Mirzakhani’s work on volumes of moduli spaces and counting simple closed curves

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Introduction

In 2008, Maryam Mirzakhani wrote three groundbreaking papers about surfaces and hyperbolic geometry. In [Mir07a], she develops a method for computing recursively the volume of moduli spaces of hyperbolic structures on surfaces with boundary. Let $S$ be a surface of genus $g$ and $n$ boundary components (such that $\chi(S) = 2 - 2g - n < 0$). We denote by $\text{Mod}(g, L_1, \ldots, L_n)$ the quotient of the space of hyperbolic metrics on $S$ with geodesic boundary components of lengths $L_1, \ldots, L_n$ by the action of the group of orientation preserving diffeomorphisms fixing setwise each boundary component. The space $\text{Mod}(g, L_1, \ldots, L_n)$ is an orbifold of dimension $6g - 6 + 2n$. It carries a natural symplectic 2-form call the Weil–Petersson symplectic form (we will denote it $\omega_{WP}$). The volume

$$\text{Vol}(g, L_1, \ldots, L_n)$$

of $\text{Mod}(g, L_1, \ldots, L_n)$ is the integral of the volume form $\omega^{3g - 3 + n}$.

**Theorem 1** (Mirzakhani 2008, [Mir07a]). The volume $\text{Vol}(g, L_1, \ldots, L_n)$ is a polynomial function in $(L_1, \ldots, L_n)$ of degree $6g - 6 + 2n$, even in each variable, whose coefficients are rational multiples of a power of $\pi$. These polynomials can be explicitly computed by induction on $3g - 3 + n$.

In particular, one can compute the constant coefficient

$$\text{Vol}(g, 0, \ldots, 0)$$

and obtain the volume of the moduli space of hyperbolic surfaces of genus $g$ with $n$ cusps. Prior to Mirzakhani, this had only be computed for punctures sphere (i.e. when $g = 0$). Note that, to our knowledge, volumes of moduli spaces of closed Riemann surfaces are not known.

In [Mir08], Mirzakhani applies this volume computation to give an asymptotic estimate of the number of simple closed geodesics of length at most $R$ on a hyperbolic surface. Let $S$ be a closed surface of genus $g \geq 2$. The mapping class group of $S$ (denoted $\text{MCG}(S)$) is the group of isotopy classes of diffeomorphisms of $S$. Let $\gamma$ be a weighted multicurve on $S$ (i.e. a formal linear combination of disjoint simple closed curves), and $[\gamma]$ the orbit of $\gamma$ under $\text{MCG}(S)$. For
any hyperbolic metric $X$ on $S$, we denote by $l_\gamma(X)$ the length of the geodesic representative of $\gamma$ for $X$. Let us define

$$s_\gamma(X, L) = \sharp \{ \gamma' \in [\gamma] | l_{\gamma'}(X) \leq L \} .$$

**Theorem 2** (Mirzakhani 2008, [Mir08]). When $L$ goes to infinity, we have

$$s_\gamma(X, L) \sim C([\gamma]) B(X) L^{6g-6+2n} ,$$

Where $C([\gamma])$ and $B(X)$ are non zero.

This is a considerable refinement of a result of Rivin [Riv01], who proved that there exist constants $C$ and $C'$ such that

$$CL^{6g-6+n} \leq s_\gamma(X, L) \leq C'L^{6g-6+2n} .$$

In her third paper [Mir07b], Mirzakhani relates coefficients of the volume polynomials to intersection numbers of tautological cohomology classes on moduli spaces. Let $\text{Mod}(g, n)$ be the moduli space of hyperbolic structures on a surface of genus $g$ with $n$ cusps. This space admits a compactification $\overline{\text{Mod}(g, n)}$ which carries $n$ cohomology classes $\psi_1, \ldots, \psi_n$ of degree 2 associated to the $n$ cusps.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a $n$-tuple of non-negative integers of length $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Let us define

$$I(g, \alpha) = \int_{\overline{\text{Mod}(g, n)}} \psi_1^{\alpha_1} \wedge \ldots \wedge \psi_n^{\alpha_n} \wedge \omega_{WP}^{3g-3+n-|\alpha|} .$$

By theorem 1, we can also write

$$\text{Vol}(g, L_1, \ldots, L_n) = \sum_{|\alpha| \leq 3g-3+n} C(g, \alpha) L_1^{\alpha_1} \ldots L_n^{\alpha_n} .$$

