Some Topics Around Serre's Modularity Conjecture

Research Seminar

Luxembourg, 2016

In this term's research seminar, we are studying some aspects of Serre's Modularity Conjecture, ranging from fundamentals to advanced subjects in the interest of (some of) the participants.

In particular, we are planning to cover:

(1) The classical Eichler-Shimura theorem. It is essential for understanding

- the construction of the Galois representation attached to a Hecke eigenform and
- the formulation of 'Serre weights' in terms of representations of $\mathrm{SL}_2(\mathbb{F}_p)$ (and generalisations thereof).

(2) The 'classical' modularity conjecture of Serre. The statement shall be explained in all details, as well as how Fermat's Last Theorem can be derived. Moreover, a sketch of the proof shall be presented (without going into full details for the deformation theory of Galois representations).

(3) The Buzzard-Diamond-Jarvis Conjecture and variants (already with a view towards the imaginary quadratic case); here we shall mostly be interested in the formulation.

(4) Generalisations of Serre's Modularity Conjecture to Bianchi modular forms. This shall include a usable treatment of Bianchi automorphic forms.

(5) Local Langlands. We just present the formulation for GL_2 over $\mathbb{Q}$.

1 Programme

1.1 17/02/2016: Introduction (Gabor)

Explain the topics.

1.2 24/02/2016: Classical Eichler-Shimura

The classical Eichler-Shimura isomorphism is

$$M_k(\Gamma) \oplus \overline{S_k(\Gamma)} \cong H^1(\Gamma, \mathbb{C}[X,Y]_{k-2})$$

for any weight $k \geq 2$ and any congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, where $\mathbb{C}[X,Y]_{k-2}$ is the $\mathrm{SL}_2(\mathbb{Z})$-module of homogeneous polynomials of degree $k-2$. It can also be seen as $\mathrm{Sym}^{k-2}(\mathbb{C}^2)$ (that's important for understanding Serre weights later on). The main point is that the Eichler-Shimura isomorphism respects Hecke operators. In order for this statement to make sense, one must define Hecke operators on cohomology via 'correspondences'. One can also see the group cohomology space $H^1(\Gamma, \mathbb{C}[X,Y]_{k-2})$ as the cohomology of the (open) modular curve $\Gamma\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half-plane.

The Eichler-Shimura isomorphism can either be written down explicitly (in terms of integrals), or it can be seen as the Hodge decomposition of de Rham cohomology (if you want to do this, please ask for references).

The speaker should present topics in a way (s)he prefers. Sources are, for instance, [Lan95], Theorem 1.1 (page 72; unfortunately only for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, but the main features are there), [Jara] or [Wiea].

Alternatively, a more general approach can be taken, for instance, by studying [Kno74]. However, Hecke operators as correspondences should not be dropped from the talk.

If one is more inclined to studying GL_2 over an imaginary quadratic field, one should also take a look at Taylor's thesis [Tay], starting on page 96, and the references to Harder therein. This can also be treated in a later talk on Bianchi modular forms.

If necessary, two sessions can be used for this talk.

1.3 The Galois representation attached to a Hecke eigenform

The construction of the Galois representation attached to a Hecke eigenform shall be sketched. It is (can be) based on the Eichler-Shimura isomorphism from the previous talk together with a (black box) comparison between singular cohomology and étale cohomology. The Eichler-Shimura relation (e.g. [DDT97], Theorem 1.29) shall be sketched (at least in weight 2) and the characteristic polynomial of Frobenius (at a place of good reduction) shall be derived from it.

No source I know of is really satisfactory. Here are some: [DDT97] (only weight 2), [DI95] (only weight 2), [Wieb] (very sketchy). Maybe a thorough internet search provides better ones.

1.4 Some local Galois representation theory

The notions of conductor and Weil-Deligne representations shall be explained. Possible sources are multiple, for instance [Wieb], [Jarb].

