p-adic Modular Forms: Serre, Katz, Coleman, Kassaei

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Serre p-adic modular forms

- Formes modulaires et fonctions zêta p-adiques, Modular Functions of One Variable III (Antwerp 1972)
- Motivation: Study special values of p-adic $L$-functions.
- Idea is to capture congruences between modular forms topologically.
Example: For $p \geq 5$, $E_{p^{m-1}(p-1)} = E_{p-1}^{p^{m-1}} \equiv 1 \pmod{p^m}$.

Example:

$$G_2 = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$G_2^* := G_2 - p^{k-1} G_2|_V,$$

Then:

$$G_2 = \sum_{m=0}^{\infty} p^m G_2^*|_V^m.$$

$G_2$ is a "$p$-adic modular form".
Serre p-adic modular forms

- $f = \sum_{n \leq 0} a_n q^n \in \mathbb{Q}_p[[q]]$.
- $v_p(f) = \inf v_p(a_n)$.
- **Definition:**
  1. For a sequence $\{f_i\}_{i \in \mathbb{N}} \in \mathbb{Q}_p[[q]]$, we say $f_i \to f$ if $v_p(f_i - f) \to \infty$.
  2. $f \in \mathbb{Q}_p[[q]]$ is a $p$-adic modular form if there exists $\{f_i \in M_{k_i}\}_{i \in \mathbb{N}}$ such that $f_i \to f$.
- **Remark:** $f \in M_k, g \in M_{k'}$,
  
  $f \equiv g \pmod{p^m} \Rightarrow k \equiv k' \pmod{p^{m-1}(p-1)}$. 

Serre pMF: Properties

- $X := \mathbb{Z}_p \times \mathbb{Z}/(p - 1)\mathbb{Z}$.
- $\{f_i \in M_{k_i}\}_{i \in \mathbb{N}}, f_i \rightarrow f$, then $\exists k \in X$, independent of $f_i, k_i$ such that $k_i \rightarrow k$.
- Elements of $X$ can be considered as characters $\mathbb{Z}_p^* \rightarrow \mathbb{C}_p^*$. 
Can define operators $U$, $V$, $T_l$.

Let $\lambda \in \mathbb{C}_p$, $v(\lambda) > 0$. Pick a $p$-adic modular form $f_0$:

1. eigenform for all $T_l$, $l \neq p$,
2. $f_0|U = 0$,
3. Note that $f_0|U = 0 \iff a_n(f_0) = 0$ whenever $p|n$.

Example: $f_0 = (1 - VU)\Delta$.

\[ f_\lambda := \sum_{n=0}^{\infty} \lambda^n f_0|V^n \text{ is a } p \text{-adic modular form.} \]

Then $a_n(f_0) = a_n(f_\lambda)$ whenever $p \nmid n$, and $f_\lambda|U = \lambda f_\lambda$.

$(UV = id)$.

This rules out a good spectral theory: cannot hope to write a modular form as a sum of eigenforms.
Katz pMF: moduli of elliptic cuves

- Approach: moduli of elliptic curves, Igusa, Deligne.
- $\mathcal{P} : \text{Sch}/\mathbb{Z}[1/N] \rightarrow \text{Sets}, \mathcal{P}(S) = \{(E/S, P)\}$, elliptic curves with $\Gamma_1(N)$-structures.
- For $N \geq 5$, $\mathcal{P}$ is representable by an affine scheme $Y = Y_1(N)_{\mathbb{Z}[1/N]}$, universal family $\mathcal{E}$.

$$
\begin{array}{c}
\mathcal{E} \\
\pi \\
\downarrow \\
Y
\end{array}
$$

- $\omega = 0^*\Omega_{\mathcal{E}/Y}$ invertible sheaf.
- $X = X_1(N)$ compactifies $Y_1(N)$, moduli scheme of ”generalized elliptic curves”, $\omega$ extends to $X$. 
Katz pMF: modular forms

