p-adic Modular Forms: Serre, Katz, Coleman, Kassaei

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Serre p-adic modular forms

- Formes modulaires et fonctions zêta p-adiques, Modular Functions of One Variable III (Antwerp 1972)
- Motivation: Study special values of p-adic L-functions.
- Idea is to capture congruences between modular forms topologically.

Serre p-adic modular forms

- Example: For $p \geq 5$, $E_{p^{m-1}(p-1)} = E_{p-1}^{p^{m-1}} \equiv 1 \pmod{p^m}$.
- Example:

$$G_2 = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$G_2^* := G_2 - p^{k-1} G_2 | V,$$

Then:

$$G_2 = \sum_{m=0}^{\infty} p^m G_2^* | V^m.$$

- G_2 is a "p-adic modular form".

Serre p-adic modular forms

- $f = \sum_{n\leq 0} a_n q^n \in \mathbb{Q}_p[[q]]$.
- $v_p(f) = \inf v_p(a_n)$.
- **Definition:**
  1. For a sequence $\{f_i\}_{i\in\mathbb{N}} \in \mathbb{Q}_p[[q]]$, we say $f_i \to f$ if $v_p(f_i - f) \to \infty$.
  2. $f \in \mathbb{Q}_p[[q]]$ is a p-adic modular form if there exists $\{f_i \in M_{k_i}\}_{i\in\mathbb{N}}$ such that $f_i \to f$.
- **Remark:** $f \in M_k, g \in M_{k'}$,

$$f \equiv g \pmod{p^m} \Rightarrow k \equiv k' \pmod{p^{m-1}(p-1)}.$$

Serre pMF: Properties

- $X := \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$.
- $\{f_i \in M_{k_i}\}_{i \in \mathbb{N}}$, $f_i \to f$, then $\exists k \in X$, independent of f_i, k_i such that $k_i \to k$.
- Elements of X can be considered as characters $\mathbb{Z}_p^* \to \mathbb{C}_p^*$.

Serre pMF: Spectral theory?

- Can define operators U, V, T_l.
- Let $\lambda \in \mathbb{C}_p$, $v(\lambda) > 0$. Pick a p-adic modular form f_0:
  1. eigenform for all T_l, $l \neq p$,
  2. $f_0|U = 0$,
  3. Note that $f_0|U = 0 \Leftrightarrow a_n(f_0) = 0$ whenever $p|n$.
- Example: $f_0 = (1 - VU)\Delta$.
- $f_\lambda := \sum_{n=0}^{\infty} \lambda^n f_0|V^n$ is a p-adic modular form.
- Then $a_n(f_0) = a_n(f_\lambda)$ whenever $p \nmid n$, and $f_\lambda|U = \lambda f_\lambda$. $(UV = id)$.
- This rules out a good spectral theory: cannot hope to write a modular form as a sum of eigenforms.

Katz pMF: moduli of elliptic cuves

- Katz (Antwerp 1972). Atkin, Swinnerton-Dyer, and Serre on congruence properties of q-expansions of modular forms.
- Approach: moduli of elliptic curves, Igusa, Deligne.
- $\mathcal{P} : Sch/\mathbb{Z}[1/N] \to Sets, \mathcal{P}(S) = \{(E/S, P)\}$, elliptic curves with $\Gamma_1(N)$-structures.
- For $N \geq 5$, $\mathcal{P}$ is representable by an affine scheme $Y = Y_1(N)_{\mathbb{Z}[1/N]}$, universal family $\mathcal{E}$.
- $$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ Y \arrow[u, bend right, "0"'] \end{tikzcd}$$
- $\omega = 0^*\Omega_{\mathcal{E}/Y}$ invertible sheaf.
- $X = X_1(N)$ compactifies $Y_1(N)$, moduli scheme of "generalized elliptic curves", ω extends to X.

Katz pMF: modular forms

- A is a $\mathbb{Z}[1/N]$-algebra, $X_A = X \otimes_{\mathbb{Z}[1/N]} A$, $\omega_A = \omega \otimes_{\mathbb{Z}[1/N]} A$.
- A modular form over A of weight k and level $\Gamma_1(N)$ is a sections $f \in H^0(X_A, \omega_A^{\otimes k})$.
- Alternatively: f is a rule assigning to each triple $(E/R, \omega, P)$ an element of R depending only on the isoclass of $(E/R, \omega, P)$, commuting with base change, and $f(E/R, \lambda w, P) = \lambda^{-k} f(E/R, w, P)$ for $\lambda \in R^\times$.
- Evaluating f at $(Tate(q), w_{can})$, get the q-expansion of f.

