Homotopy versions of Jacobi bundles

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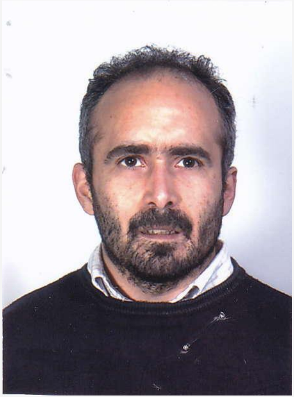
"You cannot criticize geometry. It's never wrong."

Paul Rand, 1914–1996



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Introduction

Jacobi Manifolds + Bundles

- Jacobi manifolds: Lichnerowicz 1978 & Kirillov 1976
  (Marle 1991)

  Jacobi structure: (Λ, D) such that $[D, \Lambda] = 0$ and $[\Lambda, \Lambda] = 2D \wedge \Lambda$

  $$\{f, g\} = \Lambda(f, g) + fD(g) - gD(f)$$

  The Jacobi bracket is local in the sense of Kirillov

  $$\mathrm{supp}(\{f, g\}) \subseteq \mathrm{supp}(f) \cap \mathrm{supp}(g)$$

- On the other hand Jacobi manifolds are specialisations of Poisson manifolds via the 'Poissonisation' process...

Introduction

Question: is there a *reasonable* notion of a 'Jacobi-∞ manifold'?

Why ask?

- Huebschmann (2005) – "... a first step in taming the *bracket zoo* that arose recently in topological field theory."
- Le, Oh, Tortorella & Vitagliano (2014) – deformation of coisotopic submanifolds of Jacobi manifolds.
- Grabowski (2013) – graded contact/Jacobi geometry.

L_∞-algebras

Lada + Stasheff (1993) and *super-setting* Voronov (2005).

$V = V_0 \oplus V_1$, with odd n-linear operators that satisfy:

1. the operators are *graded symmetric*

$$(a_1, \cdots, a_i, a_{i+1}, \cdots, a_n) = (-1)^{\widetilde{a}_i \widetilde{a}_{i+1}} (a_1, \cdots, a_{i+1}, a_i, \cdots, a_n),$$

2. the *generalised Jacobi identities* (Jacobiators)

$$\sum_{k+l=n} \sum_{(k,l)-\text{unshuffles}} \pm \left((a_{\sigma(1)}, \cdots, a_{\sigma(k)}), a_{\sigma(k+1)}, \cdots, a_{\sigma(k+l)} \right) = 0$$

An (k, l)-unshuffle is a permutation of the indices $1, 2, \cdots, k + l$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

L_∞-algebras

If $V = \Pi U$ is an L_∞-algebra then we have a series of brackets on U, denoted by $\{\bullet, \ldots, \bullet\}$, that are *skew-symmetric* and *even/odd* for an *even/odd* number of arguments.

$$\Pi\{x_1, \cdots, x_n\} = (-1)^{(\widetilde{x}_1(n-1)+\widetilde{x}_2(n-2)+\cdots+\widetilde{x}_{n-1})}(\Pi x_1, \cdots, \Pi x_n),$$

where $x_i \in U$.

(now closer to the original definition)

L_∞-algebras

Set $(\emptyset) = 0$ and write $\mathrm{d}x = (x)$

- $\mathrm{d}^2 = 0$
- $\mathrm{d}(x, y) = (\mathrm{d}x, y) \pm (x, \mathrm{d}y)$
- $\pm((x, y), z) \pm ((z, x), y) \pm ((y, z), x) =$
  $\pm\mathrm{d}(x, y, z) \pm (\mathrm{d}x, y, z) \pm (x, \mathrm{d}y, z) \pm (x, y, \mathrm{d}z)$
- *etc.*

Intuitively we almost have a differential Lie algebra + more brackets and higher Jacobi identities.

L_∞-algebras

Th. Voronov's higher derived bracket formalism

- A Lie (super)algebra $\mathcal{L}$
- A projector onto an abelian subalgebra $V \subset \mathcal{L}$ satisfying $\pi[a,b] = \pi[\pi a, b] + \pi[a, \pi b]$ for all $a, b \in \mathcal{L}$
- A chosen element $\Delta \in \mathcal{L}$

Get a series of brackets on the abelian subalgebra

$$(a_1, a_2, \cdots, a_n) = \pi[\cdots[[\Delta, a_1], a_2], \cdots a_n],$$

with a_i in V.

