B-representations and regular G-rings
Talk in the Forschungsseminar on p-adic Galois Representations
Wintersemester 2008/09

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2008-10-23

Contents

1 B-representations

Let G be a topological group and B a topological commutative ring with continous G-action, i.e. for all $g \in G, b_1, b_2 \in B$ we have $g(b_1 + b_2) = g(b_1) + g(b_2)$ and $g(b_1 * b_2) = g(b_1) * g(b_2)$.

Example 1.1 $B = L \supset K$ a Galois extention of fields, $G = \mathrm{Gal}(L/K)$.

Definition 1.2 *A B-representation X of G is a B-modul of finite type X equipped with a semi-linear continous G-action. Semi-linear means that for all $g \in G, b \in B, x, x_1, x_2 \in X$ we have $g(x_1 + x_2) = g(x_1) + g(x_2)$ and $g(bx) = g(b)g(x)$.*

If $B = \mathbb{F}_p$, we call it a mod-p-representation.
If $B = \mathbb{Q}_p$, we call it a p-adic representation.
If G acts trivial on B, we call it a linear representation.

Definition 1.3 *A B-representation X of G is called free if the underlying B-modul X is free.*

Definition 1.4 *A free B-representation X of G is called trivial if one of the equivalent conditions hold:*

(a) There is a Basis of X over B in X^G.

(b) $X \cong B^d$ *as* G*-modules with the natural* G*-action on* B^d*.*

Let us look at three key examples:

Example 1.5 If $F \subseteq B^G$ is a subfield, and V an F-representation of G and let G act on $X := B \otimes_F V$ as $g(b \otimes x) = g(b) \otimes g(x)$. Then X is a free B-representation (free, since V was just a vector space over F).

Example 1.6 Let $\mathbb{Z}_p(1) = T_p(\mathbb{G}_m/\mathbb{Q}_p) = \varprojlim \mu_{p^n}(\overline{\mathbb{Q}}_p)$ be the p-adic Tate-module of the multiplicative group and $\mathbb{Q}_p(1) = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Set $\mathbb{Q}_p(-1) = \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbb{Q}_p(1), \mathbb{Q}_p)$ where $G_{\mathbb{Q}_p}$ denotes the absolute Galois group of $\mathbb{Q}_p$ and define $\mathbb{Q}_p(i) = \mathbb{Q}_p(1)^{\otimes i}$ for all $i \in \mathbb{Z}$. Then the $\mathbb{Q}_p(i)$ are $G_{\mathbb{Q}_p}$-modules. It is a result of Tate that with $B := \hat{\overline{\mathbb{Q}}}_p =: C_p =: C$ the obtained B-representations $\mathbb{Q}_p(i) \otimes_{\mathbb{Q}_p} B$ are trivial if and only if $i = 0$. In later talks of the seminar we will construct a ring B_{dR} such that $\mathbb{Q}_p(i) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ is trivial for all i.

Example 1.7 Let $E/\mathbb{Q}_p$ be an elliptic curve and set $V_p = T_p(E/\mathbb{Q}_p) \otimes \mathbb{Q}_p$. Then $V_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ is trivial. In yet later talks we will construct a subring B_{cris} of B_{dR} such that $V_p \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$ is trivial if E has good reduction.

Our first goal is an interpretation of equivalence classes of free B-representations of G of rank d as cohomology classes in $H^1_{\mathrm{cont}}(G, \mathrm{GL}_d(B))$, where two free B-representations are equivalent if they only differ by a change of basis.
Before we give the proposition we recall some facts about group cohomology: Let M be any (multiplicativly written) topological G-group. Then $H^0_{\mathrm{cont}}(G, M) = M^G$ and $H^1_{\mathrm{cont}}(G, M) = Z^1(G, M)/\sim$ where $Z^1(G, M) = \{f : G \to M \text{ continous} \mid f(g_1 * g_2) = f(g_1) * (g_1 f(g_2))\}$ and $f_1 \sim f_2$ if there is an $a \in M$ such that $f_1(g) = a^{-1} f_2(g) a(g)$ for all $g \in G$.
Thus $H^1_{\mathrm{cont}}(G, M)$ is a pointet set with the distinguished point being the class of the cocylce $f(g) \equiv 1$.
We recall the famous theorem Hilbert 90:

Proposition 1.8 *Let* L/K *be a Galois extention of fields. Then*

(a) $H^1(\mathrm{Gal}(L/K), L) = 0$

(b) $H^1(\mathrm{Gal}(L/K), L^\star) = 1$

(c) $H^1(\mathrm{Gal}(L/K), \mathrm{GL}_d(L)) = 1$

Now we can formulate the proposition:

Proposition 1.9 *There is a natural bijection between equivalence classes of free* B*-representations of* G *of rank* d *and* $H^1_{\mathrm{cont}}(G, \mathrm{GL}_d(B))$*, denoted by* $X \mapsto [X]$*. Moreover* X *is trivial if and only if* $[X]$ *is the distinguished point of* $H^1_{\mathrm{cont}}(G, GL_d(B))$*.*

Remark 1.10 The proposition and Hilbert 90 imply that for L/K a Galois extention any L-representation of $\mathrm{Gal}(L/K)$ is trivial.

PROOF: Let X be a free B-representation of G of rank d and $\{e_1, \ldots, e_d\}$ a basis of X/B. Write $g(e_1, \ldots, e_d) = (e_, \ldots e_d)A(g)$. Then we get a map $\alpha : G \to \mathrm{Mat}_d(B), g \mapsto A(g)$. We have to check the following for claims:

(a) $\alpha \in Z^1_{\text{cont}}(G, \text{Mat}_d(B))$

(b) $A(g) \in \text{GL}_d(B)$ for all $g \in G$

(c) If $\{e'_1, \ldots, e'_d\}$ is another basis of X/B and P is the basechange matrix, define $A'(g)$ as above, then $A'(g) = P^{-1}A(g)g(P)$.

(d) Given $\alpha \in Z^1_{\text{cont}}(G, \text{GL}_d(B))$ there is a unique semi-linear action of G on $X = B^d$ such that $[X] = \bar{\alpha}$.

These claims are all easy to check.

∎

2 Regular (F, G)-rings

Assume now $E := B^G$ is a field and let F be a closed subfield of E. Denote by $\mathbf{Rep}_F(G)$ the category of F-representations of G. If B is a domain, then the G-action on B extends to $C := \text{Frac}(B)$ as $g(\frac{b_1}{b_2}) := \frac{g(b_1)}{g(b_2)}$.

Definition 2.1 *B is (F, G)-regular, if*

(a) B is a domain.

(b) $B^G = C^G$.

(c) If $b \neq 0$ and $Gb \subseteq Fb$ we have $b \in B^\star$.

As Hélène explained, $\mathbf{Rep}_F(G)$ is a neutral Tannakian category.

Definition 2.2 *A sub-Tannakin category of $\mathbf{Rep}_F(G)$ is a strictly full subcategory $\mathcal{C}$ wich is closed under direct sums, tensor products and duals and contains the unit representation.*

Definition 2.3 *An F-representation V of G is called B-admissible, if $B \otimes_F V$ is trivial. Let $\mathbf{Rep}^B_F(G)$ denote the full subcategory of B-admissible F-representations of G.*

We define a functor $\mathbf{Rep}_F(G) \to \text{Vec}_E : V \mapsto D_B(V) := (B \otimes_F V)^G$ and for each V a map $\alpha_V : B \otimes_E D_B(B) \to B \otimes_F V : \lambda \otimes x \mapsto \lambda x$ for $\lambda \in B, x \in D_B(V)$. α_V is a B-linear G-equivariant map, where G acts on $B \otimes_E D_B(V)$ as $g(\lambda \otimes x) = g(\lambda) \otimes x$.
This functor maps objects, wich are hard to understand (F-representations of G) to objects, wich are easy to understand (vector spaces over the field E). The next theorem is the main theorem of my talk, wich showes some properties of this functor, once we have assume B to be (F, G)-regular.