- $A$ is a $\mathbb{Z}[1/N]$-algebra, $X_A = X \otimes_{\mathbb{Z}[1/N]} A$, $\omega_A = \omega \otimes_{\mathbb{Z}[1/N]} A$.
- A modular form over $A$ of weight $k$ and level $\Gamma_1(N)$ is a sections $f \in H^0(X_A, \omega_A^\otimes k)$.
- Alternatively: $f$ is a rule assigning to each triple $(E/R, \omega, P)$ an element of $R$ depending only on the isoclass of $(E/R, \omega, P)$, commuting with base change, and $f(E/R, \lambda \omega, P) = \lambda^{-k} f(E/R, \omega, P)$ for $\lambda \in R^\times$.
- Evaluating $f$ at $(\text{Tate}(q), w_{can})$, get the $q$-expansion of $f$. 
Katz pMF: Hasse invariant

- $w \in H^0(E, \Omega^1_{E/R})$, $\eta \in H^1(E, O_E)$ its dual,
  $F_{abs}: O_E \to O_E$, $f \mapsto f^p$, induces
  $F^*_{abs}: H^1(E, O_E) \to H^1(E, O_E)$, has rank 1.

- Define the Hasse invariant $A(E/R, w)$ by
  $F^*_{abs}(\eta) = A(E/R, w) \eta$, hence $A(E/R, \lambda w) = \lambda^{1-p} A(E/R, w)$
  for $\lambda \in R^\times$.

- $A$ is a modular form of level 1 and weight $p - 1$ with
  $A(Tate(q), w_{can}) = 1$. By q-expansion principle,

  \[ A = (E_{p-1} \pmod{p}). \]
Katz pMF: going p-adic

- $p$-adic modular forms: "$H^0(X \otimes \mathbb{Z}_p, \omega \otimes k)$"?
- Recall: for $p \geq 5$, $E_{p^{m-1}(p-1)} = E_{p^{m-1}}^{p^{m-1}} \equiv 1 \pmod{p^m}$.
- A lift of the Hasse invariant should be invertible.
- Problem 1: $A$ vanishes at the supersingular points.
- Solution 1: throw away elliptic curves which are supersingular or have supersingular reduction.
Katz pMF: going rigid-analytic

- Consider $X \otimes \overline{\mathbb{Q}}_p$, $X^{ord}$ the locus corresponding to elliptic curves with good ordinary, or multiplicative, reduction.
- Problem 2: $X^{ord}$ and $SS := X \setminus X^{ord}$ both have infinitely many points, so cannot be subvarieties (Since $X$ is a curve).
- Solution 2: Forget the Zariski topology. $SS$ is isomorphic to a finite union of $p$-adic discs corresponding to supersingular $j$-invariants in char $p$. Hence $X^{ord}$ has the structure of a rigid analytic space $X_{\geq 0}$, and inherits an invertible analytic sheaf $w^{an}$.
- $p$-adic modular forms as $H^0(X_{\geq 0}, (w^{an}) \otimes k)$: these are the convergent modular forms.
- **Theorem (Katz):** Space of convergent modular forms $\cong$ Serre $p$-adic modular forms (as a Banach space and Hecke module).
- We are throwing away too many elliptic curves.
Katz pMF: modular forms

- $R_0$ ring of integers in a finite extension of $\mathbb{Q}_p$, $R$ an $R-algebra$ in which $p$ is nilpotent
- $f$ can alternatively be seen as a rule acting on $(E/R, \omega, P, Y)$ where $YE_{p-1} = 1$.

**Definition:** A $\rho$-overconvergent modular form is a rule acting on $(E/R, \omega, P, Y)$ where $YE_{p-1} = \rho \in R_0 \setminus \{0\}$.
- $YE_{p-1}(E, \omega) = \rho \Rightarrow v_p(E_{p-1}(E, \omega)) \leq r := v_p(\rho)$.
- If $r < 1$, this definition is independent of the lift of the Hasse invariant.