Theorem (Th. Voronov)

If $\Delta \in \mathcal{L}$ is Grassmann odd and $[\Delta, \Delta] = 0$ then we have an L_∞-algebra.

Aside on supermanifolds

Supermanifold = ‘manifold’ with commuting and anticommuting coordinates

$$(x^\mu,\ \xi^\alpha) := x^a$$

Such that $x^\mu x^\nu = x^\nu x^\mu$, $x^\mu \xi^\alpha = \xi^\alpha x^\mu$ and $\xi^\alpha \xi^\beta = -\xi^\beta \xi^\alpha$

Note $\xi^\alpha \xi^\alpha = -\xi^\alpha \xi^\alpha = 0$

Grassmann parity $\widetilde{x^a} = \widetilde{a} \in \{0, 1\}$

$$x^a x^b = (-1)^{\widetilde{a}\widetilde{b}} x^b x^a$$

Grassmann parity extends to tensor and tensor-like objects.

Example: Pure odd supermanifold = Grassmann algebra

Example: Vector bundle E local coordinates $(x^\mu,\ y^\alpha)$, then ΠE is the supermanifold formed by shifting the parity of the fibre coordinates $\Pi(y^\alpha) = \xi^\alpha$

L_∞-algebras

Example: *Homotopy Poisson structure* $\mathcal{P} \in C^\infty(\Pi T^* M)$, Grassmann even and $[\![\mathcal{P}, \mathcal{P}]\!] = 0$.

(Note we have shifted the parity here)

Higher Poisson bracket

$$\{f_1, f_2, \cdots, f_r\} = \pm [\![\cdots [\![[\![\mathcal{P}, f_1]\!], f_2]\!], \cdots f_r]\!]|_M$$

Leibniz rule

$$\{f_1, f_2, \cdots, f_r f_{r+1}\} = \{f_1, f_2, \cdots, f_r\} f_{r+1} \pm f_r \{f_1, f_2, \cdots, f_{r+1}\}$$

Used by; Cattaneo + Felder (2007), Khudaverdian + Voronov (2008), Mehta (2011), Braun + Lazarev (2013), Bashkirov + (A) Voronov (2014), Vitagliano (2015)

(We will use these structures later)

Kirillov Manifolds

Line bundle $L \to M$ over a manifold M

$\mathrm{Sec}(L)$ correspond to homogeneous functions of degree 1 on the principal $\mathbb{R}^{\times}$-bundle

$$P = (L^*)^{\times} := L^* \setminus \{\underline{0}\}$$

$f \in C^{\infty}(P)$ s.t. $(\mathrm{h}_s)^* f = s\, f$, where h is the action of $\mathbb{R}^{\times}$.

$$u \rightsquigarrow \iota_u,$$

where $u \in \mathrm{Sec}(L)$.

Kirillov Manifolds

Definition (Grabowski 2013)

A *principal Poisson* $\mathbb{R}^\times$*-bundle*, shortly *Kirillov manifold*, is a principal $\mathbb{R}^\times$-bundle (P, h) equipped with a Poisson structure Λ of degree -1, i.e. such that $(\mathsf{h}_s)_*\Lambda = s^{-1}\Lambda$.

Kirillov Manifolds

Theorem (Grabowski 2013)

There is a one-to-one correspondence between Kirillov brackets $[\cdot,\cdot]_L$ on a line bundle $L \to M$ and Poisson structures Λ of degree -1 on the principal $\mathbb{R}^\times$-bundle $P = (L^*)^\times$ given by

$$\iota_{[u,v]_L} = \{\iota_u, \iota_v\}_\Lambda.$$

Attitude: Jacobi manifolds and Jacobi bundles are *specialisations* and not *generalisations* of Poisson manifolds!

