Drinfeld cusp forms and their combinatorics

Gebhard Böckle
gebhard.boeckle@uni-due.de

Department of Mathematics
Universität Duisburg-Essen
45117 Essen, Germany

Workshop on
Computations with Modular Forms
Bristol, August 22, 2008
Drinfeld modular forms
  The Bruhat-Tits tree
  The Drinfeld upper half plane
Drinfeld modular forms

Outline

Drinfeld modular forms
  The Bruhat-Tits tree
  The Drinfeld upper half plane
Drinfeld modular forms

Harmonic cocycles
  Definition
  Basic properties
  The residue map

How to understand the quotient tree?

References
Drinfeld modular forms
  The Bruhat-Tits tree
  The Drinfeld upper half plane
  Drinfeld modular forms
Harmonic cocycles
  Definition
  Basic properties
  The residue map
Drinfeld modular forms
   The Bruhat-Tits tree
   The Drinfeld upper half plane
   Drinfeld modular forms
Harmonic cocycles
   Definition
   Basic properties
   The residue map
How to understand the quotient tree?

Outline
Drinfeld modular forms
   The Bruhat-Tits tree
   The Drinfeld upper half plane
   Drinfeld modular forms
Harmonic cocycles
   Definition
   Basic properties
   The residue map
How to understand the quotient tree?
References
Drinfeld modular forms
   The Bruhat-Tits tree
   The Drinfeld upper half plane
Drinfeld modular forms
Harmonic cocycles
   Definition
   Basic properties
   The residue map
How to understand the quotient tree?
References
Basic notation

$\mathbb{F}_q$ the field of $q = p^n$ elements

$K_\infty$ a local field with residue field $\mathbb{F}_q$

$\mathcal{O}_\infty$ the ring of integers of $K_\infty$

$\pi$ a uniformizer of $K_\infty$
The Bruhat-Tits tree

Definition (Bruhat-Tits tree)

\(\mathcal{T} := \text{the simplicial complex of dimension 1 with}\)

set of vertices \(\text{Vert}(\mathcal{T}) :=\)

homothety classes \([L]\) of rank 2 \(\mathcal{O}_\infty\)-lattices \(L \subset K_\infty^2\)

set of edges \(\text{Edge}(\mathcal{T}) :=\)

pairs \(([L], [L'])\) such that \(\pi L \subsetneq L' \subsetneq L\).

|\(\mathcal{T}\)| the geometric realization of \(\mathcal{T}\)
The Bruhat-Tits tree

\[ K_\infty, O_\infty, \pi, \mathbb{F}_q \]

Definition (Bruhat-Tits tree)
\[ T := \text{the simplicial complex of dimension 1 with} \]

set of vertices \( \text{Vert}(T) := \]

homothety classes \([L]\) of rank 2 \( O_\infty\)-lattices \( L \subset K_\infty^2 \)

set of edges \( \text{Edge}(T) := \]

pairs \(([L], [L'])\) such that \( \pi L \subset L' \subset L \).

\(|T|\) the geometric realization of \( T \)

Lemma
\( T \) is a \( q + 1 \)-regular tree.

Definition
Vertices \( \Lambda_i := [O_\infty \oplus \pi^i O_\infty], i \in \mathbb{Z} \).
standard vertex \( \Lambda_0 \), standard edge \( e_0 := (\Lambda_0, \Lambda_1) \)
Group action

Consider elements of \( K_\infty^2 \) as **column vectors**

⇒ have natural left action of \( GL_2(K_\infty) \) on \( K_\infty^2 \).
Group action

Consider elements of $K_\infty^2$ as column vectors \( \Rightarrow \) have natural left action of \( GL_2(K_\infty) \) on \( K_\infty^2 \).

**Definition (\( GL_2(K_\infty) \)-action on \( T \))**

\[
GL_2(K_\infty) \times T \rightarrow T : (\gamma, [L]) \mapsto [\gamma L]
\]

Set \( \Gamma_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(O_\infty) \mid c \in \pi O_\infty \right\} \).