Theorem 2.4 *Assume B to be (F, G)-regular. Then*

(1) For all $V \in \mathbf{Rep}_F(G)$ we have α_V is injective and $\dim_E D_B(V) \leqslant \dim_F(V)$. Moreover $\dim_E D_B(V) = \dim_F(V)$ iff α_V is an isomorphism iff V is B-admissible.

(2) $\mathbf{Rep}^B_F(G)$ is a sub-Tannakian category and D_B restricted to $\mathbf{Rep}^B_F(G)$ is an exact and faithfull tensor functor.

For the second part, we have to show:

(a) D_B preserves exact sequences.

(b) $V \neq 0$ implies $D_B(V) \neq 0$ (this is clear)

(c) If V is admissible, then subs and quotiens of V are also admissible.

(d) $D_B(F) \cong E$ (this is clear)

(e) If V_1, V_2 are admissible, then $V_1 \otimes V_2$ is admissible and $D_B(V_1) \otimes D_B(V_2) \cong D_B(V_1 \otimes V_2)$.

(f) If V is admissible, then $V^\star$ is admissible and $D_B(V^\star) \cong (D_B(V))^\star$.

Proof: (1) First we proof that α_V is injective: Let $C = \mathrm{Frac}(B)$. Since B is (F,G)-regular, we have $C^G = B^G = E$. So we have the commutative diagram:

$$\begin{tikzcd}
B \otimes_E D_B(V) \arrow[r, "\alpha_V"] \arrow[d, hook] & B \otimes_F V \arrow[dd, hook] \\
B \otimes_E D_C(V) \arrow[d, hook] & \\
C \otimes_E D_C(V) \arrow[r, "\alpha_V"] & C \otimes_F V
\end{tikzcd}$$

and hence for the injectivity we can restrict to the case $B = C$ a field. What we have to show is that given $h \geqslant 1, x_1, \ldots, x_h \in D_B(V)$ linear independent over E they remain linear independet over B. We use induction. For $h = 1$ there is nothing to show. So let $h \geqslant 2$ and assume

$$\sum_{i=1}^{h} \lambda_i x_i = 0, \lambda_i \in B.$$

Since B is a field, we can assume $\lambda_h = -1$, so

$$x_h = \sum_{i=1}^{h-1} \lambda_i x_i.$$

But since all x_i are G-invariant, we have for all $g \in G$:

$$\sum_{i=1}^{h-1} \lambda_i x_i = x_h = g(x_h) = g(\sum_{i=1}^{h-1} \lambda_i x_i) = \sum_{i=1}^{h-1} g(\lambda_i) x_i.$$

So by induction we have $\lambda_i = g(\lambda_i)$ for all $g \in G$ hence $\lambda_i \in B^G = E$, wich is a contradiction. Therefore α_V is injective.

For the second assertion of (1): If α_V is an isomorphism, then $\dim_E D_B(V) = \dim_F V = \mathrm{rank}_B\, B \otimes_F V$. Conversly, if $\dim_E D_B(B) = \dim_F(V)$, we choose bases $\{v_1, \ldots, v_d\}$ of V/F and $\{e_1, \ldots, e_d\}$ of $D_B(V)/E$ and write

$$e_j = \sum_{i=1}^{d} b_{ij} v_i.$$

The matrix (b_{ij}) is called period matrix, since α_V is injective, we have $b = \det((b_{ij})) \neq 0$. We have to show that b is in $B^\star$. Let $\det V = \bigwedge_F^d V = Fv$ with $v = v_1 \wedge \cdots \wedge v_d$ and

$g(v) = \eta(g)v$ with $\eta : G \to F^\star$ a character. For $e = e_1 \wedge \cdots \wedge e_d \in \bigwedge_E^d D_B(V)$ we have $e = bv$.
But also for all $g \in G$: $bv = e = g(e) = g(bv) = g(b)\eta(g)v$. Therefore $g(b) = \eta(g)^{-1}b$ for all $g \in G$. Since B is (F,G)-regular, we have $b \in B^\star$.
For the second equivalence: V is B-admissible is by definition equivalent to the existence of a B-basis $\{x_1, \dots, x_d\}$ od $B \otimes_F V$ such that $x_i \in D_B(V)$, therefore this is equivalent to α_V being surjective. Since α_V is always injective, this is equivalent to α_V being an isomorphism.