(For contact/Jacobi groupoids Bruce, Grabowska + Grabowski 2015)

Kirillov Manifolds

In local coordinates $(t,\ x^a)$ on P

$$\Lambda = \frac{1}{2t}\Lambda^{ab}(x)\frac{\partial}{\partial x^b} \wedge \frac{\partial}{\partial x^a} + \Lambda^a(x)\frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial t}$$

- If Λ is non-degenerate then we are in world of *contact geometry*
- If $P \simeq \mathbb{R}^\times \times M$ then we are in the world of *Jacobi geometry*

Note: The Jacobi bracket is a bracket on sections of a *trivial line bundle* and not a bracket on *functions*!

Higher Kirillov manifolds

The idea now is clear...

1. consider (even) line bundles over supermanifolds in terms of principal $\mathbb{R}^{\times}$-bundles
2. equip the total space with a homogeneous homotopy Poisson structure
3. the brackets restricted to homogeneous functions are the homotopy Kirillov brackets

Higher Kirillov manifolds

Consider a principal $\mathbb{R}^{\times}$-bundle $P \to M$ with action h and employ homogeneous local coordinates

$$(t, x^a)$$

$\widetilde{t} = 0$ and $\widetilde{x}^a = \widetilde{a} \in \{0, 1\}$.
The action

$$\mathsf{h} : \mathbb{R}^{\times} \times P \to P$$

at the level of coordinates is

$$\mathsf{h}_s^*(t) = s\, t, \qquad \mathsf{h}_s^*(x^a) = x^a$$

Higher Kirillov manifolds

Let us pick homogeneous local coordinates on $\Pi\mathrm{T}^*P$

$$(\underbrace{t}_{(1,0)}, \underbrace{x^a}_{(0,0)}, \underbrace{t^*}_{(0,1)}, \underbrace{x^*_b}_{(1,1)}),$$

$\widetilde{t}^* = 1$ and $\widetilde{x}^*_a = \widetilde{a} + 1$.

The graded structure defined via *phase lift*

(Grabowski 2013)

Higher Kirillov manifolds

We thus have a *double structure*

$$\begin{array}{ccc} \Pi \mathrm{T}^* P & \xrightarrow{\pi} & \Pi \mathrm{T}^* P/\mathbb{R}^\times \\ \downarrow \tau & & \downarrow \tau_0 \\ P & \xrightarrow{\pi_0} & M \end{array}$$

where τ, τ_0 are vector bundles, and π, π_0 are principal $\mathbb{R}^\times$-bundles

(Grabowski 2013)

Higher Kirillov manifolds

The supermanifold $\Pi\mathsf{T}^*P$ comes canonically equipped with a Schouten bracket which is homogeneous of degree -1

$$\begin{aligned} [\![F, G]\!] &= (-1)^{(\widetilde{a}+1)(\widetilde{F}+1)}\frac{\partial F}{\partial x^*_a}\frac{\partial G}{\partial x^a} - (-1)^{\widetilde{a}(\widetilde{F}+1)}\frac{\partial F}{\partial x^a}\frac{\partial G}{\partial x^*_a} \\ &+ (-1)^{\widetilde{F}+1}\frac{\partial F}{\partial t^*}\frac{\partial G}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial G}{\partial t^*}, \end{aligned}$$

for any F and $G \in C^\infty(\Pi\mathsf{T}^*P)$.

Higher Kirillov manifolds

Definition (Bruce + Tortorella 2016)

A *higher Kirillov manifold* is a homogeneous higher Poisson manifold; that is a triple $(P, \mathrm{h}, \mathcal{P})$, such that (P, h) is a principal $\mathbb{R}^{\times}$-bundle and $\mathcal{P} \in C^{\infty}(\Pi \mathrm{T}^{*} P)$ is a homogeneous higher Poisson structure i.e. Grassmann even, weight one, and $[\![\mathcal{P}, \mathcal{P}]\!] = 0$.

$$\mathcal{P} = \sum_{k=0} \frac{1}{k!} t^{1-k} \mathcal{P}^{a_1 \cdots a_k}(x) x^{*}_{a_k} \cdots x^{*}_{a_1} + \sum_{k=0} \frac{1}{k!} t^{1-k} \bar{\mathcal{P}}^{a_1 \cdots a_k}(x) x^{*}_{a_k} \cdots x^{*}_{a_1} t^{*}$$

(Note $\mathcal{P}$ is of degree 1 and not -1)