**Lemma**

\( GL_2(K_\infty) \) acts transitively on \( \text{Vert}(T) \) and \( \text{Edge}(T) \).

\[
\text{Vert}(T) = GL_2(K_\infty)/GL_2(O_\infty)K_\infty^*,
\]

\[
\text{Edge}(T) = GL_2(K_\infty)/\Gamma_\infty K_\infty^*.
\]
Drinfeld’s upper half plane $\Omega$

$\mathbb{C}_\infty := \hat{K}_\infty^{alg}$

**Definition (Drinfeld’s upper half plane)**

$\Omega := \mathbb{P}^1(\mathbb{C}_\infty) \setminus \mathbb{P}^1(K_\infty)$

**Definition (GL$_2(K_\infty)$-action on $\Omega$)**

$GL_2(K_\infty) \times \Omega \to \Omega : (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \gamma z = \frac{az+b}{cz+d}$
(rigid) analysis on $\Omega$

$K_\infty, \mathcal{O}_\infty, \pi, \mathbb{F}_q, \mathbb{C}_\infty, \Omega$

Let $b_1, \ldots, b_q \in \mathcal{O}_\infty$ be representatives of $\mathcal{O}_\infty/\pi$.

Proposition (reduction map)

$\exists$ a (natural) $GL_2(K_\infty)$-equivariant map

$$\rho : \Omega \rightarrow |T| \text{ such that}$$

$$\rho^{-1}(|e_0| \setminus \{\Lambda_0, \Lambda_1\}) = \{z \in \mathbb{C}_\infty \mid 1 < |z| < q\}$$

$$\rho^{-1}(\Lambda_0) = \{z \in \mathbb{C}_\infty \mid \forall_{i=1,\ldots,q} |z - b_i| = 1\}$$

Remarks

"$\Omega$ is like a tubular neighborhood of $T$"

GL$_2$(K$_\infty$)-translates of $\rho^{-1}(|e_0|)$ provide an atlas for $\Omega$.

On these charts use Laurent series type expansions to define (rigid) analytic functions on $\Omega$. 
(rigid) analysis on $\Omega$

Let $b_1, \ldots, b_q \in \mathcal{O}_\infty$ be representatives of $\mathcal{O}_\infty/\pi$.

**Proposition (reduction map)**

$\exists$ a (natural) $GL_2(K_\infty)$-equivariant map

$$\rho : \Omega \to |\mathcal{T}|$$

such that

$$\rho^{-1}(|e_0| \setminus \{\Lambda_0, \Lambda_1\}) = \{z \in \mathbb{C}_\infty \mid 1 < |z| < q\}$$

$$\rho^{-1}(\Lambda_0) = \{z \in \mathbb{C}_\infty \mid \forall i = 1, \ldots, q |z - b_i| = 1\}$$

**Remarks**

"$\Omega$ is like a tubular neighborhood of $\mathcal{T}$"
(rigid) analysis on $\Omega$

\[ K_\infty, O_\infty, \pi, F_q, C_\infty, \Omega \]

Let $b_1, \ldots, b_q \in O_\infty$ be representatives of $O_\infty/\pi$.

**Proposition (reduction map)**

\[ \exists \text{ a (natural) } GL_2(K_\infty)\text{-equivariant map} \]

\[ \rho : \Omega \to |T| \text{ such that} \]

\[ \rho^{-1}(|e_0| \setminus \{\Lambda_0, \Lambda_1\}) = \{z \in C_\infty \mid 1 < |z| < q\} \]

\[ \rho^{-1}(\Lambda_0) = \{z \in C_\infty \mid \forall i=1,\ldots,q \quad |z - b_i| = 1\} \]

**Remarks**

“$\Omega$ is like a tubular neighborhood of $T$”

$GL_2(K_\infty)$-translates of $\rho^{-1}(|e_0|)$ provide an atlas for $\Omega$. 
(rigid) analysis on $\Omega$

Let $b_1, \ldots, b_q \in \mathcal{O}_\infty$ be representatives of $\mathcal{O}_\infty/\pi$.

