Jacquet-Langlands in weight 1

In this section we will prove the main theorem of our paper. Firstly, we introduce the concept of overconvergent modular forms, which will be useful for us. Let \( f \) be a normalized eigenform; either a classical modular form or an overconvergent modular form. We define the slope of \( f \) to be the normalized \( p \)-valuation of the eigenvalue of \( U_p \) acting on \( f \).

**Theorem 1 (Coleman)** Let \( f \) be a classical modular eigenform of weight \( k \). Then the normalized \( p \)-slope of \( f \) is less than or equal to \( k-1 \).

Conversely, if \( f \) is a \( p \)-adic overconvergent modular form of weight \( k \) with normalized slope strictly less than \( k-1 \), then \( f \) is a classical modular form.

We see that there is a slight asymmetry in this result; if an overconvergent modular eigenform of weight \( k \) has normalized slope exactly \( k-1 \), then it can be either classical or non-classical. There are examples of both; we will see this in weight 1 in Section 1. The question of telling whether an overconvergent form of weight \( k \) and slope \( k-1 \) is classical or not is raised by Coleman, and is still open in general.

We now state the main theorem of this paper; that the standard Jacquet-Langlands correspondence can be extended to weight 1.

**Theorem 2** Let \( N \) be a positive odd integer and let \( b \) be either 0 or 1. If \( f \in S^N_{\text{new}}((1-10^9)!) \), then there exists an overconvergent automorphic form \( f_* \) in \( S^N_{\text{new}}((1-10^9)!) \) with the same Hecke eigenvalues as \( f \) if \( b = 0 \), and there is no extra level structure at 2, and if \( b = 1 \) then \( G = 1 + n \). Conversely, if \( f_* \) is an overconvergent modular form of weight 1, then there exists an overconvergent modular form \( f \) of weight 1 with the same Hecke eigenvalues as \( f_* \).

We note that this version of the theorem is true in more generality, for other subgroups of \( D^* \) of finite index, but we will not need this for the section on approximation eigenforms.

**Approximating eigenforms**

In this section we will give an account of how to actually find approximations to overconvergent automorphic eigenforms of weight 1, using \( \Gamma_0 \)-programs. We also indicate how this method can be generalized to find other forms.

This method is a development of the work of Gouvêa and Mazur, where they find overconvergent \( 5 \)-adic modular eigenforms of weight 0 by iterating the action of the \( U_p \) operator. This in turn builds on the work of Atkin and O’Brien which pioneered this technique for finding \( p \)-adic eigenforms for \( p = 13 \).

On the automorphic side, we will consider the space of classical modular forms \((1-10^9)!) \); this can be checked to be one-dimensional, and it is in fact generated by the \( p \)-product \( f = q \eta(q)q^{2k} \), which is necessarily a Hecke eigenform. This has Fourier expansion at \( \infty \) given by

\[
f(q) = q \prod_{n=1}^{\infty} \left( 1 - q^{2n} + q^{11n} - q^{22n} + q^{33n} + q^{44n} + q^{55n} + q^{66n} + q^{77n} + q^{88n} + q^{99n} + q^{110n} + q^{111n} + q^{112n} \right),
\]

in particular, it has 11-slope 0, which shows that it is in the interesting case left open by the theory of Coleman, where the slope is \( k-1 \). By the Ramanujan-Petersson Conjecture, the Fourier coefficients \( a_n \) of \( f(q) \) satisfy \( |a_n| \leq 2 \).

We use this fact to find such \( f \), and then to find other forms of weight 1 with the same \( p \)-product. We note here that the methods we have outlined will also work for higher weights; let \( k \) be a positive integer. We can find any automorphic forms of slope 0 using exactly this procedure; these will be classical automorphic forms, so they will be determined by a tuple of polynomials. After subtracting these out, we will be able to find forms of higher slope, and this will enable us to approximate overconvergent automorphic eigenforms of weight 1.

**Future developments**

It would be interesting if one could find simultaneous eigenforms for \( U_p \), and for some other Hecke operators. We consider the action of the \( U \) operator, the analogue of a diamond operator in the classical setting.

\[
W_f = U(1) + \mathbb{C} \cdot f
\]

The action of this operator splits the 2-dimensional eigenspace for \( U_1 \) into two one-dimensional eigenspaces. Basis elements for each of these eigenspaces are eigenforms for all of the Hecke operators \( T_p \), for \( k \) is prime not equal to 2 or 11) and \( U_1 \).

We now choose a second random element \( \gamma \), and compute \( U_1^\gamma \). This will also be congruent to a linear combination of \( h \) and \( h \) modulo \( p \); with very high probability, these two linear combinations are linearly independent, and we can now use linear algebra to find \( h \) and \( h \) modulo \( p \) from them.

Finally, to find eigenforms for all of the Hecke operators, we consider the action of the \( W \) operator, which is defined to be

\[
W_f = (U(1) + \mathbb{C} \cdot f) \cdot W
\]

We can write down similar recurrence relations for each of the \( T_p \) and the \( U \).

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