Higher Kirillov manifolds

Homotopy Poisson algebra on $C^\infty(P)$ viz

$$\{f_1, f_2, \cdots, f_r\}_{\mathcal{P}} := \pm[\![\cdot \cdots [\![[\![\mathcal{P}, f_1]\!], f_2]\!], \cdots, f_r]\!]|_P$$

Note that each r-arity bracket is of degree $(1-r) \Rightarrow$ submodule of homogeneous functions of weight one is closed.

$$\iota : \mathsf{Sec}(L) \hookrightarrow C^\infty(P)$$

We then define an L_∞-algebra on $\mathsf{Sec}(L)$ viz

$$\iota_{[\sigma_1, \sigma_2, \cdots \sigma_r]} = \{\iota_{\sigma_1}, \iota_{\sigma_2}, \cdots, \iota_{\sigma_r}\}$$

Higher Kirillov manifolds

Theorem (Bruce + Tortorella 2016)

Given a higher Kirillov manifold $(P, \mathfrak{h}, \mathcal{P})$*, then the module of sections of the corresponding even line bundle* $L \to M$ *comes equipped with the structure of an* L_∞*-algebra via the proceeding constructions.*

Higher Kirillov manifolds

$$[\sigma_1, \sigma_2, \cdots, \sigma_r] = \pm\ t^{1-r}\mathcal{P}^{a_1\cdots a_r}(x)\frac{\partial\sigma_1}{\partial x^{a_r}}\cdots\frac{\partial\sigma_r}{\partial x^{a_1}}$$

$$\pm\ t^{2-r}\bar{\mathcal{P}}^{a_1\cdots a_{r-1}}(x)\left(\frac{\partial\sigma_1}{\partial t}\ \frac{\partial\sigma_2}{\partial x^{a_{r-1}}}\cdots\ \frac{\partial\sigma_r}{\partial x^{a_1}}\pm\frac{\partial\sigma_1}{\partial x^{a_{r-1}}}\frac{\partial\sigma_2}{\partial t}\ \cdots\ \frac{\partial\sigma_r}{\partial x^{a_1}}\pm\cdots\right.$$

$$\left.\pm\frac{\partial\sigma_1}{\partial x^{a_{r-1}}}\ \frac{\partial\sigma_2}{\partial x^{a_{r-2}}}\cdots\ \frac{\partial\sigma_r}{\partial t}\right),$$

where $\sigma(t,x) = t\sigma(x)$ is a section of L

Higher Kirillov manifolds

Quasi-derivation rule which defines a series of anchors

$$\rho_k : \mathsf{Sec}(L)^k \to \mathsf{Vect}(M)$$

viz

$$[\sigma_1, \cdots, \sigma_k, f\sigma_{k+1}] = \rho_k(\sigma_1, \cdots, \sigma_k)(f)\sigma_{k+1} \pm f[\sigma_1, \cdots, \sigma_k, \sigma_{k+1}],$$

for $f \in C^\infty(M)$.

$$\rho_k(\sigma_1, \cdots, \sigma_k)(f) = \{\sigma_1, \cdots, \sigma_k, f\}$$

The anchors depend on the first order derivatives of the sections.

Higher Kirillov manifolds

Example: If $P \simeq \mathbb{R}^{\times} \times M$ and M is a manifold, then we have a classical Jacobi manifold.

$$\mathcal{P} = \frac{t^{-1}}{2}\mathcal{P}^{ab}(x)x_b^* x_a^* + \bar{\mathcal{P}}^a(x)x_a^* t^*,$$

which is the 'superisation' of the 'Poissonisation' of the classical Jacobi structure.

Example: If the homogeneous homotopy Poisson structure is concentrated in order r and the line bundle is trivial, then up to matters of conventions and ‘superisation’ the resulting structure is equivalent to the generalised Jacobi structures of Pérez Bueno (1997).