**Theorem 3** (Mirzakhani 2008, [Mir08]). The intersection number $I(g, \alpha)$ and the coefficient $C(g, \alpha)$ of the volume polynomial are related by the formula

$$C(\alpha) = \frac{1}{2^{|\alpha|}(3g-3+n)!\alpha_1! \ldots \alpha_n!} I(g, \alpha) .$$

Witten conjectured that a certain formal series whose coefficients are defined in terms of the numbers $I(\alpha)$ satisfies the so called KDV equation. This conjecture was originally proved by Kontsevich. Mirzakhani proves in [Mir07b] that the recurrence relations required to satisfy the KDV equation follow from theorem 3 and the recurrence relation of volume polynomials in theorem 1. She therefore obtains a new proof of Witten’s conjecture.

1 Background in hyperbolic geometry

1.1 The hyperbolic plane

The hyperbolic plane $\mathbb{H}^2$ is the unique complete simply connected hyperbolic surface. It can be realized as a domain in $\mathbb{C}$ in two different ways:
• As the disc \( \{ z \mid |z| < 1 \} \) with the metric \( \frac{dz d\bar{z}}{4(1-|z|^2)} \). The group PU(1, 1) acts on the disc by homographies and identifies with the group of orientation preserving isometries.

• As the upper half-plane \( \{ z \mid \text{im}(z) > 0 \} \) with the metric \( \frac{1}{\text{im}(z)} d\bar{z}dz \). Here the group of orientation preserving isometries identifies with the action of PSL(2, R) by homographies.

The hyperbolic plane can be compactified by adding a boundary at infinity denoted \( \partial_{\infty} \mathbb{H}^2 \). In the disc model, this boundary is just the unit circle. In the upper half-plane model, it identifies with the real projective line \( \mathbb{RP}^1 = \mathbb{R} \cup \{ \infty \} \). The action of \( \text{Isom}(\mathbb{H}^2) \) extends to an action on the boundary by diffeomorphisms.

1.1.1 geodesics

In both models, the geodesics of \( \mathbb{H}^2 \) are circles or lines intersecting the boundary orthogonally. Geodesics are determined by their end-points in \( \partial_{\infty} \mathbb{H}^2 \).

Two geodesics may intersect in \( \mathbb{H}^2 \), have a common end-point in \( \partial_{\infty} \mathbb{H}^2 \) or may not intersect at all in \( \mathbb{H}^2 \cup \partial_{\infty} \mathbb{H}^2 \). In that case, they have a unique common perpendicular and one can define the distance between those two geodesics as the length of the perpendicular segment joining them.

The isometry group of \( \mathbb{H}^2 \) acts transitively on unit tangent vectors in \( \mathbb{H}^2 \). In particular, it acts transitively on geodesics.

1.2 Classification of orientation-preserving isometries

Let \( g \) be an isometry of \( \mathbb{H}^2 \). We define the translation length of \( g \):

\[ l(g) = \inf_{x \in \mathbb{H}^2} d(x, g \cdot x) . \]

The translation length of \( g \) is a conjugacy invariant. It allows a classification of orientation preserving isometries in three categories.

• Elliptic isometries are those fixing a point in \( \mathbb{H}^2 \). They therefore have translation length 0. They are conjugate to a rotation of the unit disc and are classified up to conjugacy by the angle of this rotation.

• Parabolic isometries fix a unique point in \( \partial_{\infty} \mathbb{H}^2 \). They also have null translation length and are all conjugate. In the upper half-space model, the parabolic isometries fixing \( \infty \) are those of the form

\[ z \mapsto z + t . \]

• Orientation preserving isometries of non-zero translation length are called hyperbolic isometries. A hyperbolic isometry \( g \) fixes two points \( g_- \) and \( g_+ \) in \( \partial_{\infty} \mathbb{H}^2 \) and stabilizes the geodesic \( \text{axis}(g) \) between those end-points. Along this geodesic, \( g \) acts as a translation of length \( l(g) \). Two hyperbolic isometries are conjugate if and only if they have the same translation length. In the upper half-space model, a hyperbolic isometry \( g \) is conjugate to the transformation

\[ z \mapsto e^{l(g)} z . \]
1.2.1 Reflections

Let $\gamma$ be a geodesic in $\mathbb{H}^2$. There exists a unique orientation reversing isometry $\sigma_\gamma$ fixing $\gamma$ pointwise. It is called the reflection along $\gamma$. More general orientation reversing isometries are the composition of a reflection along a geodesic $\gamma$ with a hyperbolic isometry of axis $\gamma$.

