On the minimal ramification problem for semiabelian groups

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Winter School on Galois Theory

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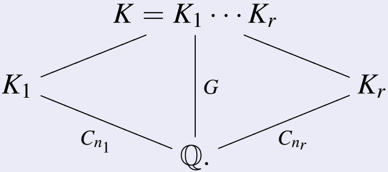

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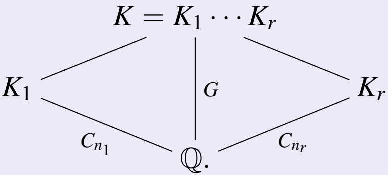

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A consequence of Minkowski's Theorem

$$\begin{array}{c} K \\ G \Big| \text{ tame} \\ \mathbb{Q} \end{array} \qquad \begin{array}{c} pO_K = \mathfrak{p}_1^e \cdots \mathfrak{p}_r^e \\ \Big| \\ p \end{array} \qquad \begin{array}{c} K \\ \Big| \\ K^{I_j} \\ \Big| \\ \mathbb{Q} \end{array} \qquad \begin{array}{c} \mathfrak{p}_j \\ \Big| \\ \mathfrak{p}_j \cap K^{I_j} \\ \text{unramified} \Big| \\ p \end{array}$$

- I_j- the inertia group $I(\mathfrak{p}_j/p)$.
- The groups $I_1, \ldots, I_r$ are conjugates.
- Each I_j is cyclic.

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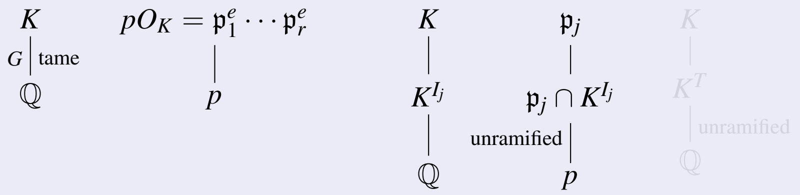

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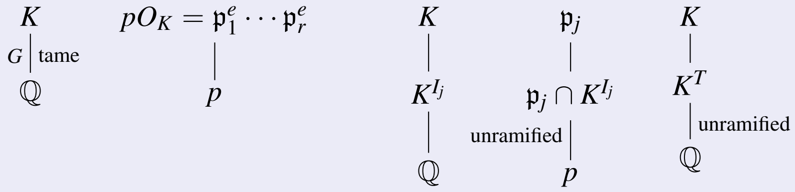

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Let $G \neq \{1\}$. Then:

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- A group G is semiabelian if and only if it is an epimorphic image of an iterated wreath product of cyclic groups $G_1, \ldots, G_r$:

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Corollary I - The Boston-Markin conjecture for nilpotent semiabelian groups

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Iterated wreath products of cyclic groups

Let $G = G_1 \wr (G_2 \wr \ldots \wr G_r)$ for non-trivial cyclic groups $G_1, \ldots, G_r$.

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- For n odd and $G = D_n$ the tame minimal ramification problem is equivalent to:

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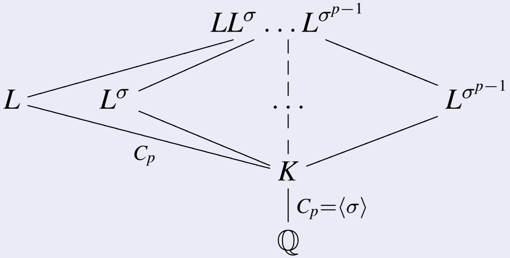

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Splitting Lemma, Kisilevsky-Sonn (2005)

Let K be a number field and r an integer. There is a number field $K_r \supseteq K$ such that for every prime $\mathfrak{p}$ of K that splits completely in K_r there is a C_{p^r}-extension of K that is ramified only at $\mathfrak{p}$ and is totally ramified there.

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There are number fields $K_1, \ldots, K_m$ containing K such that there is a C_{p^r}-extension of K that is ramified only at $\mathfrak{p}$ and is totally ramified there if and only if $\mathfrak{p}$ splits completely in at least one of the fields $K_1, \ldots, K_m$.

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Theorem II for p-groups (Recall)

Let G be a semiabelian p-group and $c := c(G)$. Then $wl(G) = c(G)$, i.e. there are cyclic groups $G_1, \ldots, G_c$ and an epimorphism $G_1 \wr (G_2 \wr \ldots \wr G_c) \to G$.

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Questions?

You can find both the slides and the paper at

`http://www-personal.umich.edu/~neftin/index.html`