(2) Let V be admissible and V' a sub-representation. Then we obtain an exact sequence of F-vectorspaces:

$$0 \to V' \to V \to V'' := V/V' \to 0$$

tensoring with B gives the exact sequence:

$$0 \to B \otimes_F V' \to B \otimes_F V \to B \otimes_F V'' \to 0$$

and since taking G-invariance is an left-exact functor, we obtain:

$$0 \to D_B(V') \to D_B(V) \to D_B(V'')$$

and we have to show, that the map from $D_B(V)$ to $D_B(V'')$ is also surjective. Let $d = \dim_F V, d' = \dim_F V', d'' = \dim_F V''$. Then from (1) we have, since V is admissible $\dim_E D_B(V) = d$ and $\dim_E D_B(V') \leqslant d'$ and $\dim_E D_B(V'') \leqslant d''$ but since $d = d' + d''$, this implies that the map from $D_B(V)$ to $D_B(V'')$ is also surjective. Hence we prooved (a) and (c).

For (d) we have the commutative diagramm:

$$\begin{tikzcd} (B \otimes_F V_1) \otimes (B \otimes_F V_2) \arrow[r, equal, "\Sigma"] & B \otimes_F (V_1 \otimes_F V_2) \\ D_B(V_1) \otimes_E D_B(V_2) \arrow[u, hook] \arrow[r, dotted, "\sigma"] & D_B(V_1 \otimes_F V_2) \arrow[u, hook] \end{tikzcd}$$

σ is induced by Σ and is therefore clearly injective. But since V_1 and V_2 are admissible, we have $\dim_E(D_B(V_1) \otimes_E D_B(V_2)) = \dim_B(B \otimes_F (V_1 \otimes_F V_2)) \geqslant \dim_E D_B(V_1 \otimes_F V_2)$, hence σ is an isomorphism.
For (e) we have to show that if V is admissible, so is $V^\star$. The case $\dim_F V = 1$ is easy: If $V = Fv$, then $D_B(V) = E(b \otimes v)$, $V^\star = Fv^\star$, $D_B(V^\star) = E(b^{-1} \otimes v^\star)$. If $\dim_F V \geqslant 1$: We observed in the proof of (1) that $\det(\alpha_{V^\star}) = \alpha_{\det V^\star}$. Hence V admissible $\Rightarrow \det V$ admissible $\Rightarrow \det V^\star$ admissible $\Rightarrow V^\star$ admissible.
Finally we have to proof $D_B(V^\star) \cong D_B(V)^\star$. We have a commutative diagramm:

$$\begin{tikzcd} B \otimes_F V^\star \arrow[r, "\cong"] & (B \otimes_F V)^\star \\ D_B(V^\star) \arrow[u, hook] \arrow[r, dotted, "\tau"] & D_B(V)^\star \arrow[u, hook] \end{tikzcd}$$

Let $f \in D_B(V^\star), t \in B \otimes_F V, g \in G$. Then $f(t) = g \circ f(t) = g(f(g^{-1}(t)))$. If we assume $t \in D_B(V)$, then $g(f(t)) = f(t)$, hence $f(t) \in E$. Therefor we get the induced homomorphism τ. From the diagramm τ is clearly injective. But since the dimensions of $D_B(V)$ and $D_B(V^\star)$ are equal, τ is an isomorphism. ∎