**Proposition (reduction map)**

$\exists$ a (natural) $GL_2(K_\infty)$-equivariant map

$$\rho: \Omega \to |\mathcal{T}|$$

such that

$$\rho^{-1}(|e_0| \setminus \{\Lambda_0, \Lambda_1\}) = \{z \in \mathbb{C}_\infty | 1 < |z| < q\}$$

$$\rho^{-1}(\Lambda_0) = \{z \in \mathbb{C}_\infty | \forall \quad |z - b_i| = 1\}$$

**Remarks**

“$\Omega$ is like a tubular neighborhood of $\mathcal{T}$”

$GL_2(K_\infty)$-translates of $\rho^{-1}(|e_0|)$ provide an atlas for $\Omega$.

On these charts use Laurent series type expansions to define (rigid) analytic functions on $\Omega$. 

$K_\infty, \mathcal{O}_\infty, \pi, \mathbb{F}_q, \mathbb{C}_\infty, \Omega$
Drinfeld modular forms

**From now on:** \( K_{\infty} := \mathbb{F}_q((\frac{1}{T})) \), \( \pi := \frac{1}{T} \).

For \( A := \mathbb{F}_q[T] \) and \( K = \text{Frac}(A) \) have

\[
GL_2(A) \hookrightarrow GL_2(K) \hookrightarrow GL_2(K_{\infty}).
\]
Drinfeld modular forms

From now on: \( K_\infty := \mathbb{F}_q((\frac{1}{T})) \), \( \pi := \frac{1}{T} \).

For \( A := \mathbb{F}_q[T] \) and \( K = \text{Frac}(A) \) have

\[
GL_2(A) \hookrightarrow GL_2(K) \hookrightarrow GL_2(K_\infty).
\]

For \( \Gamma \subset GL_2(A) \) a congruence subgroup:

Definition (Drinfeld modular form (Goss))

A Drinfeld modular form of weight \( k \) (and trivial type) for \( \Gamma \) is a rigid analytic function

\[
f : \Omega \rightarrow \mathbb{C}_\infty
\]

such that

(a) \( f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \right) = (cz + d)^k f(z) \) for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \).

(b) \( f \) has a Laurent series expansion at all cusps with vanishing principal part.
Results

One defines cusp forms (in the obvious way).
Have Hecke operators for prime ideals $0 \neq p \subset \mathbb{F}_q[T]$.
No known analog of a Petersson inner product.

Let $f$ be a Hecke eigenform with eigenvalues $a_p(f)$.

**Theorem (Goss)**

*The $a_p(f)$ are integral*

$K_f := K(\{a_p(f)\}_p)$ is finite over $K$. 

No known analog of a Petersson inner product.
Results

One defines cusp forms (in the obvious way).
Have Hecke operators for prime ideals $0 \neq p \subset \mathbb{F}_q[T]$
No known analog of a Petersson inner product.

Let $f$ be a Hecke eigenform with eigenvalues $a_p(f)$.

**Theorem (Goss)**

The $a_p(f)$ are integral

$K_f := K(\{a_p(f)\}_p)$ is finite over $K$.

**Theorem (B.)**

There is a strictly compatible system

$$\left( \rho_{f,\lambda} : \text{Gal}(\overline{K}/K) \rightarrow GL_1(\widehat{K_f}^\lambda) \right)_{\lambda \text{ finite}}$$

such that $\rho_{f,\lambda}(\text{Frob}_p) = a_p(f)$ for almost all $p$.

The sequence $(a_p(f))_p$ is given by a Hecke character.
Questions

There is no multiplicity one result!

Does multiplicity one hold for fixed weight?

Does it hold in weight 2?

Does it hold in weight 2 and for $\Gamma_0(p)$ with $p$ prime?

Possible implications for uniform boundedness of torsion points of Drinfeld modules of rank 2 over $K$. 

References
Questions

There is no multiplicity one result!
Does multiplicity one hold for fixed weight?
Does it hold in weight 2
Does it hold in weight 2 and for $\Gamma_0(p)$ with $p$ prime?