Katz pMF: Hasse invariant

- $w \in H^0(E, \Omega^1_{E/R})$, $\eta \in H^1(E, O_E)$ its dual,
  $F_{abs} : O_E \to O_E, f \mapsto f^p$, induces
  $F^*_{abs} : H^1(E, O_E) \to H^1(E, O_E)$, has rank 1.
- Define the Hasse invariant $A(E/R, w)$ by
  $F^*_{abs}(\eta) = A(E/R, w)\eta$, hence $A(E/R, \lambda w) = \lambda^{1-p} A(E/R, w)$
  for $\lambda \in R^\times$.
- A is a modular form of level 1 and weight $p - 1$ with
  $A(Tate(q), w_{can}) = 1$. By q-expansion principle,

$$A = (E_{p-1} \pmod{p}).$$

Katz pMF: going p-adic

- p-adic modular forms: "$H^0(X \otimes \mathbb{Z}_p, \omega^{\otimes k})$"?
- Recall: for $p \geq 5$, $E_{p^{m-1}(p-1)} = E_{p-1}^{p^{m-1}} \equiv 1 \pmod{p^m}$.
- A lift of the Hasse invariant should be invertible.
- Problem 1: A vanishes at the supersingular points.
- Solution 1: throw away elliptic curves which are supersingular or have supersingular reduction.

Katz pMF: going rigid-analytic

- Consider $X \otimes \overline{\mathbb{Q}}_p$, X^{ord} the locus corresponding to elliptic curves with good ordinary, or multiplicative, reduction.
- Problem 2: X^{ord} and $SS := X \setminus X^{ord}$ both have infinitely many points, so cannot be subvarieties (Since X is a curve).
- Solution 2: Forget the Zariski topology. SS is isomorphic to a finite union of p-adic discs corresponding to supersingular j-invariants in char p. Hence X^{ord} has the structure of a *rigid analytic space* $X_{\geq 0}$, and inherits an invertible analytic sheaf w^{an}.
- p-adic modular forms as $H^0(X_{\geq 0}, (w^{an})^{\otimes k})$: these are the *convergent modular forms*.
- **Theorem (Katz):** Space of convergent modular forms $\cong$ Serre p-adic modular forms (as a Banach space and Hecke module).
- We are throwing away too many elliptic curves.

Katz pMF: modular forms

- R_0 ring of integers in a finite extension of $\mathbb{Q}_p$, R an $R-algebra$ in which p is nilpotent
- f can alternatively be seen as a rule acting on $(E/R, \omega, P, Y)$ where $YE_{p-1} = 1$.
- **Definition:** A ρ-overconvergent modular form is a rule acting on $(E/R, \omega, P, Y)$ where $YE_{p-1} = \rho \in R_0 \setminus \{0\}$.
- $YE_{p-1}(E, \omega) = \rho \Rightarrow v_p(E_{p-1}(E, \omega)) \leq r := v_p(\rho)$.
- If $r < 1$, this definition is independent of the lift of the Hasse invariant.
- **Definition:** $X_{\geq r} := X \setminus \{x corresponding to E : v_p(E) > r\}$. If $r < 1$, this definition is independent of the lift of the Hasse invariant. Then ρ-overconvergent modular forms of weight k are $H^0(X_{\geq v_p(\rho)}, (w^{can})^{\otimes k})$.
- If $r < \frac{p}{p+1}$, we have a continuous action of the Hecke operators, and U is a compact operator, hence a good spectral theory.

Rigid geometry

- Tate: elliptic curves with multiplicative reduction.
- $\mathbb{Q}_p$-analytic manifold: locally ring space locally isomorphic to $\mathbb{Z}_p^n$, sheaf of locally analytic functions.
- It's totally disconnected. Too many locally constant functions.

Rigid geometry: affinoid algebras

- **Definition:** $T_n = \mathbb{Q}_p < x_1, \cdots, x_n > \subseteq \mathbb{Q}_p[[x_1, \cdots, x_n]]$ such that if $f = \sum_\alpha a_\alpha t^\alpha \in T_n$ then $a_\alpha \to 0$ in $\mathbb{Q}_p$ as $|\alpha| \to \infty$.
- These are the rigid analytic functions on $\mathbb{Z}_p^n$. An affinoid algebra is $A = T_n/I$ for some ideal I of T_n.
- Close to polynomial algebras.
- **Proposition (Tate):** T_n is Noetherian, Jacobson, UFD, regular of equidimension n, and the Nullstellensatz holds: if $\mathfrak{m}$ is a maximal ideal in T_n, then $[T_n/\mathfrak{m} : \mathbb{Q}_p] < \infty$.

Rigid geometry: "weak" G-topology

- The maximum spectrum: $X = Max(A)$.
- Since NSS holds, a morphism of $\mathbb{Q}_p$-algebras $A \to B$ induces a morphism $Max(B) \to Max(A)$.
- Admissible opens: affinoid subdomains, i.e. $U \subset X$ such that $\exists X' = Max(A') \to U \subset X$ which is universal: if $X'' = MaxA'' \to U \subset X$ then this factors through X'.
- Admissible open coverings: finite coverings by affinoid subdomains.
- Presheaf: $O_X(X(f/g)) = A < f/g >$ where $X(f/g) = Max(A < f/g >)$. By Tate acyclicity, O_X is a sheaf.