Consider $\gamma$ and $\gamma'$ two geodesics in $\mathbb{H}^2$. Then the composition $\sigma_\gamma \sigma_{\gamma'}$ is

- a rotation of angle $2\alpha$ when $\gamma$ and $\gamma'$ intersect with an angle $\alpha$,
- a parabolic isometry when $\gamma$ and $\gamma'$ have a common endpoint in $\partial \infty \mathbb{H}^2$,
- a hyperbolic isometry with translation length $2d(\gamma, \gamma')$ and axis perpendicular to $\gamma$ and $\gamma'$ when the those geodesics are disjoint in $\mathbb{H}^2 \cup \partial \infty \mathbb{H}^2$.

2 Hyperbolic structures on compact surfaces with boundary

A hyperbolic structure on a surface $S$ is a riemannian metric of curvature $-1$. If this riemannian metric is geodesically complete (i.e. geodesics run for all time), then the surface $S$ identifies to a quotient of $\mathbb{H}^2$ by a discrete group of isometries.

On a surface with boundary, one can hope to find a complete (but not geodesically complete) hyperbolic metric such that boundary components are geodesics. In that case, the surface identifies with the quotient of a convex domain in $\mathbb{H}^2$ bounded by (infinitely many) geodesics.

It is known since the beginning of the XXth century that most surfaces admit hyperbolic structures. Indeed, the uniformisation theorem of Poincaré–Koebe asserts that any oriented surface with a complex structure is either $\mathbb{C}P^1$, a quotient of $\mathbb{C}$, or a quotient of $\mathbb{H}^2$ by a discrete group of isometry. Since the only smooth quotients of $\mathbb{C}$ are cylinders and tori, this implies that all other surfaces carry hyperbolic structures, that are in one-to-one correspondence with complex structures.

If $S$ is a closed surface with boundary, one can obtain a hyperbolic metric with geodesic boundary by a “doubling” process: fix a complex structure on $S$. Take two copies $S_1$ of $S_2$ of this surface $S$ and glue them together along their boundary. One obtains a closed surface $\Sigma$ with an involution that preserves the complex structure, switches $S_1$ and $S_2$ and fixes their common boundary. Then the hyperbolic metric given by the uniformisation theorem is preserved by this involution. Therefore, the set of fixed points of this involution is geodesic, which means that the uniformisation metric restricted to $S_1$ has geodesic boundary.

We will now describe a more concrete way of building hyperbolic structures on surfaces, by assembling together smaller pieces called pairs of pants.

2.1 Hyperbolizing pairs of pants

A pair of pants is a closed surface with boundary homeomorphic to a pair of pants, i.e. the complement of three disjoint open discs in a sphere. A hyperbolic pair of pants is a pair of pants provided with a hyperbolic metric with geodesic boundary.
One can construct a hyperbolic pair of pants by gluing together two hyperbolic hexagons with angles $\pi/2$.

**Lemma 2.1.** Given $l_1, l_2, l_3 > 0$, there exist, up to isometry, a unique hyperbolic hexagon $(p_1, q_1, p_2, q_2, p_3, q_3)$ with angles $\pi/2$ such that $d(p_i, q_i) = l_i$.

**Proof.** Let $\gamma_1$ and $\gamma_2$ be two disjoint geodesics in $\mathbb{H}^2$ such that $d(\gamma_1, \gamma_2) = l_1$. On can prove first that there is a unique geodesic $\gamma_3$ disjoint from $\gamma_1$ and $\gamma_2$ such that $d(\gamma_2, \gamma_3) = l_2$ and $d(\gamma_3, \gamma_1) = l_3$. The geodesics $(\gamma_1, \gamma_2, \gamma_3)$ together with their common perpendiculars then bound the required hexagon.

**Proposition 2.2.** Given $l_1, l_2, l_3$, there is, up to isometry, a unique hyperbolic pair of pants with geodesic boundary components of length $l_1, l_2$ and $l_3$.

