PROBLEM SET
WINTER SCHOOL IN GALOIS THEORY
LUXEMBOURG, FEBRUARY 16, 2012

(1) Let G be a profinite group. Prove/Disprove the following implications:
  (a) G proabelian implies G abelian.
  (b) G prosolvable implies G solvable.

(2) Let $K = \mathbb{Q}^{\mathrm{sol}}$ be the maximal pro-solvable extension of $\mathbb{Q}$, i.e. the compositum of all finite Galois extensions with solvable Galois group. Show that K is not Hilbertian.

(3) Let F_n be the free (abstract) group on n letters. Show that F_n is residually finite, i.e. that the profinite completion map $F_n \to \hat{F}_n$ is injective.

(4) Let G be a profinite group. Prove that G is finitely generated (as a topological group) if and only if there exists d such that every finite continuous quotient of G is generated by d elements.

(5) Prove that $\mathrm{Gal}(\mathbb{F}_p)$ is projective.

(6) Let G be a profinite abelian group. Assume that

$$\{\mathrm{ord}(g) \mid g \in G\} \subseteq \mathbb{N} \cup \{\infty\}$$

is unbounded. Show that there exists $g \in G$ of infinite order.

Hint: Prove that most g (either in the Haar measure sense or in the Baire category sense) have infinite order.

(7) What is the profinite completion of F_X, where $|X| = \aleph_0$?

(8) Show that $\mathbb{Q}_p$ is not PAC.

(9) Show that any finite abelian group occurs as a Galois group over $\mathbb{Q}^{\mathrm{ab}}$, over $\mathbb{C}(x)$, and over $\mathbb{Q}$.

(10) Show that a closed subgroup of a projective group is projective. (This might be difficult...)

(11) Over a Hilbertian field, realize the groups S_n and $(\mathbb{Z}/2\mathbb{Z})^{\aleph_0}$ as Galois groups.

(12) Show that every infinite profinite group contains a non-closed subgroup.

(13) Show that any finite index subgroup of $\mathbb{Z}_p$ is open.

(14) Give an example of a profinite group Γ where (13) is wrong, i.e. for which there exists a non-close finite index subgroup.

(15) Consider the embedding problem

$$\begin{array}{ccc} & & \mathrm{Gal}(\mathbb{Q}) \\ & & \downarrow \\ \mathbb{Z}/4\mathbb{Z} & \twoheadrightarrow & \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q}). \end{array}$$

For which $d \in \{3, 5\}$ this embedding problem is solvable?

(16) For $f \in M_k(\Gamma_1(N))$ prove that $f(dz) \in M_k(\Gamma_1(dN))$.

(17) Prove that the sequence

$$1 \longrightarrow \Gamma(N) \longrightarrow \mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 1$$

is exact.

(18) Prove that $\tau(n) \equiv \sigma_{11}(n) \mod 691$.

Hint:

k	2	4	6	8	10	12
$-\frac{B_k}{2}$	$-\frac{1}{24}$	$\frac{1}{240}$	$-\frac{1}{504}$	$\frac{1}{480}$	$-\frac{1}{264}$	$\frac{691}{65520}$

(19) Prove that $\Delta = q\prod_n (1-q^n)^{24}$ is a modular form.

Hint: Consider $\frac{\Delta'(z)}{\Delta(z)}$ and use the almost modularity of G_2.