Possible implications for uniform boundedness of torsion points of Drinfeld modules of rank 2 over $K$.

There is no Ramanujan-Petersson conjecture
But each eigenvalue systems seems to have fixed weight.
What is the distribution of weights?
Questions

There is no multiplicity one result!
Does multiplicity one hold for fixed weight?
Does it hold in weight 2
Does it hold in weight 2 and for $\Gamma_0(p)$ with $p$ prime?
$\rightsquigarrow$ Possible implications for uniform boundedness of torsion points of Drinfeld modules of rank 2 over $K$.

There is no Ramanujan-Petersson conjecture
But each eigenvalue systems seems to have fixed weight.
What is the distribution of weights?

There may be $p$ not dividing the level $N$ of $f$ such that

$$\rho_{f,\lambda}(\text{Frob}_p) \neq a_p(f)$$

(because of non–ordinariness of modular curves of level $Nq$)
What happens at these $p$?
Harmonic cocycles

How to compute Drinfeld modular forms?
Let $M$ be a $K[GL_2(A)]$-module with $\dim_K(M)$ finite.
Harmonic cocycles

How to compute Drinfeld modular forms?
Let $M$ be a $K[GL_2(A)]$-module with $\dim_K(M)$ finite.

**Definition**
The $K$-vector space $C_{\text{har}}(\Gamma, M)$ of $M$-valued $\Gamma$-invariant harmonic cocycles is the set of maps

$$c : \text{Edge}(T) \to M : e \mapsto c(e),$$

such that:

1. For all edges $e$ one has $c(-e) = -c(e)$.
2. For all vertices $v$ one has $\sum_{e \to v} c(e) = 0$,
   where the sum is over all edges $e$ ending at $v$.
3. For all $\gamma \in \Gamma$ and $e \in \text{Edge}(T)$ one has $c(\gamma e) = \gamma c(e)$. 
Basic properties

Proposition (automatic cuspidality; Teitelbaum)

Given $M$ there exists a finite subset $Z$ of $\Gamma \backslash T$ such that any $c \in \mathcal{C}_{\text{har}}(\Gamma, M)$ vanishes on all edges $e$ not in a class of $Z$. 
Basic properties

Proposition (automatic cuspidality; Teitelbaum)
Given $M$ there exists a finite subset $Z$ of $\Gamma \backslash T$ such that any $c \in C_{\text{char}}(\Gamma, M)$ vanishes on all edges $e$ not in a class of $Z$.

Definition
A simplex $t \in \text{Vert}(T) \cup \text{Edge}(T)$ is $\Gamma$-stable iff

$$\text{Stab}_\Gamma(t) = \{1\}.$$  

Proposition
There are only finitely many $\Gamma$-stable orbits of simplices.

Theorem (Teitelbaum)
Suppose $\Gamma$ is $p'$-torsion free. Then:

- Any $\Gamma$-invariant harmonic cocycle is determined by its values on the $\Gamma$-stable orbits of edges.
- The only relations are those coming from $\Gamma$-stable vertices.
Remark:

The space $C_{\text{har}}(\Gamma, M)$ has an interpretation in terms of relative group homology. Let $\Gamma_v \subset \Gamma$ be the stabilizers of a set of representatives for the cusps. Then:

$$C_{\text{har}}(\Gamma, M) \cong H_1(\Gamma, \Gamma_v, M).$$
The residue map

\[ \mathcal{T}, \text{Edge}(\mathcal{T}), \Gamma, \text{Char}(\Gamma, M) \]

Recall:
A Drinfeld modular form \( f \) is a rigid analytic function on \( \Omega \).
\( \Omega \) is a tubular neighborhood of \( \mathcal{T} \) via \( \rho \).
The residue map

\[ \mathcal{T}, \text{Edge}(\mathcal{T}), \Gamma, \text{Char}(\Gamma, M) \]

