} \Delta G = G \otimes G & \text{i.e.} & \Delta G^\alpha_\beta = \sum_{\sigma=1}^n G^\alpha_\sigma \otimes G^\sigma_\beta \\ \Delta \bar{G} = \bar{G} \otimes \bar{G} & \text{i.e.} & \Delta (G^{-1})^\alpha_\beta = \sum_{\sigma=1}^n \bar{G}^\alpha_\sigma \otimes \bar{G}^\beta_\sigma \\ \epsilon G = \epsilon \bar{G} = I & \text{i.e.} & \epsilon(G^\alpha_\beta) = \epsilon(\bar{G}^\alpha_\beta) = \delta^\alpha_\beta \end{array}$$

The determinant and its inverse are group like elements ($\Delta t = t \otimes t$ and $\epsilon(t) = 1$). To get $Fun(Aut\,\mathfrak{g})$ in these coordinates divide by the ideal generated by relations

$$C^\mu_{\sigma\tau} G^\sigma_\alpha G^\tau_\beta - C^\rho_{\alpha\beta} G^\mu_\rho$$

$\forall \mu, \alpha, \beta = 1, \ldots, \dim\,\mathfrak{g}$ (which is a Hopf ideal!). Generators in the quotient still denoted G^α_β, $(G^{-1})^\alpha_\beta$.

First define an obvious degenerate pairing $Fun(Aut\mathfrak{g}) \otimes \mathfrak{g} \to \mathbf{k}$ by

$$\langle f, X \rangle := \frac{d}{dt} f(\exp(t\,\mathrm{ad}\;X))|_{t=0}$$

This can be understood as $(\mathrm{Ad}\,X)(f)$ as $\mathrm{Ad}\,X$ is a tangent vector at $\mathrm{Inn}\,\mathfrak{g} \subset \mathrm{Aut}\,\mathfrak{g}$. This gives the formulas which make sense over all fields:

Proposition.

$$\langle G^i_j, X \rangle = (\mathrm{ad}\;X)^i_j$$

$$\langle G^i_j, e_k \rangle = (\mathrm{ad}\;e_k)^i_j = -C^i_{jk}$$

$$\langle \bar{G}^i_j, e_k \rangle = (-\,\mathrm{ad}\;e_k)^i_j = -C^i_{kj}$$

A **Hopf paring** between Hopf algebras H and U is linear map $\langle\ ,\ \rangle\colon H \otimes U \to \mathbf{k}$ such that

$$\begin{array}{l}\langle fg, D\rangle = \langle f \otimes g, \Delta(D)\rangle, \\ \langle 1_H, D\rangle = \epsilon(D), \\ \langle \Delta f, D \otimes E\rangle = \langle f, DE\rangle, \\ \epsilon(f) = \langle f, 1_U\rangle, \\ \langle Sf, D\rangle = \langle f, SD\rangle.\end{array}$$

for all $f, g \in H$ and $D, E \in U$.

Lemma. There exists a unique degenerate *Hopf pairing* $H \otimes U(\mathfrak{g}) \to \mathbf{k}$ extending the pairing above (Δ on $U(\mathfrak{g})$ standard).
It is given by formulas

$$\langle f, X_1 X_2 \ldots X_r \rangle =$$

$$\frac{d}{dt_n}\frac{d}{dt_{n-1}}\cdots\frac{d}{dt_1} f(\exp(t_1 \,\mathrm{ad}\, X_n)\cdots exp(t_n \,\mathrm{ad}\, X_1))|_{t_1=0,\ldots,t_n=0}$$

This degenerate pairing induces a left **Hopf** action $\blacktriangleright : H \otimes U(\mathfrak{g}) \to U(\mathfrak{g})$

$$f \blacktriangleright D = D_{(1)} \langle f, D_{(2)} \rangle$$

This action moreover extends to the unique action $\blacktriangleright : (U \sharp H) \otimes U \to U$ such that $U \sharp 1 \cong U$ acts by multiplication in U.