**Proof.** Let $(p_1, q_1, p_2, q_2, p_3, q_3)$ be a hyperbolic hexagon such that $d(p_i, q_i) = l_i/2$. Let $(p'_1, q'_1, p'_2, q'_2, p'_3, q'_3)$ be another copy of the same hexagon. Then, gluing $(q_1, p_2)$ with $(q'_1, p'_2)$, $(q_2, p_3)$ with $(q'_2, p'_3)$ and $(q_3, p_1)$ with $(q'_3, p'_1)$, one obtains the required hyperbolic pair of pants.

Let us now prove that any hyperbolic pair of pants with geodesic boundary components $c_1, c_2, c_3$ of length $l_1, l_2$ and $l_3$ is constructed this way. Let $p_i, q_i$ be two points on the boundary component of length $l_i$, such that $d(p_1, q_1)$ (resp. $d(p_2, q_2), d(p_3, q_3)$) minimizes the distance between a point in $c_i$ (resp. $c_2, c_3$) and a point in $c_2$ (resp. $c_3, c_1$). Then the geodesics $(p_1, q_2), (p_2, q_3)$ and $(p_3, q_1)$ cut the pair of pants into two hexagons with angles $\pi/2$ and the required side-lengths.

### 2.2 Hyperbolizing compact surfaces with boundaries

Recall that compact surfaces with boundaries are topologically classified by their genus and their number of boundary components. Let $S$ be a compact surface of genus $g$ with $n$ boundary components. Then the Euler characteristic $\chi(S)$ of $S$ is equal to $2 - 2g - n$. It is negative unless $S$ is a sphere, a disc, a cylinder or a torus. In particular, a pair of pants has Euler characteristic $-1$.

A surface $S$ of negative Euler characteristic can always be cut along a finite number of disjoint simple closed curves to obtain a collection of pairs of pants. Though there are several combinatorial ways of doing so, it will always require $3g - 3 + n$ curves, and the surface will always be cut into $-\chi(S) = 2g - 2 + n$ pairs of pants.

Let us fix a pair of pants decomposition of $S$, that is, a family of disjoint simple closed curves $(c_i)_{1 \leq i \leq 3g - 3 + n}$ cutting $S$ into pairs of pants. Assume now that $S$ is provided with a hyperbolic metric with geodesic boundary. Then every curve $c_i$ is freely homotopic to a unique closed geodesic of length $l_i$. What’s more, those geodesics are still disjoint. Up to homotopy, one can thus assume that the curves $c_i$ are disjoint simple closed geodesics. Therefore, those curves cut the surface $S$ into hyperbolic pairs of pants.

Conversely, one can recover a hyperbolic metric on $S$ such that the curves $c_i$ and the boundary components of $S$ have prescribed lengths by taking hyperbolic pairs of pants with suitable boundary lengths, and gluing them along their boundary components in the proper combinatorial way. In doing so, there is a choice, for any curve $c_i$, of a “twist parameter” $\tau_i$ that tells you how to glue the pair of pants on one side of $c_i$ to the pair of pants on the other side. We will come back to this in the next section.
2.3 Punctures and cusps

We define a surface of genus $g$ with $n$ boundary components and $k$ punctures as the complement of $k$ points in a compact surface of genus $g$ with $n$ boundary components. The uniformisation theorem implies that such a surface can carry a complete hyperbolic metric such that each puncture is a cusp, meaning that the curve going around the puncture is freely homotopic to arbitrarily short curves. The prototype of a cusp is the quotient of $\mathbb{H}^2$ by a parabolic isometry.

A hyperbolic surface of genus $g$ with $n$ geodesic boundary components and $k$ cusps can be seen as a limit of hyperbolic surfaces of genus $g$ with $n+k$ geodesic boundary components such that the lengths of $k$ of those boundary components go to 0. Therefore one can consider a hyperbolic surface with cusps as a hyperbolic surface with some “geodesic boundary components of length 0”. With this convention, all what we will say from now on applies to hyperbolic metrics with cusps on surfaces with punctures.