Recall:
A Drinfeld modular form \( f \) is a rigid analytic function on \( \Omega \).
\( \Omega \) is a tubular neighborhood of \( \mathcal{T} \) via \( \rho \).
\( \rho^{-1} \) of the inner part of an edge \( e \) is an annulus \( A(e) \).
For \( f \) of weight 2 define

\[
\text{Res}_2 : \text{Edge}(\mathcal{T}) \to \mathbb{C}_\infty : e \mapsto \text{Res}_{A(e)}(fdz).
\]
The residue map

$\mathcal{T}$, $\text{Edge}(\mathcal{T})$, $\Gamma$, $C_{\text{har}}(\Gamma, M)$

Recall:
A Drinfeld modular form $f$ is a rigid analytic function on $\Omega$.
$\Omega$ is a tubular neighborhood of $\mathcal{T}$ via $\rho$.
$\rho^{-1}$ of the inner part of an edge $e$ is an annulus $A(e)$.
For $f$ of weight 2 define

$$\text{Res}_2 : \text{Edge}(\mathcal{T}) \to \mathbb{C}_\infty : e \mapsto \text{Res}_{A(e)}(fdz).$$

Theorem (Teitelbaum)

$\text{Res}_2$ defines an isomorphism from the $\mathbb{C}_\infty$ vector space of Drinfeld cusp forms of weight 2 and level $\Gamma$ to $C_{\text{har}}(\Gamma, K) \otimes_K \mathbb{C}_\infty$.

An analogous theorem holds in weight $k$ with $M \approx \text{Sym}^{k-2}$. 
On the proof of Teitelbaum’s theorem:
It suffices to prove it for $\Gamma = \Gamma(N)$ with $N \in \mathbb{F}_q[T] \setminus \mathbb{F}_q$.

*Injectivity:* Using a $\pi$-adic measure theory, Teitelbaum constructs an explicit section for

$$\text{Res}_2 : S_2(\Gamma(N)) \rightarrow C_{\text{har}}(\Gamma, \mathbb{C}_\infty).$$
On the proof of Teitelbaum’s theorem:
It suffices to prove it for $\Gamma = \Gamma(N)$ with $N \in \mathbb{F}_q[T] \setminus \mathbb{F}_q$.

Injectivity: Using a $\pi$-adic measure theory, Teitelbaum constructs an explicit section for

$$\text{Res}_2 : S_2(\Gamma(N)) \to \text{Char}(\Gamma, \mathbb{C}_\infty).$$

Surjectivity: Compute $\dim S_2(\Gamma(N))$ via Riemann-Roch and a canonical line bundle on $\Gamma(N) \setminus \Omega$.
Express $\dim \text{Char}(\Gamma(N), \mathbb{C}_\infty)$ as the number of stable orbits of edges minus the number of stable orbits of vertices.
Show that the dimensions are equal.
How to understand the quotient tree?

**Proposition**

The quotient tree $GL_2(\mathbb{F}_q[T]) \backslash \mathcal{T}$ is represented by the half line with vertices $\{\Lambda_i\}_{i \geq 0}$.

There are no $GL_2(\mathbb{F}_q[T])$-stable simplices of $\mathcal{T}$. For $i \geq 1$, the stabilizer of $\Lambda_{i+1}$ is strictly larger than that of $\Lambda_i$. 

References
How to understand the quotient tree?

**Proposition**

The quotient tree $GL_2(\mathbb{F}_q[T]) \backslash T$ is represented by the half line with vertices $\{\Lambda_i\}_{i \geq 0}$.

There are no $GL_2(\mathbb{F}_q[T])$-stable simplices of $T$. For $i \geq 1$, the stabilizer of $\Lambda_{i+1}$ is strictly larger than that of $\Lambda_i$.

**For general $\Gamma$:**

Consider $\Gamma \backslash T$ as a finite ‘covering’ of the above half line.

The stabilizers of simplices of the ‘cover’ have a similar monotonicity property as those of $GL_2(\mathbb{F}_q[T]) \backslash T$.

Stable simplices can only be found above $\Lambda_i$ for small $i$ (depending on $\Gamma$).

Using the above idea, one can show all ‘basic properties’ on harmonic cocycles we quoted.
References


J.-P. Serre, *Trees*