Let $f_\nu = e_\sigma \bar{G}^\sigma_\nu \in H\sharp U$. From the expressions for ▶ we can compute some useful commutators in the smash product:

$$[\bar{G}^\lambda_\mu, e_\nu] = C^\rho_{\mu\nu}\bar{G}^\alpha_\rho$$

$$[\bar{G}^\alpha_\mu, f_\nu] = C^\alpha_{\rho\nu}\bar{G}^\rho_\mu$$

$$[G^\alpha_\mu, e_\nu] = -C^\alpha_{\rho\nu}G^\rho_\mu$$

$$[G^\alpha_\mu, f_\nu] = -C^\rho_{\mu\nu}G^\alpha_\rho$$

Now consider the tensor product representation of $\mathrm{Aut}\,\mathfrak{g}$ on $T(\mathfrak{g})$. It induces a unique representation on $U(\mathfrak{g})$. For monomials, in terms of matrices (where $\phi(e_\alpha) = e_\beta M^\beta_\alpha$),

$$e_{\sigma_1}\cdots e_{\sigma_k} \mapsto e_{\tau_1}\cdots e_{\tau_k} M^{\tau_1}_{\sigma_1}\cdots M^{\tau_k}_{\sigma_k}$$

This induces the right $\mathrm{Aut}\,\mathfrak{g}$-coaction on $U(\mathfrak{g})$ given by

$$\rho : e_{\sigma_1}\cdots e_{\sigma_k} \mapsto e_{\tau_1}\cdots e_{\tau_k} \otimes G^{\tau_1}_{\sigma_1}\cdots G^{\tau_k}_{\sigma_k}$$

Theorem. (Stojić, ZŠ) Algebra U satisfy the compatibilities with the left H-action ρ and the right H-coaction ρ making U into a braided commutative Yetter-Drinfeld module algebra.
Proof is long but mainly straightforward. We show here few easy parts. The identity $\rho(XY) = X_{[0]}Y_{[0]} \otimes Y_{[1]}X_{[1]}$ can be proved in coordinates using induction and the identity

$$C^{\mu}_{\sigma\tau}G^{\sigma}_{\alpha}G^{\tau}_{\beta} - C^{\rho}_{\alpha\beta}G^{\mu}_{\rho}.$$

Therefore $\rho : U \to U \otimes H^{op}$ is an algebra map.

First we prove the YD condition for the algebra generators only. The YD condition for generators $\bar{G}^\rho_\alpha$ and e_σ is

$$(\bar{G}^\rho_\alpha \blacktriangleright e_\sigma) \otimes \bar{G}^\beta_\rho G^\sigma_\nu = (\bar{G}^\beta_\rho \blacktriangleright e_\nu)_{[0]} \otimes (\bar{G}^\beta_\rho \blacktriangleright e_\nu)_{[1]} \bar{G}^\rho_\alpha$$

Prove and substitute inside the relation $\bar{G}^\rho_\alpha \blacktriangleright e_\sigma = C^\rho_{\alpha\sigma} + \delta^\rho_\alpha e_\sigma$:

$$(C^\rho_{\alpha\sigma} + \delta^\rho_\alpha e_\sigma) \otimes \bar{G}^\beta_\rho G^\sigma_\nu = \rho(C^\rho_{\alpha\sigma} + \delta^\rho_\alpha e_\sigma)(1 \otimes \bar{G}^\rho_\alpha)$$

This candidate identity then reduces to easy cancelations (using $C^\mu_{\sigma\tau} \bar{G}^\sigma_\alpha \bar{G}^\tau_\beta = C^\rho_{\alpha\beta} \bar{G}^\mu_\rho$, which also holds). Similarly for other combinations of generators.

This check on generators is sufficient. Namely, if $\rho : U \to U \otimes H^{op}$ is an algebra map in order to check the YD module property it is sufficient to check it on the algebra generators of H and U.
Indeed, it is clear that YD condition is linear. Thus check it for products (in H and in U) whenever factors satisfy it. For the products XY in U compute

$$\begin{aligned}
(f_{(1)} & \blacktriangleright (XY)_{[0]}) \otimes f_{(2)}(XY)_{[1]} = (f_{(1)} \blacktriangleright (X_{[0]} Y_{[0]})) \otimes f_{(2)} Y_{[1]} X_{[1]} \\
& = (f_{(1)} \blacktriangleright X_{[0]})(f_{(2)} \blacktriangleright Y_{[0]}) \otimes f_{(3)} Y_{[1]} X_{[1]} \\
& = (f_{(1)} \blacktriangleright X_{[0]})(f_{(3)} \blacktriangleright Y_{[0]})_{[0]} \otimes (f_{(3)} \blacktriangleright Y_{[0]})_{[1]} f_{(2)} X_{[1]} \\
& = (f_{(2)} \blacktriangleright X)_{[0]}(f_{(3)} \blacktriangleright Y)_{[0]} \otimes (f_{(3)} \blacktriangleright Y_{[0]})_{[1]} f_{(2)} \blacktriangleright X)_{[1]} f_{(1)} \\
& = ((f_{(2)} \blacktriangleright X)(f_{(3)} \blacktriangleright Y))_{[0]} \otimes ((f_{(2)} \blacktriangleright X)(f_{(3)} \blacktriangleright Y))_{[1]} f_{(1)} \\
& = (f_{(2)} \blacktriangleright (XY))_{[0]} \otimes (f_{(2)} \blacktriangleright (XY))_{[1]} f_{(1)},
\end{aligned}$$