**Definition:** $X_{\geq r} := X \setminus \{x \text{corresponding to } E : v_p(E) > r\}$. If $r < 1$, this definition is independent of the lift of the Hasse invariant. Then $\rho$-overconvergent modular forms of weight $k$ are $H^0(X_{\geq v_p(\rho)}, (w_{\text{can}}) \otimes k)$.
- If $r < \frac{p}{p+1}$, we have a continuous action of the Hecke operators, and $U$ is a compact operator, hence a good spectral theory.
Rigid geometry

- Tate: elliptic curves with multiplicative reduction.
- $\mathbb{Q}_p$-analytic manifold: locally ring space locally isomorphic to $\mathbb{Z}_p^n$, sheaf of locally analytic functions.
- It’s totally disconnected. Too many locally constant functions.
Rigid geometry: affinoid algebras

- **Definition:** $T_n = \mathbb{Q}_p < x_1, \cdots, x_n > \subseteq \mathbb{Q}_p[[x_1, \cdots, x_n]]$ such that if $f = \sum a_\alpha t^\alpha \in T_n$ then $a_\alpha \to 0$ in $\mathbb{Q}_p$ as $|\alpha| \to \infty$.

- These are the rigid analytic functions on $\mathbb{Z}_p^n$. An affinoid algebra is $A = T_n/I$ for some ideal $I$ of $T_n$.

- Close to polynomial algebras.

- **Proposition (Tate):** $T_n$ is Noetherian, Jacobson, UFD, regular of equidimension $n$, and the Nullstellensatz holds: if $m$ is a maximal ideal in $T_n$, then $[T_n/m : \mathbb{Q}_p] < \infty$. 
Rigid geometry: "weak" $G$-topology

- The maximum spectrum: $X = \text{Max}(A)$.
- Since NSS holds, a morphism of $\mathbb{Q}_p$-algebras $A \to B$ induces a morphism $\text{Max}(B) \to \text{Max}(A)$.
- Admissible opens: affinoid subdomains, i.e. $U \subset X$ such that $\exists X' = \text{Max}(A') \to U \subset X$ which is universal: if $X'' = \text{Max}A'' \to U \subset X$ then this factors through $X'$.
- Admissible open coverings: finite coverings by affinoid subdomains.
- Presheaf: $\mathcal{O}_X(X(f/g)) = A < f/g >$ where $X(f/g) = \text{Max}(A < f/g >)$. By Tate acyclicity, $\mathcal{O}_X$ is a sheaf.
Rigid geometry: ”strong” G-topology

- There exists a Grothendieck topology on $X$ satisfying:
  1. $G_0$: $\emptyset$ and $X$ are admissible opens.
  2. $G_1$: if $U \subset X$ admissible open, $V \subset X$, and there exists an admissible covering $\{U_i\}$ of $U$ such that $V \cap U_i$ is admissible open in $X$ for all $i$, then $V$ is admissible open.
  3. $G_2$: if $\{U_i\}_{i \in I}$ be some covering of an admissible open $U$ such that $U_i$ is admissible open for all $i$, and if it has a refinement, then it is an admissible covering.

- $O_X$ extends uniquely to a sheaf $O_X$ in the strong topology.
Definition:

1. An affinoid variety over $\mathbb{Q}_p$ is a pair $(X, O_X)$, $X = \text{Max}(A)$ for an affinoid $\mathbb{Q}_p$-algebra $A$ equipped with the strong topology and a sheaf $O_X$ with respect to it. Write $\text{Sp}(A) = (\text{Max}(A), O_{\text{Max}(A)})$.

2. A rigid analytic variety is a set $X$ with a Grothendieck topology satisfying $G0 - G2$ and a sheaf $O_X$ of $\mathbb{Q}_p$-algebras such that there exists an admissible covering $X = \bigcup_{i \in I} X_i$ where each $(X_i, O_X|_{X_i})$ is isomorphic to an affinoid variety.
Rigid geometry: "an" functor

- **Proposition:** $X = \bigcup_{i \in I} X_i$ a set, $X_i$ has Grothendieck topology satisfying $G_0 - G_2$ for each $i$, and compatible, then there exists a Grothendieck topology on $X$ satisfying:
  1. $X_i$ is admissible open in $X$, restricts to the Grothendieck topology on $X_i$.
  2. $G_0 - G_2$.
  3. \{$X_i\}_{i \in I}$ is an admissible covering of $X$.