1.5 The formulation of the 'classical' modularity conjecture by Serre

I propose to follow Serre's original article [Ser87], complemented by Edixhoven's survey [Edi97]. Emphasis should be put on the definition of the Serre weight and the level. Moreover, a sketch how Serre's conjecture implies Fermat's Last Theorem shall be given.

1.6 Sketch of the proof of Serre's Modularity Conjecture

Following the fundamental paper of Khare and Wintenberger [KW09], the proof of Serre's Modularity Conjecture shall be sketched. The MathSciNet review of [KW09] can be helpful. If the speaker prefers, more modern proofs (relying on stronger modularity lifting theorems) can be presented (instead of the original one). In any case the role of the two main tools:

- modularity lifting theorems,
- embedding into a compatible system

shall be clearly pointed out.

1.7 Statement of the Buzzard-Diamond-Jarvis (BDJ) Conjecture

In this talk, the generalisation of Serre's modularity conjecture to Hilbert modular forms by Buzzard, Diamond and Jarvis [BDJ10] shall be presented. For simplicity, we can (but need not) restrict to real quadratic base fields F.

One main point is that the Jacquet-Langlands correspondence allows one to view Hilbert modular forms in the first cohomology of Shimura curves (over F), as well as their Galois representations. After having this, the formulation of the BDJ conjecture looks very similar to the classical Serre conjecture, when one views Serre weights as representations. This should be clearly explained. Of course, the recipe for Serre weights is more complicated than in the classical case, but the level is still given by the conductor.

1.8 Variations of BDJ

It turns out that the (weight part of the) BDJ conjecture is more accessible if one works with certain rank two unitary groups. This is explained in [BLGG13], for instance. The weight part (of the accordingly modified) BDJ conjecture is proved in [GLS15]. The main points should be sketched.

1.9 Generalisations to Bianchi modular forms

(to be filled in)

1.10 Local Langlands

(to be filled in)

References

[BDJ10] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, *On Serre's conjecture for mod ℓ Galois representations over totally real fields*, Duke Math. J. **155** (2010), no. 1, 105–161. MR 2730374 (2012k:11067)

[BLGG13] Thomas Barnet-Lamb, Toby Gee, and David Geraghty, *Serre weights for rank two unitary groups*, Math. Ann. **356** (2013), no. 4, 1551–1598. MR 3072811

[DDT97] Henri Darmon, Fred Diamond, and Richard Taylor, *Fermat's last theorem*, Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993), Int. Press, Cambridge, MA, 1997, pp. 2–140. MR 1605752 (99d:11067b)

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[Edi97] Bas Edixhoven, *Serre's conjecture*, Modular forms and Fermat's last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 209–242. MR 1638480

[GLS15] Toby Gee, Tong Liu, and David Savitt, *The weight part of Serre's conjecture for GL(2)*, Forum Math. Pi **3** (2015), e2, 52. MR 3324938

[Jara] Fazer Jarvis, *Topics in algebraic number theory 1*.

[Jarb] ———, *Topics in algebraic number theory 2*.

[Kno74] Marvin I. Knopp, *Some new results on the Eichler cohomology of automorphic forms*, Bull. Amer. Math. Soc. **80** (1974), 607–632. MR 0344454 (49 #9193)

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serre's modularity conjecture. I*, Invent. Math. **178** (2009), no. 3, 485–504. MR 2551763 (2010k:11087)

[Lan95] Serge Lang, *Introduction to modular forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 222, Springer-Verlag, Berlin, 1995, With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original. MR 1363488 (96g:11037)

[Ser87] Jean-Pierre Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230. MR 885783 (88g:11022)

[Tay] Richard Taylor, *On congruences between modular forms*, Available on the internet.

[Wiea] Gabor Wiese, *Computational arithmetic of modular forms*, Lecture Notes available on the internet.

[Wieb] ———, *Galois representations*, Lecture Notes available on the internet.