3 Teichmüller spaces and moduli spaces

3.1 The Teichmüller–Fricke space

Let $S$ be a closed surface of genus $g \geq 2$. The Teichmüller space of $S$, denoted $\mathcal{T}(S)$, is the space of complex structures on $S$ modulo isotopies. It is named after Oswald Teichmüller who proved that it is homeomorphic to $\mathbb{R}^{6g-6}$. By the uniformisation theorem, it is also homeomorphic to the space of hyperbolic metrics on $S$ modulo isotopies, often called the Fricke space. However, to be consistent with Mirzakhani’s terminology, we will consider the Teichmüller space as a space of hyperbolic metrics and generalize its definition:

Définition 3.1. Let $S$ be a compact surface of genus $g$ with $n$ boundary components (labeled from 1 to $n$). Fix some real numbers $L_1, \ldots, L_n \geq 0$. We define the Teichmüller space

$$\mathcal{T}(S, L_1, \ldots, L_n)$$

as the space of hyperbolic metrics on $S$ such that the $i$-th boundary component is a geodesic of length $L_i$, modulo the action of the isotopies of $S$ (fixing setwise each boundary component).

3.2 Fenchel–Nielsen coordinates

The Teichmüller space $\mathcal{T}(S, L_1, \ldots, L_n)$ is homeomorphic to $\mathbb{R}^{6g-6+2n}$. We will prove it by giving explicit coordinates called the Fenchel–Nielsen coordinates.

Let us fix a pants decomposition $(c_i)_{1 \leq i \leq 3g-3+n}$ of the surface $S$. Considerations of the previous section show that $\mathcal{T}(S, L_1, \ldots, L_n)$ is parametrized by then lengths $\lambda_i$ of the geodesics homotopic to $c_i$, together with a twist parameter $\tau_i$ for each curve $c_i$. Giving a proper definition of $\tau_i$ is neither easy nor canonical. What is intrinsically define (once you fixed the pants decomposition) is a flow $\phi_{t,i}$ that cuts a hyperbolic metric along the geodesic homotopic to $c_i$ and glue it back after twisting the “right side” of the geodesic so that a point on the right geodesic has moved “forward” by a length $t$. (“Right side” and “forward” depend on an orientation of $c_i$, but the result is the same if you switch orientation).
Since for $i \neq j$, the geodesics homotopic to $c_i$ and $c_j$ are disjoint, it is clear that the flow $\phi_i$ and $\phi_j$ commute. To obtain coordinates, we only need to choose continuously, for any data of geodesic lengths $(\lambda_i)$, a standard way of gluing the pairs of pants along $c_i$. We can then decide that this will be a point with all $\tau_i = 0$, and that the flow $\phi_{i,t}$ will add $t$ to the coordinate $\tau_i$.

An important point here is that the coordinates $\tau_i$ have values in $\mathbb{R}$. Consider a hyperbolic metric with coordinates $(\lambda_i, \tau_i)$. When one applies $\phi_{i,t}$ the right side of of the curve $c_i$ makes a full turn, so we may think that we get the same hyperbolic metric. Indeed, the initial metric and its image by $\phi_{i,t}$ are isometric, but this isometry is not an isotopy and therefore the two points do not coincide on Teichmüller space. However, they will coincide in the moduli space.

3.3 Mapping class group and moduli space

Dénfinition 3.2. The Mapping class group of a surface $S$ with $n$ boundary components, denoted $\text{MCG}(S)$, is the quotient of the group of homeomorphisms of $S$ fixing setwise each boundary component by the subgroup of homeomorphisms isotopic to the identity.

Here is a fundamental example of an element of the mapping class group of $S$. Let $a$ be a simple closed curve in $S$ (not homotopic to a boundary component). Cut the surface along the curve $a$, then glue it back after applying a full-turn twist on the right side of $a$. We obtain a homeomorphism of $S$ that is not in the connected component of the identity. The isotopy class of this element in the mapping class group is called the Dehn twist along $a$. We will denote it $T_a$.

A “generic” element of $\text{MCG}(S)$ is not a Dehn twist, but Dehn twists generate the mapping class group. More precisely, Dehn proved that the mapping class group of a closed surface of genus $g$ can be generated by $2g(g - 1)$ Dehn twists (this number was improved later by Lickorish (see [?])).

The natural action of homeomorphisms of $S$ on hyperbolic metrics induces an action of $\text{MCG}(S)$ on $\mathcal{T}(S, L_1, \ldots, L_n)$ which is properly discontinuous. This leads to the following definition:

Dénfinition 3.3. Let $S$ be a compact surface of genus $g$ with $n$ boundary components (labeled from 1 to $n$). Fix some real numbers $L_1, \ldots, L_n \geq 0$. We define the moduli space

$$\text{Mod}(S, L_1, \ldots, L_n)$$

as the space of hyperbolic metrics on $S$ such that the $i$-th boundary component is a geodesic of length $L_i$, modulo the action of the whole group of homeomorphisms of $S$ (fixing setwise each boundary component).