3 Potentially semi-stable l-adic representations

We try now to give an alternative description of potentially semi-stable l-adic representations. This part is quite sketchy.
For $E/\mathbb{C}$ an elliptic curve, you can find a $q \in \mathbb{C}^\star$ with $|q| < 1$ such that $E(\mathbb{C}) \cong \mathbb{C}^\star/q^{\mathbb{Z}}$. For K a local field with residue characteristic $p > 0$ and E/K an elliptic curve with multiplicative reduction, a result of Tate shows that $E(K^{\mathrm{sep}}) \cong (K^{\mathrm{sep}})^\star/q^{\mathbb{Z}}$ for some $q \in \mathfrak{m}_K$ the maximal ideal of O_K. Hence $E_{l^n-\mathrm{tors}}(K^{\mathrm{sep}}) =< \zeta_{l^n}, q^{\frac{1}{l^n}} >$ where $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ acts on ζ_{l^n} via a cyclotomic character χ_{cycl} and on $q^{\frac{1}{l^n}}$ as $\sigma(q^{\frac{1}{l^n}}) = q^{\frac{1}{l^n}}\zeta_{l^n}^{i_\sigma}$. Therefore G_K acts on $T_l(E)$ via $\begin{pmatrix} \chi_{\mathrm{cycl}} & \star \\ 0 & 1 \end{pmatrix}$. Set $V := T_l(E)(-1) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. We have

$$0 \to \mathbb{Q}_l \to V \to \mathbb{Q}_l(-1) \to 1$$

and G_K acts on V via $\begin{pmatrix} 1 & \star \\ 0 & \chi_{\mathrm{cycl}}^{-1} \end{pmatrix}$. Write $\mathbb{Q}_l(-1) = \mathbb{Q}_l t^{-1}$ and let $u \in V$ be any lift of t^{-1} and define $B_l := \mathbb{Q}_l[u]$ where we let G_K act on $1, u, u^2, \ldots$ via

$$\begin{pmatrix} 1 & \star & \star & \star & \cdots \\ 0 & \chi_{\mathrm{cycl}}^{-1} & \star & \star & \cdots \\ 0 & 0 & \chi_{\mathrm{cycl}}^{-2} & \star & \cdots \\ 0 & 0 & 0 & \chi_{\mathrm{cycl}}^{-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and we have a map $N : B_l \to B_l(-1) := B_l \otimes \mathbb{Q}_l(-1) : g(u) \mapsto g'(u) \otimes t^{-1}$.
Our aim is the description of potentially semi-stable l-adic representation. We want to give a functor

$$\mathbf{Rep}_{\mathbb{Q}_l}(G_K) \to \mathbf{Rep}_K(WD)$$

where $\mathbf{Rep}_K(WD)$ is the category of Weil-Deligne-Representations. The objects of $\mathbf{Rep}_K(WD)$ are pairs (D, N) where D is a $\mathbb{Q}_l$-vectorspace with action of G_K such that I_K acts trivial after a finite extention and $N : D \to D(-1)$ is a nilpotent endomorphism. The morphisms of $\mathbf{Rep}_K(WD)$ between (D, N) and (D', N') are $\mathbb{Q}_l$-linear endomorphisms $\eta : D \to D'$, who commute with G_K and such that the diagramm

$$\begin{tikzcd} D \arrow[r, "\eta"] \arrow[d, "N"] & D' \arrow[d, "N'"] \\ D(-1) \arrow[r, "\eta(-1)"] & D'(-1) \end{tikzcd}$$

commutes.

Theorem 3.1 *The map*

$$V \mapsto \lim_{\leftarrow H \subseteq I_K \, open} (B_l \otimes_{\mathbb{Q}_l} V)^H$$

defines an equivalence of categories

$$\mathbf{Rep}_{\mathbb{Q}_l}^{p.st.}(G_K) \to \mathbf{Rep}_K(WD)$$

between the category of potentially semi-stable $\mathbb{Q}_l$-representations of G_K and the category of Weil-Deligne-representations over K with quasi-inverse

$$(D, N) \mapsto V_l(D, N) := \mathrm{Kern}(N : B_l \otimes_{\mathbb{Q}_l} D \to (B_l \otimes_{\mathbb{Q}_l} D)(-1)).$$

One main ingredients of the proof is the observation that B_l is $(\mathbb{Q}_l, H)$-regular and the Theorem 2.4.

References

[FO] **Jean-Marc Fontaine, Yi Ouyang** Theory of p-adic Galois representations, *Preprint*