- There exists a functor $an$ from the category of schemes $X$ over $\mathbb{Q}_p$ locally of finite type to the category of rigid analytic varieties $X$ over $\mathbb{Q}_p$.
- $(Spec A)^{an} = Sp(A)$.
- There exists a functor $an$ from $O_X$-modules to $O_{X^{an}}$-modules which is exact, faithful, takes coherent sheaves to coherent sheaves, and $\mathcal{F}^{an} = 0 \iff \mathcal{F} = 0$. 

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Back to classical modular forms

- We have a nice theory of overconvergent modular forms, with analytic tools at our disposal.
- How can we get back to the classical case?
- Fix $p$, $N$ coprime.

**Definition:** Let $v_p$ be the $p$-adic valuation in $\mathbb{Q}_p$, normalized so that $v_p(p) = 1$. For a $p$-adic modular form $f$ over $\mathbb{Q}_p$, the slope $v_p(f) := v_p(a_p(f))$.

**Proposition:** Let $f$ be a classical eigenform for $U := U_p$. Then $f$ has slope at most $k - 1$

**Proof (sketch):** If $f$ is a newform, a computation shows that $v_p(f) = \frac{k-2}{2}$. If $f$ is an oldform, it’s in the span of $g(z)$ and $g(pz)$ for some $g$, and this span is stable under $U_p$. On this space $U_p$ has the characteristic polynomial $x^2 - a_p(g) + p^{k-1}$, of which $a_p(f)$ is a root.

**Theorem (Hida):** if $f$ is a $p$-adic $U$-eigenform of weight $k \geq 2$ and slope 0, then $f$ is classical.
Theorem (Coleman, 96): Let $f$ be a $p$-adic overconvergent modular form of level $\Gamma_1(Np)$ weight $k$, with slope $\nu_p(f) < k - 1$, and which is a generalized eigenvector for $U$. Then $f$ is classical.

Theorem (Coleman, 96): Let $f$ be a $p$-adic overconvergent Hecke eigenform of weight of level $\Gamma_1(N)$ and $k \geq 2$ and slope $k - 1$ such that $f \not\in \theta^{k-1}M_{2-k}$. Then $f$ is classical.

In fact we have a stronger result due to Kassaei:

Theorem (Kassaei 06): Let $f$ be an overconvergent modular form of level $\Gamma_1(Np^m)$ of weight $k$, defined over $K$, a finite extension of $\mathbb{Q}_p$. Let $R(x) \in K[x]$ whose roots in $\mathbb{C}_p$ have valuation $< k - 1$. If $R(U)f$ is classical, then so is $f$.

Coleman’s theorem follows from this by taking $R(x) = (x - \lambda)^n$ (since 0 a classical).
Coleman, Kassaei

- **Proof idea:** This relies on a result of Buzzard and Taylor on analytic continuation of modular forms. For simplicity, take \( m = 1 \), assume \( Uf = af \), \( \nu_p(a) < k - 1 \).

1. \( Z^\infty \) and \( Z^0 \) are the connected components of \( X_1(Np)_K^{an} \) which contain the cusp \( \infty \) and 0 respectively.
2. Buzzard: Can extend \( f \) to \( U_1 \) the rigid analytic part of \( X_1(Np)_K^{an} \) whose noncuspidal points correspond to \( (E, i, P) \), \( i \) a \( \Gamma_1(N) \)-structure and \( P \) a point of order \( p \), and either \( E \) has supersingular reduction or \( E \) has ordinary reduction and \( P \) generates the canonical subgroup of \( E \) (equivalently \( (E, i, P) \in Z^\infty \)).
3. Using a gluing lemma, show that \( f \) extends to the complement \( Z^0 \), so that \( f \) is defined on all of \( X_1(Np)_K^{an} \).
4. By rigid-analytic GAGA, \( f \) is classical.