The moduli space is thus the quotient of the Teichmüller space by the mapping class group.

If we fix a pants decomposition $(c_i)_{1 \leq i \leq 3g-3+n}$ of $S$, and associated Fenchel–Nielsen coordinates $(\lambda_i, \tau_i)$ on $\mathcal{T}(S, L_1, \ldots, L_n)$, it is clear, from previous remarks, that the Dehn twist along $c_i$ preserves all coordinates except the coordinate $\tau_i$ which is translated by $\lambda_i$.

However, other Dehn twists are much more difficult to express in Fenchel–Nielsen coordinates, and in general, it is impossible to write an element of the mapping class group in those coordinates.
3.4 Weil–Petersson symplectic geometry

The Teichmüller space $T(S, L_1, \ldots, L_n)$ carries several geometric structures that are preserved by the mapping class group and therefore induce a geometry on the moduli space. Here we are particularly interested in a symplectic form called the Weil–Petersson symplectic form.

A way of defining the Weil–Petersson symplectic form $\omega_{WP}$ is to identify the space $T(S, L_1, \ldots, L_n)$ with a character variety. More precisely, let $h$ be a hyperbolic metric on $S$ with geodesic boundary components of lengths $L_1, \ldots, L_n$. Then $(S, h)$ is isometric to the quotient of a convex domain in $\mathbb{H}^2$ by an action of $\pi_1(S)$ given by a representation $j: \pi_1(S) \to \text{Isom}(\mathbb{H}^2)$ which is discrete, faithful and sends the homotopy class of a curve circling around the $i$-th boundary component to a hyperbolic isometry of translation length $L_i$. This representation is well defined up to conjugation by an isometry of $\mathbb{H}^2$, and $T(S, L_1, \ldots, L_n)$ thus identifies with the space of such representations modulo conjugation.

Now, Goldman defined, in a very general setting, a symplectic structure on certain character varieties of surface groups. This symplectic form, in our special situation, is the Weil–Petersson symplectic form. It is naturally invariant by the mapping class group, and therefore induces a symplectic form on the moduli space $\text{Mod}(S, L_1, \ldots, L_n)$.

The Weil–Petersson symplectic is also the symplectic form associated to a natural Kähler metric on $T(S, L_1, \ldots, L_n)$. Using this point of view, Wolpert proved that $\omega_{WP}$ has a very nice expression in Fenchel–Nielsen coordinates.

**Theorem 4** (Wolpert, [Wol82]). Let $(c_i)_{1 \leq i \leq 3g-3+n}$ be a pants decomposition of $S$ and $(\lambda_i, \tau_i)$ associated Fenchel–Nielsen coordinates on $T(S, L_1, \ldots, L_n)$. Then we have

$$\omega_{WP} = \sum_{i=1}^{3g-3+n} d\lambda_i \wedge d\tau_i.$$  

For our (and Mirzakhani’s) purpose, one can take this as a definition of $\omega_{WP}$ and admit (which is by no mean obvious) that this symplectic form is invariant under the mapping class group.

This symplectic form induces a volume form on $\text{Mod}(S, L_1, \ldots, L_n)$, and we can finally define properly the volume of those moduli spaces:

**Définition 3.4.** The volume $\text{Vol}(S, L_1, \ldots, L_n)$ of the moduli space $\text{Mod}(S, L_1, \ldots, L_n)$ is the integral of the volume form associated to the symplectic form $\omega_{WP}$:

$$\text{Vol}(S, L_1, \ldots, L_n) = \int_{\text{Mod}(S, L_1, \ldots, L_n)} \omega_{WP}^{3g-3+n}.$$  

It follows from Wolpert’s theorem that in Fenchel–Nielsen coordinates, the volume form associated to $\omega_{WP}$ is translation invariant:

$$\omega_{WP}^{3g-3+n} = d\lambda_1 \wedge d\tau_1 \wedge \ldots d\lambda_{3g-3+n} \wedge d\tau_{3g-3+n}.$$  

To compute the total volume of the moduli space, it would therefore suffice to describe a fundamental domain of the action of the mapping class group in Fenchel–Nielsen coordinates. But this is precisely where the difficulty is. We cannot find such a fundamental domain because we cannot express the action of the mapping class group in Fenchel–Nielsen coordinates.
References