Rigid geometry: "strong" G-topology

- There exists a Grothendieck topology on X satisfying:
  1. G0: $\emptyset$ and X are admissible opens.
  2. G1: if $U \subset X$ admissible open, $V \subset X$, and there exists an admissible covering $\{U_i\}$ of U such that $V \cap U_i$ is admissible open in X for all i, then V is admissible open.
  3. G2: if $\{U_i\}_{i \in I}$ be some covering of an admissible open U such that U_i is admissible open for all i, and if it has a refinment, then it is an admissible covering.
- O_X extends uniquely to a sheaf O_X in the strong topology.

Rigid geometry: rigid analytic varieties

- **Definition:**
  1. An affinoid variety over $\mathbb{Q}_p$ is a pair (X, O_X), $X = Max(A)$ for an affinoid $\mathbb{Q}_p$-algebra A equipped with the strong topology and a sheaf O_X with respect to it. Write $Sp(A) = (Max(A), O_{Max(A)})$.
  2. A rigid analytic variety is a set X with a Grothendieck topology satisfying $G0 - G2$ and a sheaf O_X of $\mathbb{Q}_p$-algebras such that there exists an admissible covering $X = \bigcup_{i \in I} X_i$ where each $(X_i, O_X|_{X_i})$ is isomorphic to an affinoid variety.

Rigid geometry: "an" functor

- **Proposition:** $X = \bigcup_{i \in I} X_i$ a set, X_i has Grothendieck topology satisfying $G0 - G2$ for each i, and compatible, then there exists a Grothendieck topology on X satisfying:
  1. X_i is admissible open in X, resticts to the Grothendieck topology on X_i.
  2. $G0 - G2$.
  3. $\{X_i\}_{i \in I}$ is an admissible covering of X.
- There exists a functor *an* from the category of schemes X over $\mathbb{Q}_p$ locally of finite type to the category of rigid analytic varietiyes X over $\mathbb{Q}_p$.
- $(SpecA)^{an} = Sp(A)$.
- There exists a functor *an* from O_X-modules to $O_{X^{an}}$-modules which is exact, faithful, takes coherent sheaves to coherent sheaves, and $\mathcal{F}^{an} = 0 \Leftrightarrow \mathcal{F} = 0$.

Back to classical modular forms

- We have a nice theory of overconvergent modular forms, with analytic tools at our disposal.
- How can we get back to the classical case?
- Fix p, N coprime.
- **Definition:** Let v_p be the p-adic valuation in $\overline{\mathbb{Q}}_p$, normalized so that $v_p(p) = 1$. For a p-adic modular form f over $\overline{\mathbb{Q}}_p$, the slope $v_p(f) := v_p(a_p(f))$.
- **Proposition:** Let f be a classical eigenform for $U := U_p$. Then f has slope at most $k - 1$
- *Proof (sketch):* If f is a newform, a computation shows that $v_p(f) = \frac{k-2}{2}$. If f is an oldform, it's in the span of $g(z)$ and $g(pz)$ for some g, and this span is stable under U_p. On this space U_p has the characteristic polynomial $x^2 - a_p(g) + p^{k-1}$, of which $a_p(f)$ is a root.
- **Theorem (Hida):** if f is a p-adic U-eigenform of weight $k \geq 2$ and slope 0, then f is classical.

Coleman, Kassaei

- **Theorem (Coleman, 96):** Let f be a p-adic overconvergent modular form of level $\Gamma_1(Np)$ weight k, with slope $v_p(f) < k-1$, and which is a generalized eigenvector for U. Then f is classical.
- **Theorem (Coleman, 96):** Let f be a p-adic overconvergent Hecke eigenform of weight of level $\Gamma_1(N)$ and $k \geq 2$ and slope $k-1$ such that $f \notin \theta^{k-1} M_{2-k}$. Then f is classical.
- In fact we have a stronger result due to Kassaei:
- **Theorem (Kassaei 06):** Let f be an overconvergent modular form of level $\Gamma_1(Np^m)$ of weight k, defined over K, a finite extension of $\mathbb{Q}_p$. Let $R(x) \in K[x]$ whose roots in $\mathbb{C}_p$ have valuation $< k-1$. If $R(U)f$ is classical, then so is f.
- Coleman's theorem follows from this by taking $R(x) = (x-\lambda)^n$ (since 0 a classical).

Coleman, Kassaei

- *Proof idea:* This relies on a result of Buzzard and Taylor on analytic continuation of modular forms. For simplicity, take $m = 1$, assume $Uf = af$, $v_p(a) < k - 1$.
  1. Z^∞ and Z^0 are the connected components of $X_1(Np)_K^{an}$ which contain the cusp ∞ and 0 respectively.
  2. Buzzard: Can extend f to U_1 the rigid analytic part of $X_1(Np)_K^{an}$ whose noncuspidal points correspond to (E, i, P), i a $\Gamma_1(N)$-structure and P a point of order p, and either E has supersingular reduction or E has ordinary reduction and P generates the canonical subgroup of E (equivalentely $(E, i, P) \in Z^\infty$).
  3. Using a gluing lemma, show that f extends to the complement Z^0, so that f is defined on all of $X_1(Np)_K^{an}$.
  4. By rigid-analytic GAGA, f is classical.