Similarly, for the products fg in H we compute

$$\begin{array}{rcl}
(fg)_{(1)}X_{[0]} & \otimes & (fg)_{(2)}X_{[1]} = (f_{(1)} \blacktriangleright (g_{(1)} \blacktriangleright X_{[0]})) \otimes f_{(2)}(g_{(2)}X_{[1]}) \\
& = & (f_{(2)} \blacktriangleright (g_{(2)} \blacktriangleright X)_{[0]}) \otimes (f_{(2)} \blacktriangleright (g_{(2)} \blacktriangleright X)_{[1]})g_{(1)} \\
& = & (f_{(2)} \blacktriangleright (g_{(2)} \blacktriangleright X))_{[0]} \otimes (f_{(2)} \blacktriangleright (g_{(2)} \blacktriangleright X))_{[1]} f_{(1)} g_{(1)} \\
& = & ((fg)_{(2)} \blacktriangleright X)_{[0]} \otimes ((fg)_{(2)} \blacktriangleright X)_{[1]}(fg)_{(1)}
\end{array}$$

Braided commutativity:

$$X_{[0]}(X_{[1]} \blacktriangleright Y) = YX$$

Products on the left (recall $\rho(XX') = X_{[0]}X'_{[0]} \otimes X'_{[1]}X_{[1]}$):

$$X_{[0]}X'_{[0]}(X'_{[1]} \blacktriangleright (X_{[1]} \blacktriangleright Y)) = X_{[0]}(X_{[1]} \blacktriangleright Y)X' = YXX'$$

Products on the right

$$\begin{aligned} X_{[0]}(X_{[1]} \blacktriangleright (YY')) &= X_{[0]}(X_{[1]} \blacktriangleright Y)(X_{[2]} \blacktriangleright Y') \\ &= YX_{[0]}(X_{[1]} \blacktriangleright Y') \\ &= YY'X \end{aligned}$$

Theorem. (Stojić, ZŠ) Consider the smash product $H\sharp U = Fun(\mathrm{Aut}\,\mathfrak{g})\sharp U(\mathfrak{g})$ induced from

$$\blacktriangleright : Fun(\mathrm{Aut}\,\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{g}).$$

Inclusion $\alpha : U \to U\sharp 1 \hookrightarrow U\sharp H$ and algebra map $\beta = i \circ \rho : U^{op} \to U\sharp H$ (for $i : U \otimes H \to U\sharp H$ identification), make $H\sharp U$ into a U-bimodule via

$$u.a.v = \alpha(u)\beta(v)a, \quad u, v \in U, a \in U\sharp H.$$

In particular, the images of α and β commute in $H\sharp U$. It has a coproduct $\Delta : H\sharp U \to (H\sharp U) \otimes_U (H\sharp U)$ which is coassociative map of U-bimodules with counit $\epsilon : H\sharp U \to U$ in a way which is a part of a canonical left bialgebroid structure $(H\sharp U, \alpha, \beta, \Delta, \epsilon)$ over U.
Above structure can furthermore be completed to a structure of a *symmetric Hopf algebroid* in the sense of G. Böhm.