Higher Kirillov manifolds

Example: Any semisimple Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ comes with a canonical 3-cocycle

$$C_{ijk} = k_{il} Q^l_{kj} + k_{jl} Q^l_{ki} + k_{kl} Q^l_{ij},$$

where $k_{ij} = Q^k_{il} Q^l_{kj}$ is the Killing metric

$\mathfrak{g}^*$ comes with a canonical linear Poisson structure. $C[3]$ can also be considered as a cocycle of the Lichnerowicz complex

$$\mathcal{P} := t^{-1}\Lambda[2] + t^{-3}C[3]E[1] + t^{-2}C[3]t^*$$

provides the trivial principle bundle $\mathbb{R}^\times \times \mathfrak{g}^* \to \mathfrak{g}^*$ with an order 4 homotopy Jacobi structure. Here $E[1] = \xi^i y_i$.

The associated homotopy BV-algebra

Differential forms on P are identified with functions on $\Pi \mathrm{T} P$.

Let us pick homogeneous local coordinates

$$(\underbrace{t}_{(1,0)}, \underbrace{x^a}_{(0,0)}, \underbrace{\mathrm{d}t}_{(0,1)}, \underbrace{\mathrm{d}x^b}_{(-1,1)}),$$

Grassmann parity of the fibre coordinates is assigned as $\widetilde{\mathrm{d}t} = 1$ and $\widetilde{\mathrm{d}x^a} = \widetilde{a} + 1$.

$$\mathrm{d} = \mathrm{d}x^a \frac{\partial}{\partial x^a} + \mathrm{d}t \frac{\partial}{\partial t},$$

is homogeneous and of degree -1 with respect to the action of $\mathbb{R}^{\times}$.

The associated homotopy BV-algebra

Send the homogeneous higher Poisson structure to its interior derivative $\mathcal{P} \rightsquigarrow i_{\mathcal{P}}$ viz

$$t^* \leftrightsquigarrow \frac{\partial}{\partial \mathrm{d}t}, \qquad x_a^* \leftrightsquigarrow \frac{\partial}{\partial \mathrm{d}x^a},$$

(with an overall minus sign)

Definition (Bruce + Tortorella 2016)

The *higher Koszul–Brylinski* operator on a higher Kirillov manifold is the differential operator (Lie derivative)

$$L_{\mathcal{P}} := [\mathrm{d}, i_{\mathcal{P}}].$$

Note: $[L_{\mathcal{P}}, L_{\mathcal{P}}] = L_{[\![\mathcal{P},\mathcal{P}]\!]} = 0$

The associated homotopy BV-algebra

Define a homotopy BV-algebra:

$$(\omega_1, \omega_2, \cdots, \omega_r)_{\mathcal{P}} := [\cdots[[L_{\mathcal{P}}, \omega_1], \omega_2], \cdots \omega_r](1)$$

for all $\omega_i \in C^\infty(\Pi \mathsf{T} P)$.

The brackets closes on $\mathcal{A}_0(P) := C^\infty(\Pi \mathsf{T} P/\mathbb{R}^\times) = C^\infty(\Pi J_1^*(L))$.

The associated homotopy BV-algebra

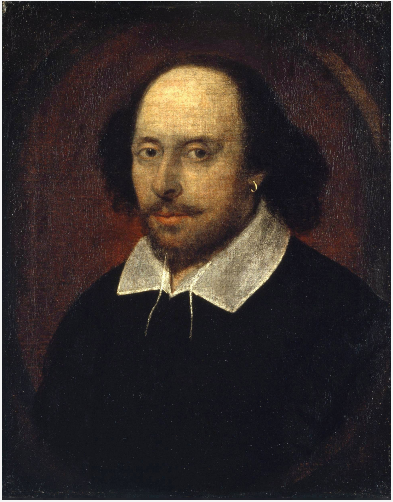
Theorem (Bruce + Tortorella 2016)

Given any higher Kirillov manifold $(P, \mathsf{h}, \mathcal{P})$ there is canonically a homotopy BV-algebra on the $\mathbb{R}^{\times}$-invariant differential forms $\mathcal{A}_0(P)$ generated by the higher Koszul–Brylinski operator $L_{\mathcal{P}}$.

Generalises Vaisman 2000

Take home message

- We do have a good notion of Kirillov and Jacobi structures up to homotopy via principle $\mathbb{R}^{\times}$-bundles and homogeneous homotopy Poisson structures.
- Trying to work directly with brackets is clumsy and could miss the geometry.



"I can no other answer make but thanks, and thanks and ever thanks..."

William Shakespeare, Twelfth Night, Act III, Scene III