- Δ, ϵ are U-bimodule maps
- $(\Delta \otimes_U id) \circ \Delta = (id \otimes_U \Delta) \circ \Delta$ (coassociativity)
- $(\epsilon \otimes_U id) \circ \Delta \cong id \cong (id \otimes_U \epsilon) \circ \Delta$ (counitality) with identification $(H\sharp U) \otimes_U U \cong (H\sharp U)$
- $h \otimes \hat{f} \mapsto \epsilon(h\alpha(\hat{f}))$ defines an action $(H\sharp U) \otimes U \to U$ extending the left regular action $U \otimes U \to U$

Finally, as U is noncommutative, $(H\sharp U)\otimes_U (H\sharp U)$ is a U-bimodule but not an algebra componentwise. The subset of all $\sum b_i \otimes b_i' \in (H\sharp U)\hat{\otimes}_U(H\sharp U)$ satisfying

$$\sum_i b_i \otimes b_i'\alpha(a) = \sum_i b_i\beta(a)\otimes b_i', \ \forall a \in U,$$

is called **Takeuchi product** and is an algebra. It is natural (from many points of view) to require that Δ corestricts to the Takeuchi's product and this corestriction is an algebra map. This finishes the structure and axioms of the Hopf algebroid. In our case, the antipode of the Hopf algebroid is *not* an involution!

Generalization for Lie algebroids. Recall

Let $\mathcal{M}$ be a smooth manifold. A **Lie algebroid** over M a smooth vector bundle $A \to \mathcal{M}$, equipped with additional structure:

- a $\mathbb{k}$-Lie bracket $[,]$ on the space of sections of A;
- a map of vector bundles $a : A \to T\mathcal{M}$, called the **anchor map**, such that

$$[X, fY] = f[X, Y] + a(X)(f)Y$$

  for all sections X, Y of A and smooth function f on $\mathcal{M}$, where we by a slight abuse of notation denote by a the induced map of sections.

Let $\mathcal{O}$ be a commutative algebra over **k**, and L – a symmetric $\mathcal{O}$-bimodule. One says that $\mathcal{O}$ is a **Lie-Rinehart algebra** i f there is a **k**-linear Lie bracket [,] on L, and a morphism of $\mathcal{O}$-modules $a : L \to Der_{\mathbf{k}}(\mathcal{O})$, such that

$$[X, fY] = f[X, Y] + a(X)(f)Y. \tag{22}$$

Here $Der_{\mathbf{k}}(\mathcal{O})$ denotes the (Lie algebra of) **k**-linear derivations of $\mathcal{O}$.

The universal enveloping algebra $U(L)$ of a Lie algebroid or Lie-Rinehart algebra is the tensor algebra $T_{\mathcal{O}}L$ over module $\mathcal{O}$ modulo the ideal, generated by the ideal of the relations

$$XY - YX = [X, Y], \tag{23}$$

$$XfY - fXY = a(X)(f)Y. \tag{24}$$

We want to repeat the construction of the Hopf algebroid $Fun(\mathrm{Aut}\,\mathfrak{g})\sharp U(\mathfrak{g})$ with a Lie algebra $\mathfrak{g}$ replaced by a Lie algebroid or even a Lie-Rinehart algebra L.

$U(L)$ has a comultiplication over $\mathcal{O}$ inherited from $T_{\mathcal{O}}L$ There is a PBW map from the symmetric algebra over $\mathcal{O}$ to the universal enveloping but is not respecting the coproduct! In presence of a Lie algebroid connection it can be corrected but (Sharygin, ZŠ, 2009).

Choose a basis e_α of ΓL as a $C^\infty(\mathcal{M})$-module. The automorphism of L as a Lie algebroid is an automorphism as a vector bundle, given by a matrix M with entries in $C^\infty(\mathcal{M})$ such that it commutes with the anchor map a and preserves the bracket. In terms of M,

$$a(e_\alpha) = M_\alpha^\beta a(e_\beta)$$

$$M_\mu^\alpha M_\nu^\beta C_{\alpha\beta}^\rho - M_\mu^\sigma a(e_\sigma)(M_\nu^\rho) + M_\nu^\sigma a(e_\sigma)(M_\mu^\rho) = C_{\alpha\beta}^\gamma M_\gamma^\rho$$

These are algebraic conditions on M *and derivatives* of M in the setup of infinite-dimensional geometry over a *ring* $C^\infty(\mathcal{M})$.

We do not know how to do such differential algebra. However, the matrix function G – if the duals of $U(L)^*$ and $\hat{S}_{\mathcal{O}}(L^*)$ are properly identified via the transpose of the corrected coexponential map – has a meaning (passing between left and right invariant vector fields);
Therefore take the $\mathcal{O}$-subalgebra generated by these matrix elements. This enables constructions of structure maps (including the pairing) like in Hopf algebroid (but some maps exist only dually).

- THANKS!